# Lattices generated by orbits of flats under finite affine-symplectic groups

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**Abstract** Let $ASG(2\nu, \mathbb{F}_q)$ be the $2\nu$-dimensional affine-symplectic space over the finite field $\mathbb{F}_q$ and let $ASp_{2\nu}(\mathbb{F}_q)$ be the affine-symplectic group of degree $2\nu$ over $\mathbb{F}_q$. For any two orbits $M'$ and $M''$ of flats under $ASp_{2\nu}(\mathbb{F}_q)$, let $\mathcal{L}'$(resp. $\mathcal{L}''$) be the set of all flats which are joins ( resp. intersections) of flats in $M'$ (resp. $M''$) such that $M'' \subseteq \mathcal{L}'$ (resp. $M' \subseteq \mathcal{L}''$) and assume the join (resp. intersection) of the empty set of flats in $ASG(2\nu, \mathbb{F}_q)$ is $\emptyset$ (resp. $\mathbb{F}_q^{(2\nu)}$). Let $\mathcal{L} = \mathcal{L}' \cap \mathcal{L}''$. By ordering $\mathcal{L}', \mathcal{L}''$ ,$\mathcal{L}$ by ordinary or reverse inclusion, six lattices are obtained. This article discusses the relations between different lattices, and computes their characteristic polynomial.

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## 1. Introduction

We first recall some terminologies and definitions about finite posets lattices[1,2].

Let $P$ be a poset. For $a, b \in P$, we say *a covers b*, denoted by $b < \cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If $P$ has the minimum (resp.maximum) element, then we denote it by 0 (rep.1) and say that $P$ is a finite poset with 0(resp.1). Let $P$ be a finite poset with 0. By a rank function on $P$, we mean a function $r$ from $P$ to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b < \cdot a$.

A poset $P$ is said to be a lattice if both $a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$ exist for any two elements $a, b \in P$. Let $P$ be a finite lattice with 0. By an atom in $P$, we mean an element in $P$ *covering* 0. We say $P$ is atomic if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice $P$ is said to be a geometric lattice if $P$ admits a rank function $r$ satisfying

$$r(a \wedge b) + r(a \vee b) \le r(a) + r(b), \ \forall \ a, b \in P.$$

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Let $P$ be a finite poset with 0 and 1. The polynomial

$$\mathcal{X}(P,t) = \sum_{a \in P} \mu(0,a) t^{r(1)-r(a)}$$

is called the characteristic polynomial of $P$, $\mu$ is the *Möbius* function, $r$ is the function on $P$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two lattices. If there exists a bijection $\sigma$ from $\mathcal{L}$ to $\mathcal{L}'$ such that

$$\sigma(a \vee b) = \sigma(a) \vee \sigma(b), \quad \sigma(a \wedge b) = \sigma(a) \wedge \sigma(b), \quad \forall a, b \in \mathcal{L},$$

then $\sigma$ is said to be an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$. In this case we call $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, denoted by $\mathcal{L} \simeq \mathcal{L}'$. It is well know that two isomorphic lattices have the same characteristic polynomial.

In the following we introduce the concepts of affine-symplectic spaces. Notation and terminology will be adopted from Wan's book[3].

Suppose $\mathbb{F}_q$ is a finite field with $q$ elements, where $q$ is a prime power. Let $\mathbb{F}_q^{(2\nu)}$ be the $2\nu$-dimensional row vector space over $\mathbb{F}_q$ and let

$$K_\nu = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The symplectic group of degree $2\nu$ over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrices $T$ over $\mathbb{F}_q$ satisfying $TK_\nu T^t = K_\nu$. The vector space $\mathbb{F}_q^{(2\nu)}$ together with the right multiplication action of $Sp_{2\nu}(\mathbb{F}_q)$ is called the $2\nu$-dimensional symplectic space over $\mathbb{F}_q$. Let $P$ be an $m$-dimensional subspace of $\mathbb{F}_q^{(2\nu)}$, denote also by $P$ an $m \times 2\nu$ matrix of rank $m$ whose rows span the subspace $P$ and call the matrix $P$ a matrix representation of the subspace $P$. An $m$- dimensional subspace $P$ is said to be of type $(m,s)$ if $PK_\nu P^t$ is of rank $2s$. It is known that subspaces of type $(m,s)$ exist if and only if $2s \leq m \leq \nu + s$.

