Decompositions of K_v into the graphs with 7 points,7 edges and a 5-circle*

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Abstract. Let K_v be the complete graph with v vertices, where any two distinct vertices x and y are joined by exactly one edge $\{x, y\}$. Let G be a finite simple graph. A G-design of K_v, denoted by $(v, G, 1)$-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly one block of $\mathcal{B}$. In this paper, the discussed graphs are $G_i, i = 1, 2, 3, 4$, where G_i are the four graphs with 7 points,7 edges and a 5-circle. We obtain the existence spectrum of $(v, G_i, 1)$-GD.

Keywords: G-design, G-holey design, G-incomplete design.

1 Introduction

A *complete graph* of order v, denoted by K_v, is a graph with v vertices, where any two distinct vertices x and y are joined by exactly one edge $\{x, y\}$. A *t-partite graph* is one whose vertex set can be partitioned into t subsets $X_1, X_2, \cdots, X_t$, such that two ends of each edge lie in distinct subsets respectively. Such a partition $(X_1, X_2, \cdots, X_t)$ is called a *t-partition* of the graph. A *complete t-partite graph* is a t-partite graph with t-partition $(X_1, X_2, \cdots, X_t)$, in which each vertex of X_i is joined to each vertex of

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X_j by one time (where $i \neq j$). Such a graph is denoted by $K_{n_1,n_2,\cdots,n_t}$ if $|X_i| = n_i$ $(1 \leq i \leq t)$.

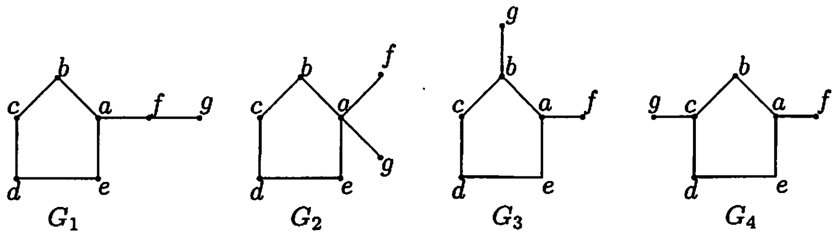
Let $(X_1, X_2, \cdots, X_t)$ be the t-partition of $K_{n_1,n_2,\cdots,n_t}$, and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G}=\{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $K_{n_1,n_2,\cdots,n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design*, denoted by G-$HD(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$HD(1^{v-w}w^1)$ is called an *incomplete G-design*, denoted by G-$ID(v;w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a $(v, G, 1)$-GD is a G-$HD(1^v)$ or a G-$ID(v;w)$ with $w = 0$ or 1.

Let G be a finite simple graph. A *G-design* of K_v, denoted by $(v, G, 1)$-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly one block of $\mathcal{B}$. It is well known that if there exists a $(v, G, 1)$-GD, then

$$v(v-1) \equiv 0 \pmod{2e(G)} \quad \text{and} \quad v - 1 \equiv 0 \pmod{d},$$

where $e(G)$ denotes the number of edges in G and d is the greatest common divisor of the degrees of the vertices of G. For the path P_k and the star $K_{1,k}$, the existence problems of their graph designs have been solved (see [1] and [2]). For the graphs which have four vertices or less, the problem of their graph designs has already been solved (see [3]). For some graphs which have k vertices (where $k = 5, 6$), the problem of their graph designs has already been researched (see [4]-[11]). In [12], the existence problem of G-decomposition for graphs G with 7 points, 7 edges and an even-circle has been completely solved.

In this paper, the discussed graphs are G_i $(i = 1, 2, 3, 4)$, where G_i are the four graphs with 7 points,7 edges and a 5-circle, they are listed as follows.



For convenience, the graphs G_1-G_4 above are denoted by (a, b, c, d, e, f, g). we obtain the existence spectrum of $(v, G_i, 1)$-GD.

