Hamilton cycles in a family of graphs which includes the generalized Petersen graphs

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Abstract

It is well known that the Petersen graph does not contain a Hamilton cycle. In 1983 Alspach completely determined which Generalized Petersen graphs are Hamiltonian [1]. In this paper we define a larger class of graphs which includes the Generalized Petersen graphs as a special case, and determine which graphs in this larger class are Hamiltonian, and which are 1-factorable. We call this larger class *spoked Cayley graphs.*

keywords:

Hamilton cycle, Hamiltonian, generalized Petersen graph, spoked Cayley graph, I-graph, Petersen graph, vertex-transitive, Tait coloring, 1-factorization, Y-graph

1 Introduction

Hamilton cycles and the *Petersen graph* P, both featured in the early history of the four colour problem, and have both become popular concepts of graph theory (see [9] and [12]). The Petersen graph became of interest as a cubic bridgeless graph which has no 1-factorization, and consequently no Hamilton cycle (see [9]). The *generalized Petersen Graphs*, first defined by Watkins [14], provided more examples of cubic, bridgeless graphs and have also been studied in this regard.

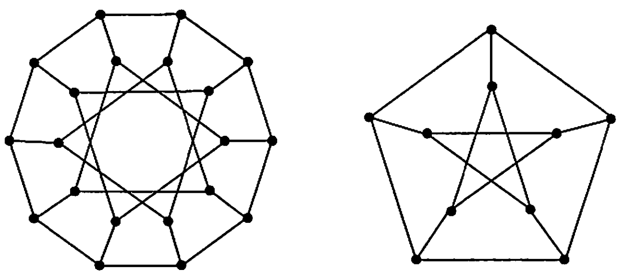
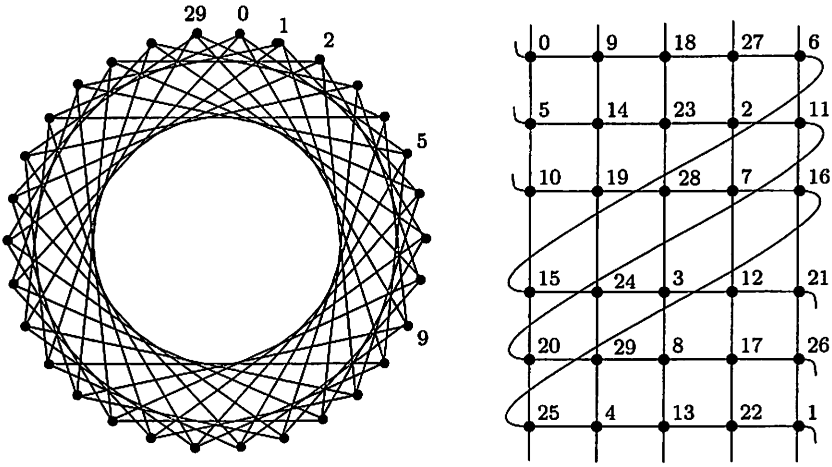


Figure 1: GP$(10,3)$ and GP$(5,2) = P$

The *generalized Petersen graph* $GP(n,k)$, where $n \geq 3$ and $1 \leq k < n$, has vertices $u_0, u_1, \ldots, u_{n-1}$, $v_0, v_1, \ldots, v_{n-1}$, and edges $\{u_i, u_{i+1}\}, \{v_i, v_{i+k}\}$ and $\{u_i, v_i\}$ for each i with $0 \leq i < n$. The edges of type $\{u_i, v_i\}$ are usually called *spokes* (see Figure 1). Castagna and Prins showed in 1972 that $P \cong \text{GP}(5,2) \cong \text{GP}(5,3)$ is the only cubic generalized Petersen graph which has no 1-factorization. Ten years later, after partial results were found by various authors (see [8]), Alspach [1] completely determined which generalized Petersen graphs are Hamiltonian (Theorem 1.1 below). In this paper we generalize this family of graphs further to a new class, which we call *spoked Cayley graphs* and determine which *spoked Cayley graphs on Abelian groups* are Hamiltonian, and also which are 1-factorable. But first we define a *Cayley graph on an Abelian group.*