A coset of $\mathbb{F}_q^{(2\nu)}$ relative to a subspace $P$ of type $(m,s)$ is called a $(m,s)$-flat. The dimension of a flat $U + x$ is defined to be the dimension of the subspace $U$, denoted by $\dim(U + x)$. In particular, $(0,0)$-flats are points, $(1,0)$-flats are lines. A flat $F_1$ is said to be incident with a flat $F_2$, if $F_1$ contains or is contained in $F_2$. The point set $\mathbb{F}_q^{(2\nu)}$ with all the flats and the incidence relation among them defined above is said to be the $2\nu$-dimensional affine-symplectic space, which is denoted by $ASG(2\nu, \mathbb{F}_q)$.

Let $\mathcal{L}_k$ denote the set of all flats in affine space $AG(k, \mathbb{F}_q)$ including the empty set. If we partially order $\mathcal{L}_k$ by reverse inclusion, then $\mathcal{L}_k$ is a lattice (see[4]).

Let $F_1, F_2$ be two flats in $ASG(2\nu, \mathbb{F}_q)$. The set of points belonging to both $F_1$ and $F_2$ is called the intersection of $F_1$ and $F_2$, which is denoted by $F_1 \cap F_2$. It follows that the intersection of all flats containing two given flats $F_1$ and $F_2$ is the minimum flat containing both $F_1$ and $F_2$, which is called the join of $F_1$ and $F_2$ and is denoted by $F_1 \cup F_2$.

The set of matrices of the form $\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}$, where $T \in Sp_{2\nu}(\mathbb{F}_q)$ *and* $v \in \mathbb{F}_q^{(2\nu)}$, forms a group under matrix multiplication, which is denoted by $ASp_{2\nu}(\mathbb{F}_q)$ and called the affine-symplectic group of degree $2\nu$ over $\mathbb{F}_q$. Define the action of $ASp_{2\nu}(\mathbb{F}_q)$ on $ASG(2\nu, \mathbb{F}_q)$ as follows:

$$ASG(2\nu, \mathbb{F}_q) \times ASp_{2\nu}(\mathbb{F}_q) \to ASG(2\nu, \mathbb{F}_q)$$

$$\left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) \mapsto xT + v.$$

The above action induces an action on the set of flats in $ASG(2\nu, \mathbb{F}_q)$; i.e., a flat $P + x$ is carried by

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in ASG(\mathbb{F}_q)$$

into the flat $PT + (xT + v)$. It is known that $(m, s)$-flats exist if and only if $2s \leq m \leq \nu + s$, and that the set of flats of the same type form an orbit under $ASp_{2\nu}(\mathbb{F}_q)$. Denote the orbit of $(m, s)$-flats by $M(m, s; 2\nu)$.

For any two orbits $M(m_1, s_1; 2\nu)$ and $M(m_2, s_2; 2\nu)$ of flats under $ASp_{2\nu}(\mathbb{F}_q)$, let $\mathcal{L}'(m_1, s_1; 2\nu)$ (resp.$\mathcal{L}''(m_2, s_2; 2\nu)$) and $M(m_1, s_1; 2\nu) \subset \mathcal{L}'(m_1, s_1; 2\nu)$ (resp.$M(m_2, s_2; 2\nu) \subset \mathcal{L}''(m_2, s_2; 2\nu)$) and assume the join (resp.intersection) of the empty set of flats in $ASG(2\nu, \mathbb{F}_q)$ is $\emptyset$ (resp.$\mathbb{F}_q^{(2\nu)}$). Let $\mathcal{L}(m_1, s_1; m_2, s_2; 2\nu) = \mathcal{L}'(m_1, s_1; 2\nu) \cap \mathcal{L}''(m_2, s_2; 2\nu)$. By ordering $\mathcal{L}'(m_1, s_1; 2\nu)$, $\mathcal{L}''(m_2, s_2; 2\nu)$, $\mathcal{L}(m_1, s_1; m_2, s_2; 2\nu)$ by ordinary or reverse inclusion, six posets are obtained, denoted by $\mathcal{L}'_O(m_1, s_1; 2\nu)$, $\mathcal{L}'_R(m_1, s_1; 2\nu)$, $\mathcal{L}''_O(m_2, s_2; 2\nu)$, $\mathcal{L}''_R(m_2, s_2; 2\nu)$, $\mathcal{L}_O(m_1, s_1; m_2, s_2; 2\nu)$, $\mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$, respectively. In this article, we discusses the inclusion relations between different lattices, and compute the characteristic polynomial of these lattices.

The results on the lattices generated by orbits of subspaces under finite classical groups can be found in Wang and Feng[4], Gao and You[5], Huo Liu and Wan[6-8], Huo and Wan[9], Orlik and Solomon[10], Wang and Guo[11],Guo and Nan[12].

## 2. Preliminaries

We begin with six useful propositions.