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, m and non-negative integer w, if there exist G-$HD(h^m)$, G-$ID(h+w; w)$ and $(w, G, 1)$-GD (or $(h+w, G, 1)$-GD), then there exists $(mh+w, G, 1)$-GD, too.*

Proof. Let $X = (Z_h \times Z_m) \bigcup W$, where W is a w-set. Suppose there exist

$$G\text{-}HD(h^m) = (Z_h \times Z_m, \mathcal{A}),$$
$$G\text{-}ID(h+w; w) = ((Z_h \times \{i\}) \bigcup W, \mathcal{B}_i),\ i \in Z_m \text{ or } i \in Z_m \setminus \{0\},$$

and

$$(w, G, 1)\text{-}GD = (W, \mathcal{C}) \text{ or } (h+w, G, 1)\text{-}GD = ((Z_h \times \{0\}) \bigcup W, \mathcal{D}),$$

then (X, Ω) is a $(mh+w, G, 1)$-GD, where

$$\Omega = \mathcal{A} \bigcup (\bigcup_{i=0}^{m-1} \mathcal{B}_i) \bigcup \mathcal{C} \text{ or } \mathcal{A} \bigcup (\bigcup_{i=1}^{m-1} \mathcal{B}_i) \bigcup \mathcal{D}.$$

Note that

$$|\Omega| = \frac{\binom{mh+w}{2}}{e(G)} = \begin{cases} \frac{\binom{m}{2}h^2}{e(G)} + \frac{m(\binom{h}{2}+wh)}{e(G)} + \frac{\binom{w}{2}}{e(G)} \\ \frac{\binom{m}{2}h^2}{e(G)} + \frac{(m-1)(\binom{h}{2}+wh)}{e(G)} + \frac{\binom{w+h}{2}}{e(G)} \end{cases}$$
$$= \begin{cases} |\mathcal{A}| + \sum_{i=0}^{m-1} |\mathcal{B}_i| + |\mathcal{C}| \\ |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{D}| \end{cases}.$$

□

The necessary conditions for the existence of $(v, G_i, 1)$-GD are $v(v-1) \equiv 0 \pmod{14}$ and $v \geq 7$, i.e., $v \equiv 0, 1 \pmod{7}$ and $v \geq 7$. For G_2, we can prove that $(7, G_2, 1)$-GD doesn't exist (see Lemma 5.3), so by Theorem 2.1

and the following tables, we only need to give the constructions of HD,ID and GD for the pointed orders.

(Table 2.1) For $G_i, i = 1,3,4$

$v(mod\ 14)$	HD	ID	GD
0	7^{2t+1}	(14;7)	7
1	7^{2t+1}	(15;8)	8
7	7^{2t+1}	$(7;0)^*$	7
8	7^{2t+1}	$(8;1)^*$	8

(Table 2.2) For G_2

$v(mod\ 14)$	HD	ID	GD
0	7^{2t+1}	(14;7)	14
1	7^{2t+1}	(15;8)	8
7	7^{2t-1}	(21;14)	14
8	7^{2t+1}	$(8;1)^*$	8

$*$: G_i-$ID(7;0) = (7, G_i, 1)$-GD, G_i-$ID(8;1) = (8, G_i, 1)$-GD.

3 Holey G_i-designs

A *quasigroup* is an ordered pair $(Q, \cdot)$, which is a set Q with a binary operation $\cdot$ such that the equations $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for every pair of elements a, b in Q. It is well known that the multiplication table of a quasigroup is equivalent with a Latin square. A quasigroup is said to be *idempotent* (or *symmetric*) if the identity $x \cdot x = x$ (or $x \cdot y = y \cdot x$) holds for all $x \in Q$ (or $x, y \in Q$). It is well known that there exists an idempotent quasigroup of order v if and only if $v \neq 2$ and there exists an idempotent symmetric quasigroup of order v if and only if v is odd.