Given a finite Abelian group A and a subset $S \subseteq A \setminus \{0\}$, the *Cayley graph* Cay$(A, S)$ is the graph with vertex set A and edge set $\{\{a, a+s\} | a \in A,\ s \in S\}$. If $S = \{s, t\}$ where $2s \neq 0, 2t \neq 0$ and $s \neq \pm t$, then Cay(A, S) is a 4-regular graph. Figure 2 (a) shows a circular representation of the the Cayley graph Cay$(Z_{30}, \{9, 5\})$, and Figure 2 (b) shows the (isomorphic) *pseudo-cartesian product* $C_5 \times_3 C_6$. *Pseudo-cartesian products* were defined in [7], see also [5] and [6]. Note, for clarity of the figure, some of the edges are not fully drawn.

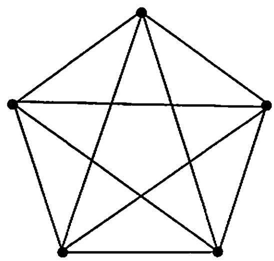


(a) (b)

Figure 2: $\mathrm{Cay}(Z_{30}, \{9, 5\}) \cong C_5 \times_3 C_6$.

In this paper, all groups under consideration will be Abelian, so we use additive notation for group operations. Note however, that the following definition also applies in the case of non-Abelian groups. Given a finite group A and two non-zero elements $s, t \in A$, not necessarily distinct, we define the *spoked Cayley graph* $\mathrm{SCay}(A, s, t)$ as follows. For each element $a \in A$, the graph has two vertices, labelled a and a'. The edges of the graph are $\{a, a+s\}, \{a, a'\}$ and $\{a', (a+t)'\}$ for each $a \in A$. The use of the same labels for group elements and (half of the) vertices should not cause confusion. The edges of type $\{a, a'\}$ are again called *spokes*. Figure 3 shows $\mathrm{Cay}(Z_5, \{1, 2\}) \cong K_5$ and $\mathrm{SCay}(Z_5, 1, 2) \cong P$. If each spoke $\{a, a'\}$ of the spoked Cayley graph $\mathrm{SCay}(A, s, t)$, is contracted to a single vertex a, we obtain the Cayley graph $\mathrm{Cay}(A, \{s, t\})$. Compare also Figures 2 and 4.

Spoked Cayley graphs might also have been called *further generalized Petersen graphs* as the generalized Petersen graph $GP(n, k)$ is the spoked Cayley graph $\mathrm{SCay}(Z_n, 1, k)$. Thus we have generalized the family of generalized Petersen graphs in two ways: firstly by considering A an arbitrary finite group rather than just a cyclic group, and secondly by allowing both s and t to be any non-zero elements. Spoked Cayley graphs on cyclic groups are also studied in [13]. Alspach [1] completely determined which generalized Petersen graphs contain a Hamilton cycle.



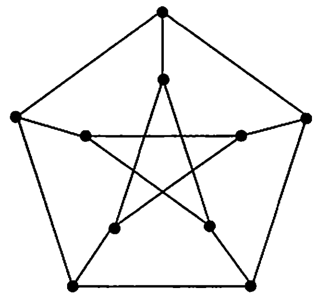


Figure 3: $K_5 \cong \mathrm{Cay}(Z_5, \{1,2\})$ and $\mathrm{SCay}(Z_5, 1, 2) \cong P$

Theorem 1.1 *The generalized Petersen graph $GP(n,k)$ is Hamiltonian if and only if it is neither*

(1) $GP(n, \pm 2) \cong GP(n, (n \pm 1)/2)$ *where* $n \equiv 5 \pmod 6$, *nor*

(2) $GP(n, n/2)$ *where* $n \equiv 0 \pmod 4$ *and* $n \geq 8$.

In this paper we prove the following result.