**Proposition 2.1 [3,13].** For any two flats $F_1 = V_1 + x_1$ and $F_2 = V_2 + x_2$

in $ASG(2\nu,\mathbb{F}_q)$, where $V_1$ and $V_2$ are subspaces of $\mathbb{F}_q^{(2\nu)}$, $x_1,x_2\in\mathbb{F}_q^{(2\nu)}$. The following (i) and (ii) hold :

(i) $F_1\cap F_2\neq\emptyset$ if and only if $x_2-x_1\in V_1+V_2$, and if $F_1\cap F_2\neq\emptyset$, then $F_1\cap F_2=V_1\cap V_2+x$, where $x\in F_1\cap F_2$.

(ii) $F_1\cup F_2=(V_1+V_2+<x_2-x_1>)+x_1$, and

$$\dim F_1+\dim F_2=\begin{cases}\dim(F_1\cup F_2)+\dim(F_1\cap F_2), & \text{if } F_1\cap F_2\neq\emptyset,\\ \dim(F_1\cup F_2)+\dim(F_1\cap F_2)-1, & \text{if } F_1\cap F_2=\emptyset.\end{cases}$$

**Proposition 2.2**[12]. Let $2s\le m<\nu+s$, and $m\ge 1$. Then for any subspace $P$ of *type* $(m+1,s)$, there exist two subspaces $P_1,P_2$ of *type*$(m,s)$ such that $P=P_1+P_2$.

**Proposition 2.3**[12]. Let $2s\le m\le\nu+s, 2s_1\le m_1\le\nu+s_1$. Then for any $(m,s)$-flat F, there exist $(m_1,s_1)$-flats $F_1,F_2,\dots,F_l$ such that $F=F_1\cup F_2\cup\dots\cup F_l$ if and only if $0\le s-s_1\le m-m_1$.

**Proposition 2.4**[12]. For $0\le m_1\le 2\nu-1$, $\mathcal{L}'(m_1,s_1;2\nu)$ consists of $\emptyset,\mathbb{F}_q^{(2\nu)}$ and all $(m,s)$-flats in $ASG(2\nu,\mathbb{F}_q)$ satisfying $2s\le m\le\nu+s$ and $0\le s-s_1\le m-m_1$.

**Proposition 2.5**[12]. For $0\le m_2\le 2\nu-1,\mathcal{L}''(m_2,s_2;2\nu)$ consists of $\emptyset,\mathbb{F}_q^{(2\nu)}$ and all $(m,s)$-flats *in* $ASG(2\nu,\mathbb{F}_q)$ satisfying $2s\le m\le\nu+s$ *and* $0\le s_2-s\le m_2-m$.

**Proposition 2.6**[12]. Let $1\le m_1<m_2\le 2\nu-1$. Then $\mathcal{L}(m_1,s_1;m_2,s_2,;2\nu)$ consists of $\emptyset,\mathbb{F}_q^{(2\nu)}$ *and all* $(m,s)$-flats in $ASG(2\nu,\mathbb{F}_q)$ satisfying

$$2s\le m\le\nu+s,\ 0\le s-s_1\le m-m_1,\ 0\le s_2-s\le m_2-m.$$

**Proposition 2.7**[14].The number of $(m,s)$-flats in $ASG(2\nu,\mathbb{F}_q)$ is equal to $N(m,s;2\nu)q^{2\nu-m}$, where $N(m,s;2\nu)$ is the number of type $(m,s)$ subspaces in $\mathbb{F}_q^{(2\nu)}$ (see[3]).

## 3. The inclusion relations between different lattices

**Theorem 3.1.** Let $2s\le m\le\nu+s$, $2s_1\le m_1\le\nu+s_1$. Then $\mathcal{L}'_R(m,s;2\nu)\subset\mathcal{L}'_R(m_1,s_1;2\nu)$ if and only if $m-m_1\ge s-s_1\ge 0$.

**Proof.** Suppose that $m-m_1\ge s-s_1\ge 0$.In order to prove $\mathcal{L}'_R(m,s;2\nu)\subset$

$\mathcal{L}'_R(m_1, s_1; 2\nu)$ it is sufficient to testify that $M(m, s; 2\nu) \subset \mathcal{L}'_R(m_1, s_1; 2\nu)$. For any $(m, s)$-flat $F$, by Proposition 2.3, there exist $(m_1, s_1)$-flats $F_1, F_2, \ldots, F_l$, such that $F = F_1 \cup F_2 \cup \ldots \cup F_l$. Clearly, $M(m, s; 2\nu) \subset \mathcal{L}'_R(m_1, s_1; 2\nu)$. Therefore, $\mathcal{L}'_R(m, s; 2\nu) \subset \mathcal{L}'_R(m_1, s_1; 2\nu)$.