Suppose $(I_n, \cdot)$ is an idempotent symmetric quasigroup on the set $I_n = \{1, 2, \cdots, n\}$. Now, let's construct a G-$HD(e^n)$, where $e = |E(G)|$. Let the element set be $Z_e \times I_n$, and the automorphism group of the block set be Z_e. It is enough to construct a base block for any $i, j \in I_n$ and $i < j$. In a base block of a G-$HD(e^n)$, all edges can be partitioned into three types: $\{(x, i), (x + d, j)\}$, $\{(x, i), (x + d, i \cdot j)\}$ and $\{(x, i \cdot j), (x + d, j)\}$, denoted by

$d(i,j), d(i,i\cdot j)$ and $d(i\cdot j,j)$ briefly, where $x\in Z_e$, $i,j\in I_n$. We have the following Lemma.

Lemma 3.1 [8] *Let n be odd and $(I_n,\cdot)$ be an idempotent symmetric quasigroup on the set $I_n=\{1,2,\cdots,n\}$. The block set $\mathcal{A}=\{A_{i,j}: i,j\in I_n$ and $i<j\}$ can be taken as a base of a G-HD(e^n) if and only if the following conditions hold,*

(1) *For any given block $A_{i,j}$ in $\mathcal{A}$, the differences $d(i,i\cdot j)$ and $-d(i\cdot j,j)$ both appear or not in $A_{i,j}$;*

(2) $\{d:\exists d(i,j)\}\bigcup\{d:\exists d(i,i\cdot j)\}\bigcup\{d:\exists d(i\cdot j,j)\}=Z_e$.

Lemma 3.2 *There exists a G_k-$HD(7^{2t+1})$ for $k=1,2,3,4$ and $t>0$.*

Proof. The number of the edges of G_k is 7. Suppose $(I_{2t+1},\cdot)$ is an idempotent symmetric quasigroup on the set $I_{2t+1}=\{1,2,\cdots,2t+1\}$. Let $X=Z_7\times I_{2t+1}$ and

$$\mathcal{B}_1=(2_{i\cdot j},1_i,3_{i\cdot j},0_i,0_j,1_j,4_{i\cdot j}),$$
$$\mathcal{B}_2=(2_{i\cdot j},1_i,3_{i\cdot j},0_i,0_j,1_j,6_j),$$
$$\mathcal{B}_3=(0_j,2_{i\cdot j},6_i,1_{i\cdot j},0_i,3_{i\cdot j},1_j),$$
$$\mathcal{B}_4=(1_{i\cdot j},0_i,2_{i\cdot j},6_i,6_j,5_j,1_j),$$

where $1\le i<j\le 2t+1$. We can verify that each $\mathcal{B}_k$ mod $(7,-)$ gives the expected G_k-$HD(7^{2t+1})$ by Lemma 3.1 for $k=1,2,3,4$. □

4 Incomplete G_i-designs

Lemma 4.1 *There exist G_1-$ID(7+w;w)$ for $w=7,8$.*

Proof. Let $X=Z_7\bigcup\{\infty_1,\infty_2,\cdots,\infty_w\}$ and G_1-$ID(7+w;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}|=w+3$. The family $\mathcal{B}$ consists of the following blocks.

$w=7$:

$(0,2,\infty_1,5,\infty_6,1,\infty_3)$, $(1,2,\infty_4,6,\infty_2,4,\infty_1)$, $(6,\infty_5,0,\infty_4,1,5,\infty_2)$,
$(3,\infty_3,4,2,\infty_5,0,\infty_2)$, $(4,\infty_5,5,3,6,0,\infty_1)$, $(5,4,\infty_6,3,1,0,\infty_3)$,
$(2,\infty_7,1,\infty_6,6,3,\infty_4)$, $(\infty_1,3,4,\infty_7,6,1,\infty_5)$, $(\infty_2,4,\infty_4,5,2,3,\infty_7)$,
$(\infty_3,5,\infty_7,0,6,2,\infty_6)$.

$w=8$:

$(0,\infty_1,3,4,\infty_3,\infty_4,1)$, $(1,\infty_6,3,6,\infty_5,4,0)$, $(2,1,0,5,\infty_8,3,\infty_4)$,
$(3,\infty_5,2,\infty_2,1,\infty_3,5)$, $(4,\infty_1,6,0,\infty_2,2,\infty_3)$, $(\infty_8,6,5,2,0,1,\infty_1)$,
$(6,1,\infty_7,0,\infty_6,\infty_4,2)$, $(\infty_5,4,6,\infty_7,5,0,3)$, $(\infty_6,4,\infty_8,3,5,2,6)$,
$(\infty_7,2,\infty_1,5,4,3,\infty_2)$, $(5,1,\infty_3,6,\infty_2,\infty_4,4)$.