Theorem 1.2 *A connected spoked Cayley graph on an Abelian group is Hamiltonian if and only if it is none of the following.*

(1) $\mathrm{SCay}(Z_n, s, \pm 2s)$ *where* $n \equiv 5 \pmod 6$ *and* $\gcd(s,n) = 1$,

(2) $\mathrm{SCay}(Z_n, s, n/2)$ *where* $n \equiv 0 \pmod 4$, $n \geq 8$ *and* $\gcd(s,n) = 1$, *and*

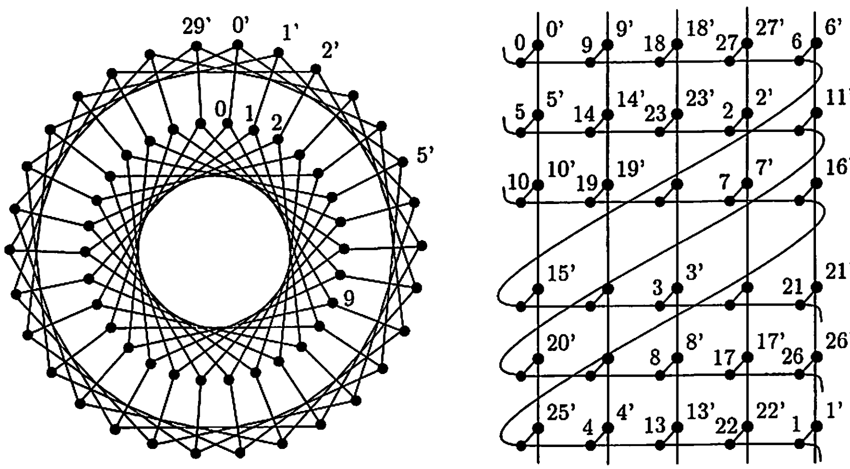
(3) $\mathrm{SCay}(Z_n, s, n/2)$ *where* $n \equiv 2 \pmod 4$ *and* $\gcd(s,n) = 2$,

$\cong \mathrm{SCay}(Z_{n/2} \times Z_2, (a,0), (0,1))$ *where $n/2$ is odd and* $\gcd(a, n/2) = 1$.

We wish to consider only simple graphs, without loops. Consequently, if $2s = 2t = 0$, then $\mathrm{SCay}(A, s, t)$ is 2-regular. If $2s \neq 0$ and $2t \neq 0$, then $\mathrm{SCay}(A, s, t)$ is 3-regular. Otherwise $\mathrm{SCay}(A, s, t)$ is not regular. The exceptions in Class (1) of Theorem 1.2 are 3-regular, but those in Classes (2) and (3) are not regular. Let $\langle s, t \rangle$ denote the subgroup generated by s and t. Each component of $\mathrm{SCay}(A, s, t)$ is isomorphic to $\mathrm{SCay}(\langle s, t \rangle, s, t)$. So, $\mathrm{SCay}(A, s, t) \cong \mathrm{SCay}(A, t, s)$ is connected if and only if $A = \langle s, t \rangle$. Since we are interested in Hamilton cycles, for the remainder of this paper, except where otherwise stated, all the spoked Cayley graphs we consider will be connected.

2 Preliminaries

We introduce the *spoke product* as a convenient way to represent a connected spoked Cayley graph on an Abelian group (see Figure 4). We will work with *spoke products* to obtain our result.



(a) (b)

Figure 4: $\mathrm{SCay}(Z_{30}, 9, 5) \cong C_5 \ast_3 C_6$.

We define the *spoke product* $C_\alpha \ast_\gamma C_\beta$ for integers α, β and γ where $\alpha > 0$ and $0 \le \gamma < \beta$ as follows. For each element $(i,j) \in Z_\alpha \times Z_\beta$, there are two vertices, labelled (i,j) and $(i,j)'$. There are edges of four types:

(i)	$\{(i,j),(i,j)'\}$	for $i \in Z_\alpha$, $j \in Z_\beta$,	
(ii)	$\{(i,j)',(i,j+1)'\}$	for $i \in Z_\alpha$, $j \in Z_\beta$,	
(iii)	$\{(i,j),(i+1,j)\}$	for $i \in Z_\alpha \backslash \{\alpha-1\}$, $j \in Z_\beta$,	and
(iv)	$\{(\alpha-1,j),(0,\gamma+j)\}$	for $j \in Z_\beta$.	

We shall follow the convention that vertices (i,j) and $(i,j)'$ both lie in column i and row j. Figure 4(b) shows the spoke product $C_5 \ast_3 C_6$. The labelling of this figure illustrates the following isomorphism between spoked Cayley graphs and spoke products.