Conversely, since $\mathcal{L}'_R(m, s; 2\nu) \subset \mathcal{L}'_R(m_1, s_1; 2\nu)$ and $M(m, s; 2\nu) \subset \mathcal{L}'_R(m, s; 2\nu)$, we get $M(m, s; 2\nu) \subset \mathcal{L}'_R(m_1, s_1; 2\nu)$. It follows that $Q + x \in \mathcal{L}'_R(m_1, s_1; 2\nu)$, for any flat $Q + x \in M(m, s; 2\nu)$. In addition, there exists a flat $P + y \in M(m_1, s_1; 2\nu)$, such that $P + y \subset Q + x$. Since $y \in Q + x$, we have $x = y - q$, where $q \in Q$. Hence, $Q + x = Q + y - q = Q + y$ is valid. From $P + y \subset Q + y$, we obtain $P \subset Q$. $Q + y$ is $(m, s)$-flat and $P + y$ is $(m_1, s_1)$-flat.

If $m = m_1$, $P + y = Q + y$, then $s = s_1$, $m - m_1 \geq s - s_1 \geq 0$ is true.

If $m > m_1$, let $m - m_1 = t$, $P$ of type $(m, s)$, we know $s_1 \geq s - t$ and $s - s_1 \leq t$, then $m - m_1 = t \geq s - s_1$. We obtain $m - m_1 \geq s - s_1 \geq 0$.

Hence the desired result follows. □

**Theorem 3.2.** Let $2s \leq m \leq \nu + s$, $2s_2 \leq m_2 \leq \nu + s_2$, then $\mathcal{L}''_R(m, s; 2\nu) \subset \mathcal{L}''_R(m_2, s_2; 2\nu)$ if and only if $m_2 - m \geq s_2 - s \geq 0$ (see[15]).

**Theorem 3.3.** Let $2s'_1 \leq m'_1 \leq \nu + s'_1$, $2s'_2 \leq m'_2 \leq \nu + s'_2$, $2s_1 \leq m_1 \leq \nu + s_1$ and $2s_2 \leq m_2 \leq \nu + s_2$. Then $\mathcal{L}_R(m'_1, s'_1; m'_2, s'_2; 2\nu) \subset \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$ if and only if $m_1 - s_1 \leq m'_1 - s'_1 \leq m'_2 - s'_2 \leq m_2 - s_2$.

**Proof** Suppose $\mathcal{L}_R(m'_1, s'_1; m'_2, s'_2; 2\nu) \subset \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$. For any $(m, s)$-flat $P + x \in \mathcal{L}_R(m'_1, s'_1; m'_2, s'_2; 2\nu)$, where $m'_1 - s'_1 \leq m - s \leq m'_2 - s'_2$. Clearly, $P + x \in \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$, where $m_1 - s_1 \leq m - s \leq m_2 - s_2$. Hence $m_1 - s_1 \leq m'_1 - s'_1 \leq m'_2 - s'_2 \leq m_2 - s_2$ as desired.

Conversely, for any $(m, s)$-flat $Q + y \in \mathcal{L}_R(m'_1, s'_1; m'_2, s'_2; 2\nu)$, then $m'_1 - s'_1 \leq m - s \leq m'_2 - s'_2$. Because $m_1 - s_1 \leq m'_1 - s'_1 \leq m'_2 - s'_2 \leq m_2 - s_2$, so $Q + y \in \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$, we can conclude that $\mathcal{L}_R(m'_1, s'_1; m'_2, s'_2; 2\nu) \subset \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$.□

## 4. Characteristic polynomial

Firstly, let $F \in \mathcal{L}'_R(m_1, s_1; 2\nu)$, and the rank function of $\mathcal{L}'_R(m_1, s_1; 2\nu)$ is

$$r'_R(F) = \begin{cases} 2\nu - m_1 + 1, & \text{if } F = \emptyset, \\ 2\nu - \dim F_1, & \text{otherwise.} \end{cases}$$

(see[12]).

**Theorem 4.1.** Let $2s_1 \leq m_1 \leq \nu + s_1$. Then

$$\mathcal{X}(\mathcal{L}_R^{'}(m_1,s_1;2\nu),t)=\sum_{s_1^{'}=0}^{s_1-1}\sum_{k=2s_1^{'}}^{\nu+s_1^{'}} q^{2\nu-k}N(k,s_1^{'};2\nu)\mathcal{X}(\mathcal{L}_k,t)+$$

$$\sum_{s_1^{'}=s_1}^{\nu}\sum_{k=m_1-s_1+s_1^{'}+1}^{\nu+s_1^{'}} q^{2\nu-k}N(k,s_1^{'};2\nu)\mathcal{X}(\mathcal{L}_k,t),$$

where $N(k,s_1^{'};2\nu)$ is the number of type $(k,s_1^{'})$ subspaces in $\mathbb{F}_q^{(2\nu)}$.