□

Lemma 4.2 *There exist G_2-$ID(7+w;w)$ for $w=7,8,14$.*

Proof. Let $X = Z_7 \bigcup \{\infty_1,\infty_2,\cdots,\infty_w\}$ and G_2-$ID(7+w;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}| = w+3$. The family $\mathcal{B}$ consists of the following blocks.

$w=7$:

$(0,\infty_7,2,1,\infty_6,\infty_1,\infty_5)$, $(1,\infty_7,4,6,5,\infty_2,3)$, $(2,\infty_6,3,4,0,\infty_3,\infty_4)$,
$(3,2,\infty_5,4,\infty_2,\infty_3,\infty_7)$, $(\infty_2,0,1,4,5,2,6)$, $(5,\infty_4,1,6,0,\infty_6,\infty_7)$,
$(6,\infty_1,3,0,\infty_4,\infty_6,\infty_7)$, $(\infty_1,1,\infty_5,6,2,4,5)$, $(4,\infty_4,3,5,2,\infty_3,\infty_6)$,
$(\infty_3,5,\infty_5,3,6,0,1)$.

$w=8$:

$(0,1,\infty_6,6,\infty_5,\infty_1,5)$, $(6,1,\infty_1,2,0,\infty_8,3)$, $(2,\infty_8,5,3,\infty_6,\infty_3,6)$,
$(3,0,\infty_6,5,\infty_5,\infty_4,4)$, $(4,5,1,2,\infty_5,\infty_6,6)$, $(5,\infty_4,1,3,\infty_2,\infty_7,2)$,
$(1,\infty_7,2,4,\infty_8,\infty_2,\infty_5)$, $(\infty_1,4,0,\infty_7,3,5,6)$, $(\infty_2,4,\infty_3,5,6,0,2)$,
$(\infty_3,6,\infty_7,4,1,0,3)$, $(\infty_4,2,3,\infty_8,0,4,6)$.

$w=14$:

$(\infty_1,3,\infty_{14},6,1,2,4)$, $(1,2,\infty_{11},5,\infty_{12},\infty_{14},\infty_2)$, $(\infty_8,6,5,\infty_5,0,2,4)$,
$(\infty_4,0,\infty_7,2,5,4,6)$, $(4,\infty_3,6,2,\infty_{14},\infty_5,\infty_6)$, $(6,3,4,5,\infty_1,\infty_9,\infty_{10}$
$(\infty_9,1,\infty_8,3,0,4,5)$, $(0,\infty_{11},1,4,\infty_{12},\infty_{13},\infty_1)$, $(\infty_2,0,2,\infty_9,3,5,6)$,
$(\infty_3,0,6,\infty_6,1,3,5)$, $(3,\infty_{12},6,4,\infty_{13},\infty_{11},\infty_4)$, $(\infty_7,3,2,\infty_4,1,4,6)$,
$(\infty_6,2,\infty_2,4,0,3,5)$, $(5,\infty_{13},1,0,\infty_{14},\infty_7,\infty_8)$, $(\infty_{10},0,5,3,1,4,2)$,
$(\infty_5,3,\infty_{10},5,1,2,6)$, $(2,4,\infty_{11},6,\infty_{13},\infty_{12},\infty_3)$.

□

Lemma 4.3 *There exist G_3-$ID(7+w;w)$ for $w=7,8$.*

Proof. Let $X = Z_7 \bigcup \{\infty_1,\infty_2,\cdots,\infty_w\}$ and G_3-$ID(7+w;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}| = w+3$. The family $\mathcal{B}$ consists of the following blocks.