**Proof** For convenience, we write $V=\mathbb{F}_q^{(2\nu)},\mathcal{L}'=\mathcal{L}_R^{'}(m_1,s_1;2\nu),\mathcal{L}_1=\mathcal{L}_R^{'}(\mathbb{F}_q^{(2\nu)})$, where $\mathcal{L}_R^{'}(\mathbb{F}_q^{(2\nu)})$ is all flats in $ASG(2\nu,\mathbb{F}_q)$.

For $m_1$-dimensional $U+x\in\mathcal{L}^{'}$, define

$$\mathcal{L}^{'U+x}=\{W+y\in\mathcal{L}^{'}|W+y\geq U+x\}$$

$$\mathcal{L}_1^{U+x}=\{W+y\in\mathcal{L}_1|W+y\geq U+x\}.$$

Clearly, $\mathcal{L}^{'V}=\mathcal{L}^{'}$. For $U+x\in\mathcal{L}^{'}\setminus V$, by Proposition 2.4 we get $\mathcal{L}^{'U+x}=\mathcal{L}_1^{U+x}$. Therefore, the characteristic polynomial of $\mathcal{L}^{'}$ is

$$\mathcal{X}(\mathcal{L}^{'V},t)=\mathcal{X}(\mathcal{L}^{'},t)=\sum_{U+x\in\mathcal{L}'}\mu(V,U+x)t^{r^{'}(\emptyset)-r^{'}(U+x)}.$$

$\mathcal{L}_1^V=\mathcal{L}_1$,

$$\mathcal{X}(\mathcal{L}_1^V,t)=\mathcal{X}(\mathcal{L}_1,t)=\sum_{U+x\in\mathcal{L}_1}\mu(V,U+x)t^{r^{'}(\emptyset)-r^{'}(U+x)}.$$

From *Möbius* inversion formula

$$t^{2\nu-m_1+1}=\sum_{U+x\in\mathcal{L}^{'V}}\mathcal{X}(\mathcal{L}^{'U+x},t)=\sum_{U+x\in\mathcal{L}'}\mathcal{X}(\mathcal{L}^{'U+x},t),$$

$$t^{2\nu-m_1+1}=\sum_{U+x\in\mathcal{L}_1^V}\mathcal{X}(\mathcal{L}_1^{U+x},t)=\sum_{U+x\in\mathcal{L}_1}\mathcal{X}(\mathcal{L}_1^{U+x},t).$$

Thus,

$$\begin{aligned}\mathcal{X}(\mathcal{L}^{'},t)=\mathcal{X}(\mathcal{L}^{'V},t)&=t^{2\nu-m_1+1}-\sum_{U+x\in\mathcal{L}'\setminus V}\mathcal{X}(\mathcal{L}^{'U+x},t)\\&=\sum_{U+x\in\mathcal{L}_1}\mathcal{X}(\mathcal{L}_1^{U+x},t)-\sum_{U+x\in\mathcal{L}'\setminus V}\mathcal{X}(\mathcal{L}^{'U+x},t)\\&=\sum_{U+x\in(\mathcal{L}_1\setminus\mathcal{L}'\cup V)}\mathcal{X}(\mathcal{L}_1^{U+x},t).\end{aligned}$$

By Proposition 2.4, $U+x \in (\mathcal{L}_1 \backslash \mathcal{L}' \cup V)$ if and only if $\{U+x \in \mathcal{L}_1 | U+x$ is $(m_1', s_1')$-flat, $s_1' - s_1 < 0\} \cup \{U+x \in \mathcal{L}_1 | U+x$ is $(m_1', s_1')$-flat, $s_1' - s_1 \geq 0, m_1 - m_1' < s_1 - s_1'\}$.

Thus, $\mathcal{X}(\mathcal{L}_R'(m_1, s_1; 2\nu), t) = \sum_{s_1'=0}^{s_1-1} \sum_{m_1'=2s_1'}^{\nu+s_1'} q^{2\nu-m_1'} N(m_1', s_1'; 2\nu) \mathcal{X}(\mathcal{L}_1^{U+x}, t) +$

$$\sum_{s_1'=s_1}^{\nu} \sum_{m_1'=m_1-s_1+s_1'+1}^{\nu+s_1'} q^{2\nu-m_1'} N(m_1', s_1'; 2\nu) \mathcal{X}(\mathcal{L}_1^{U+x}, t),$$

where $N(m_1', s_1'; 2\nu)$ is the number of $(m_1', s_1')$-flats in $ASG(2\nu, \mathbb{F}_q)$.

It is a routine to show that $\mathcal{L}_1^{U+x} \simeq \mathcal{L}_k$ where $k$=dim$(U+x)$. Hence both the lattices $\mathcal{L}_1^{U+x}$ and $\mathcal{L}_k$ have the same characteristic polynomial.

Then

$$\mathcal{X}(\mathcal{L}_R'(m_1, s_1; 2\nu), t) = \sum_{s_1'=0}^{s_1-1} \sum_{k=2s_1'}^{\nu+s_1'} q^{2\nu-k} N(k, s_1'; 2\nu) \mathcal{X}(\mathcal{L}_k, t) +$$

$$\sum_{s_1'=s_1}^{\nu} \sum_{k=m_1-s_1+s_1'+1}^{\nu+s_1'} q^{2\nu-k} N(k, s_1'; 2\nu) \mathcal{X}(\mathcal{L}_k, t).$$

as desired. □

Secondly, let $F \in \mathcal{L}_R''(m_2, s_2; 2\nu)$, and the rank function of $\mathcal{L}_R''(m_2, s_2; 2\nu)$ is

$$r_R''(F) = \begin{cases} 0, & \text{if } F = \mathbb{F}_q^{(2\nu)}, \\ m_2 + 2, & \text{if } F = \emptyset, \\ m_2 - \dim F + 1, & \text{otherwise.} \end{cases}$$

(see[12]).

**Theorem 4.2.** Let $2s_2 \leq m_2 \leq \nu + s_2$. Then

$$\mathcal{X}(\mathcal{L}_R''(m_2, s_2; 2\nu), t) = \sum_{s_2'=s_2+1}^{\nu} \sum_{k=2s_2'}^{\nu+s_2'} q^{2\nu-k} N(k, s_2'; 2\nu) \mathcal{X}(\mathcal{L}_k, t) +$$

$$\sum_{s_2'=0}^{s_2} \sum_{k=m_2-s_2+s_2'+1}^{\nu+s_2'} q^{2\nu-k} N(k, s_2'; 2\nu) \mathcal{X}(\mathcal{L}_k, t)$$

where $N(k, s_2'; 2\nu)$ is the number of type $(k, s_2')$ subspaces in $\mathbb{F}_q^{(2\nu)}$.

**Proof** We write $V = \mathbb{F}_q^{(2\nu)}, \mathcal{L}'' = \mathcal{L}_R''(m_2, s_2; 2\nu), \mathcal{L}_2 = \mathcal{L}_R''(\mathbb{F}_q^{(2\nu)})$. For any $U + x \in \mathcal{L}''$, define

$$\mathcal{L}''^{U+x} = \{W + y \in \mathcal{L}'' | W + y \geq U + x\},$$

$$\mathcal{L}_2^{U+x} = \{W + y \in \mathcal{L}_2 | W + y \geq U + x\}.$$

It is easy to see that $\mathcal{L}''^V = \mathcal{L}''$. For $U + x \in \mathcal{L}'' \setminus V$, by Proposition 2.5 we get $\mathcal{L}''^{U+x} = \mathcal{L}_2^{U+x}$. Therefore, the characteristic polynomial of $\mathcal{L}''$ is

$$\mathcal{X}(\mathcal{L}''^V, t) = \mathcal{X}(\mathcal{L}'', t) = \sum_{U+x\in\mathcal{L}''} \mu(V, U + x) t^{r''(\emptyset) - r''(U+x)}.$$

$\mathcal{L}_2^V = \mathcal{L}_2,$

$$\mathcal{X}(\mathcal{L}_2^V, t) = \mathcal{X}(\mathcal{L}_2, t) = \sum_{U+x\in\mathcal{L}_2} \mu(V, U + x) t^{r''(\emptyset) - r''(U+x)}.$$

From *Möbius* inversion formula

$$t^{m_2+2} = \sum_{U+x\in\mathcal{L}''^V} \mathcal{X}(\mathcal{L}''^{U+x}, t) = \sum_{U+x\in\mathcal{L}''} \mathcal{X}(\mathcal{L}''^{U+x}, t),$$

$$t^{m_2+2} = \sum_{U+x\in\mathcal{L}_2^V} \mathcal{X}(\mathcal{L}_2^{U+x}, t) = \sum_{U+x\in\mathcal{L}_2} \mathcal{X}(\mathcal{L}_2^{U+x}, t).$$