$w=7$:

$(\infty_2,2,\infty_6,4,5,0,\infty_3)$, $(\infty_4,4,1,\infty_2,3,6,\infty_7)$, $(\infty_3,1,\infty_5,2,4,6,0)$,
$(1,5,\infty_3,0,\infty_7,\infty_1,2)$, $(0,\infty_6,6,\infty_2,4,\infty_1,1)$, $(5,\infty_1,6,\infty_7,3,0,4)$,
$(2,3,4,\infty_5,0,\infty_1,\infty_3)$, $(\infty_5,3,0,\infty_4,5,6,\infty_1)$, $(3,6,2,\infty_4,1,\infty_6,0)$,
$(6,5,\infty_7,2,1,4,\infty_6)$.

$w=8$:

$(\infty_4,1,\infty_3,2,5,6,\infty_1)$, $(\infty_3,3,\infty_1,2,4,5,\infty_4)$, $(0,\infty_1,4,3,1,\infty_3,5)$,
$(3,\infty_6,4,\infty_4,2,\infty_2,6)$, $(1,6,\infty_5,4,\infty_2,\infty_6,\infty_1)$, $(0,6,2,1,4,\infty_4,\infty_3)$,
$(6,3,0,\infty_7,5,4,\infty_8)$, $(\infty_8,5,3,\infty_7,4,6,\infty_6)$, $(5,\infty_2,0,\infty_8,1,4,6)$,
$(\infty_5,2,\infty_6,0,5,3,\infty_2)$, $(\infty_7,2,0,\infty_5,1,6,\infty_8)$.

□

Lemma 4.4 *There exist G_4-$ID(7+w;w)$ for $w=7,8$.*

Proof. Let $X = Z_7 \bigcup \{\infty_1,\infty_2,\cdots,\infty_w\}$ and G_4-$ID(7+w;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}|=w+3$. The family $\mathcal{B}$ consists of the following blocks.

$w=7$:

$(0,\infty_4,2,4,\infty_5,\infty_1,\infty_3)$, $(\infty_6,5,\infty_5,3,0,2,1)$, $(1,\infty_7,2,0,5,\infty_1,\infty_5)$,
$(3,\infty_7,6,0,4,\infty_4,\infty_6)$, $(3,2,\infty_1,6,\infty_2,5,4)$, $(1,2,5,\infty_1,3,\infty_6,\infty_2)$,
$(4,\infty_3,6,3,\infty_6,\infty_2,\infty_5)$, $(4,1,\infty_2,2,6,\infty_7,0)$, $(\infty_4,1,\infty_3,5,4,6,3)$,
$(0,\infty_7,5,6,1,\infty_3,\infty_4)$.

$w=8$:

$(\infty_7,1,\infty_2,4,3,6,2)$, $(\infty_1,2,4,\infty_3,3,5,6)$, $(\infty_3,0,4,\infty_7,5,6,\infty_1)$,
$(\infty_4,1,2,\infty_5,4,6,\infty_3)$, $(5,\infty_4,0,1,4,2,\infty_1)$, $(\infty_6,0,3,\infty_8,4,1,\infty_4)$,
$(5,\infty_2,6,1,\infty_5,\infty_6,\infty_8)$, $(6,0,2,\infty_6,3,5,\infty_4)$, $(\infty_8,2,6,\infty_1,1,5,\infty_6)$,
$(3,\infty_2,0,\infty_7,2,5,\infty_8)$, $(\infty_5,3,1,5,0,6,\infty_3)$.

□

5 Graph designs

Lemma 5.1 *There exist $(v,G_1,1)$ for $v=7,8$.*

Proof.