Thus,

$$\begin{aligned}\mathcal{X}(\mathcal{L}'', t) = \mathcal{X}(\mathcal{L}''^V, t) &= t^{m_2+2} - \sum_{U+x\in\mathcal{L}''\setminus V} \mathcal{X}(\mathcal{L}''^{U+x}, t) \\ &= \sum_{U+x\in\mathcal{L}_2} \mathcal{X}(\mathcal{L}_2^{U+x}, t) - \sum_{U+x\in\mathcal{L}''\setminus V} \mathcal{X}(\mathcal{L}''^{U+x}, t) \\ &= \sum_{U+x\in(\mathcal{L}_2\setminus\mathcal{L}''\cup V)} \mathcal{X}(\mathcal{L}_2^{U+x}, t).\end{aligned}$$

By Proposition 2.5, $U + x \in (\mathcal{L}_2 \setminus \mathcal{L}'' \cup V)$ if and only if $\{U + x \in \mathcal{L}_2 | U + x \text{ is } (m_2', s_2')\text{−flat}, s_2 - s_2' < 0\} \cup \{U + x \in \mathcal{L}_2 | U + x \text{ is } (m_2', s_2')\text{−flat}, s_2 - s_2' \geq 0, m_2 - m_2' < s_2 - s_2'\}$.

Thus,

$$\mathcal{X}(\mathcal{L}_R''(m_2, s_2; 2\nu), t) = \sum_{s_2'=s_2+1}^{\nu} \sum_{m_2'=2s_2'}^{\nu+s_2'} q^{2\nu - m_2'} N(m_2', s_2'; 2\nu) \mathcal{X}(\mathcal{L}_1^{U+x}, t) +$$

$$\sum_{s_2'=0}^{s_2} \sum_{m_2'=m_2-s_2+s_2'+1}^{\nu+s_2'} q^{2\nu - m_2'} N(m_2', s_2'; 2\nu) \mathcal{X}(\mathcal{L}_1^{U+x}, s_2'; 2\nu), t).$$

It is a routine to show that $\mathcal{L}_2^{U+x} \simeq \mathcal{L}_k$ where $k$=dim$(U + x)$. Hence both the lattices $\mathcal{L}_2^{U+x}$ and $\mathcal{L}_k$ have the same characteristic polynomial.

$$\mathcal{X}(\mathcal{L}_R''(m_2, s_2; 2\nu), t) = \sum_{s_2'=s_2+1}^{\nu} \sum_{k=2s_2'}^{\nu+s_2'} q^{2\nu-k} N(k, s_2'; 2\nu) \mathcal{X}(\mathcal{L}_k, t) +$$

$$\sum_{s_2'=0}^{s_2} \sum_{k=m_2-s_2+s_2'+1}^{\nu+s_2'} q^{2\nu-k} N(k, s_2'; 2\nu) \mathcal{X}(\mathcal{L}_k, s_2'; 2\nu), t). \square$$

Thirdly, let $F \in \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$, and the rank function of $\mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu)$ is

$$r_R(F) = \begin{cases} 0, & \text{if } F = \mathbb{F}_q^{(2\nu)}, \\ m_2 - m_1 + 2, & \text{if } F = \emptyset. \\ m_2 - \dim F + 1, & \text{otherwise.} \end{cases}$$

(see[12]).

**Theorem 4.3.** Let $2s_1 \leq m_1 \leq \nu + s_1$, $2s_2 \leq m_2 \leq \nu + s_2$, then

$$\mathcal{X}(\mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu), t) = \sum_{s=s_1}^{s_2} \sum_{k=m_2-s_2+s+1}^{m_1+s-s_1} q^{2\nu-k} N(k, s; 2\nu) \mathcal{X}(\mathcal{L}_k, t) +$$
$$\sum_{s=0}^{s_1} \sum_{k=2s}^{\nu+s} q^{2\nu-k} N(k, s; 2\nu) \mathcal{X}(\mathcal{L}_k, t) + \sum_{s=s_2+1}^{\nu} \sum_{k=2s}^{\nu+s} q^{2\nu-k} N(k, s; 2\nu) \mathcal{X}(\mathcal{L}_k, t).$$

where $N(k, s; 2\nu)$ is the number of type $(k, s)$ subspaces in $\mathbb{F}_q^{(2\nu)}$.