$v=7$: $X=(Z_3\times Z_2)\bigcup\{\infty\}$

$(0_0,2_0,\infty,1_1,0_1,2_1,1_0)$ mod (3,–)

$v=8$: $X=Z_8$

$(4,6,7,2,1,0,3)$, $(5,2,4,7,0,3,1)$, $(6,3,4,5,1,2,0)$, $(7,5,6,0,1,3,2)$. □

Theorem 5.2 *There exist $(v,G_1,1)$ if and only if $v(v-1)\equiv 0$ (mod 14) and $v\geq 7$.*

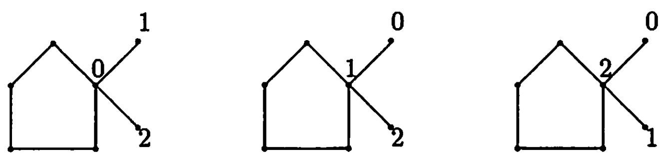
Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.1 and Lemma 5.1. □

Lemma 5.3 *There exists no $(7,G_2,1)$.*

Proof. Let $X = Z_7$. If there exists a $(7, G_2, 1)$-$GD = (X, \mathcal{B})$, then the sum of degrees of any vertex is six and $|\mathcal{B}|$=3. By the structure of the graph G_2, we know that the degrees of the vertices of the graph G_2 are 4, 2, 2, 2, 2, 1, 1 respectively. If a vertex occurs i times in 1-degree position, j times in 2-degree position and k times in 4-degree position, then we call $T = 1^i 2^j 4^k$ the *type* of the degree-distribution of the vertex. It is easy to know that the type of the degree-distribution of the vertices in X is merely $2^1 4^1$, 2^3, or $1^2 4^1$. Suppose the numbers of the vertices of the three types are u, v, w respectively, since there are seven vertices in X and there are six 1-degree positions, twelve 2-degree positions and three 4-degree positions in the blocks of $\mathcal{B}$, therefore we have the following equations:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 12 \\ 3 \end{pmatrix}$$

We can get the only solution is $u = 0$, $v = 4$ $w = 3$. Without loss of generality, suppose the three vertices which have the type $1^2 4^1$ are 0, 1, 2, then the three blocks must be as follows:



Obviously, it is impossible to exist a $(7, G_2, 1)$ under this condition. □

Lemma 5.4 *There exist* $(v, G_2, 1)$ *for* $v = 8, 14$.

Proof.

$v = 8$: $X = Z_8$

$(0,5,3,6,7,4,2)$, $(1,6,4,7,2,5,0)$, $(2,4,1,7,5,6,3)$, $(3,0,6,5,4,7,1)$.

$v = 14$: $X = Z_{13} \bigcup \{\infty\}$

$(1,0,6,2,4,9,\infty)$ mod 13 □

Theorem 5.5 *There exist* $(v, G_2, 1)$ *if and only if* $v(v-1) \equiv 0$ (*mod* 14) *and* $v > 7$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.2, Lemma 5.3 and Lemma 5.4. □

Lemma 5.6 *There exist* $(v, G_3, 1)$ *for* $v = 7, 8$.

Proof.

$v = 7$: $X = (Z_3 \times Z_2) \bigcup \{\infty\}$

$(2_1, 1_0, 0_0, \infty, 1_1, 2_0, 0_1)$ mod (3,–)

$v = 8$: $X = Z_8$

$(0, 4, 1, 5, 6, 3, 2)$, $(1, 6, 4, 7, 0, 2, 3)$, $(7, 2, 3, 4, 5, 1, 0)$, $(3, 5, 2, 6, 7, 1, 0)$. □

Theorem 5.7 *There exist* $(v, G_3, 1)$ *if and only if* $v(v-1) \equiv 0$ (*mod* 14) *and* $v \geq 7$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.3 and Lemma 5.6. □

Lemma 5.8 *There exist* $(v, G_4, 1)$ *for* $v = 7, 8$.

Proof.

$v = 7$: $X = (Z_3 \times Z_2) \bigcup \{\infty\}$

$(0_1, \infty, 0_0, 1_0, 1_1, 2_0, 2_1)$ mod (3,–)

$v = 8$: $X = Z_8$

$(0, 5, 7, 2, 6, 3, 1)$, $(5, 6, 1, 3, 4, 2, 0)$, $(2, 4, 6, 7, 0, 1, 3)$, $(3, 7, 4, 1, 5, 2, 0)$. □

Theorem 5.9 *There exist* $(v, G_4, 1)$ *if and only if* $v(v-1) \equiv 0$ (*mod* 14) *and* $v \geq 7$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.4 and Lemma 5.8. □

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