**Proof** We write $V = \mathbb{F}_q^{(2\nu)}, \mathcal{L} = \mathcal{L}_R(m_1, s_1; m_2, s_2; 2\nu), \mathcal{L}_0 = \mathcal{L}_R(\mathbb{F}_q^{(2\nu)})$. For any $U + x \in \mathcal{L}$, define

$$\mathcal{L}^{U+x} = \{W + y \in \mathcal{L} | W + y \geq U + x\},$$

$$\mathcal{L}_0^{U+x} = \{W + y \in \mathcal{L}_0 | W + y \geq U + x\}.$$

Clearly, $\mathcal{L}^V = \mathcal{L}$. For $U+x \in \mathcal{L} \backslash V$, we get $\mathcal{L}^{U+x} = \mathcal{L}_0^{U+x}$ by Proposition 2.6. Therefore, the characteristic polynomial of $\mathcal{L}$ is

$$\mathcal{X}(\mathcal{L}^V, t) = \mathcal{X}(\mathcal{L}, t) = \sum_{U+x \in \mathcal{L}} \mu(V, U + x) t^{r(\emptyset) - r(U+x)}.$$

$\mathcal{L}_0^V = \mathcal{L}_0$,

$$\mathcal{X}(\mathcal{L}_0^V, t) = \mathcal{X}(\mathcal{L}_0, t) = \sum_{U+x \in \mathcal{L}_0} \mu(V, U + x) t^{r(\emptyset) - r(U+x)}.$$

From *Möbius* inversion formula

$$t^{m_2 - m_1 + 2} = \sum_{U+x \in \mathcal{L}^V} \mathcal{X}(\mathcal{L}^{U+x}, t) = \sum_{U+x \in \mathcal{L}} \mathcal{X}(\mathcal{L}^{U+x}, t),$$

$$t^{m_2 - m_1 + 2} = \sum_{U+x \in \mathcal{L}_0^V} \mathcal{X}(\mathcal{L}_0^{U+x}, t) = \sum_{U+x \in \mathcal{L}_0} \mathcal{X}(\mathcal{L}_0^{U+x}, t).$$

Thus,

$$\mathcal{X}(\mathcal{L},t)=\mathcal{X}(\mathcal{L}^V,t)=t^{m_2-m_1+2}-\sum_{U+x\in\mathcal{L}\setminus V}\mathcal{X}(\mathcal{L}^{U+x},t)$$

$$=\sum_{U+x\in\mathcal{L}_0}\mathcal{X}(\mathcal{L}_0^{U+x},t)-\sum_{U+x\in\mathcal{L}\setminus V}\mathcal{X}(\mathcal{L}^{U+x},t)$$
$$=\sum_{U+x\in(\mathcal{L}_0\setminus\mathcal{L}\cup V)}\mathcal{X}(\mathcal{L}_0^{U+x},t).$$

By Proposition 2.6, $U+x\in(\mathcal{L}_0\setminus\mathcal{L}\cup V)$ if and only if $\{U+x\in\mathcal{L}_0|U+x \text{ is } (m,s)\text{–flat}, s_1\le s\le s_2, m-m_1<s-s_1, m_2-m<s_2-s\}\cup\{U+x\in\mathcal{L}_0|U+x \text{ is } (m,s)\text{–flat}, s<s_1\}\cup\{U+x\in\mathcal{L}_0|U+x \text{ is } (m,s)\text{–flat}, s>s_2\}$

Thus,

$$\mathcal{X}(\mathcal{L}_R(m_1,s_1;m_2,s_2;2\nu),t)=\sum_{s=s_1}^{s_2}\sum_{m=m_2-s_2+s+1}^{m_1+s-s_1}q^{2\nu-m}N(m,s;2\nu)\mathcal{X}(\mathcal{L}_0^{U+x},t)\cdot$$
$$\sum_{s=0}^{s_1}\sum_{m=2s}^{\nu+s}q^{2\nu-m}N(m,s;2\nu)\mathcal{X}(\mathcal{L}_0^{U+x},t)+\sum_{s=s_2+1}^{\nu}\sum_{m=2s}^{\nu+s}q^{2\nu-m}N(m,s;2\nu)\mathcal{X}(\mathcal{L}_0^{U+x}$$

It is a routine to show that $\mathcal{L}_0^{U+x}\simeq\mathcal{L}_k$ where $k$=dim$(U+x)$. Hence both the lattices $\mathcal{L}_0^{U+x}$ and $\mathcal{L}_k$ have the same characteristic polynomial.

We obtain

$$\mathcal{X}(\mathcal{L}_R(m_1,s_1;m_2,s_2;2\nu),t)=\sum_{s=s_1}^{s_2}\sum_{k=m_2-s_2+s+1}^{m_1+s-s_1}q^{2\nu-k}N(k,s;2\nu)\mathcal{X}(\mathcal{L}_k,t)+$$
$$\sum_{s=0}^{s_1}\sum_{k=2s}^{\nu+s}q^{2\nu-k}N(k,s;2\nu)\mathcal{X}(\mathcal{L}_k,t)+\sum_{s=s_2+1}^{\nu}\sum_{k=2s}^{\nu+s}q^{2\nu-k}N(k,s;2\nu)\mathcal{X}(\mathcal{L}_k,t).\ \Box$$

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