Lemma 2.1 *A connected spoked Cayley graph* $\mathrm{SCay}(A, s, t)$ *on an Abelian group* A*, is isomorphic to the spoke product* $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ *where*

$$\sigma = |A/\langle s\rangle|, \qquad \tau = |A/\langle t\rangle|, \qquad k = |\langle s\rangle \cap \langle t\rangle|, \quad \textit{and}$$
$$\textit{if } k = 1, \textit{ then } \pi = 0, \quad \textit{or}$$
$$\textit{if } k > 1, \textit{ then } \pi \textit{ is given by } \pi\sigma t = \tau s \textit{ so that } \gcd(\pi, k) = 1.$$

The function f *which maps vertices* $is + jt \mapsto (i, j)$ *and* $(is + jt)' \mapsto (i, j)'$*, where* $0 \leq i < \tau$ *and* $0 \leq j < k\sigma$*, is such an isomorphism.*

Proof. Group isomorphism theorems such as $|\langle s,t\rangle| \cdot |\langle s\rangle \cap \langle t\rangle| = |\langle s\rangle| \cdot |\langle t\rangle|$ imply $|A| = k\sigma\tau$, $|\langle s\rangle| = k\tau$ and $|\langle t\rangle| = k\sigma$. We also note that

$$\langle \sigma t\rangle = \langle \tau s\rangle = \langle s\rangle \cap \langle t\rangle \quad \text{is a cyclic group of order } k. \tag{1}$$

Since $A = \langle s, t\rangle$, each $a \in A$ may be uniquely expressed as $a = is + jt$ where $0 \leq i < |A/\langle t\rangle| = \tau$ and $0 \leq j < |\langle t\rangle| = k\sigma$. Thus f is a bijection. To see that f preserves incidence, we consider its effect on the three kinds of edges. If $a = is + jt$, then $\{a, a'\} \mapsto \{(i, j), (i, j)'\}$, $\{a', (a + t)'\} \mapsto \{(i, j)', (i, j + 1 \pmod{k\sigma})'\}$, and provided $i \neq \tau - 1$, $\{a, a + s\} \mapsto \{(i, j), (i + 1, j)'\}$. If $a = (\tau - 1)s + jt$, the adjacent vertex $\tau s + jt$ maps to $(0, r + j)$ for some r, since $\tau s \in \langle t\rangle$. If $k = 1$, then $\tau s = 0$, so $\pi = 0$. If $k > 1$, then $\tau s = \pi\sigma t$ for some π coprime to k, by equation (1). Thus $r = \pi\sigma$ as required. □

For each $0 \leq i < \tau$, the connected spoked Cayley graph $\mathrm{SCay}(A, s, t)$ has a cycle of length $|\langle t\rangle|$ on the primed vertices corresponding to coset $\langle t\rangle + is$. We call this $|\langle t\rangle|$-cycle i. Similarly, for each $0 \leq j < \sigma$, $\mathrm{SCay}(A, s, t)$ has a cycle of length $|\langle s\rangle|$ on the unprimed vertices corresponding to coset $\langle s\rangle + jt$. We call this $|\langle s\rangle|$-cycle j. In the spoke product $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$, we define *vertical cycle* i for each $0 \leq i < \tau$, as the cycle on the primed vertices in column i. We also define *horizontal cycle* j for each $0 \leq j < \sigma$, as the cycle on the unprimed vertices in the k rows $j, \pi\sigma + j, 2\pi\sigma + j, \ldots, (k - 1)\pi\sigma + j$ where all row numbers are computed mod $k\sigma$. Thus the isomorphism of Lemma 2.1 maps $|\langle t\rangle|$-cycle i to vertical cycle i for each $0 \leq i < \tau$, and maps $|\langle s\rangle|$-cycle j to horizontal cycle j for each $0 \leq j < \sigma$.

If $k > 1$, then for each $0 \leq l < k$ we define the *l-th box* B_l of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ as the induced subgraph on vertices (i, j) and $(i, j)'$ for $0 \leq i < \tau$ and $l\sigma \leq j < (l + 1)\sigma$. If $k = 1$, we define the *0-th box* B_0, of $C_\tau \circledast_0 C_\sigma$ as $C_\tau \circledast_0 C_\sigma$ with the following edges removed: those connecting column $\tau - 1$ with column 0 and those connecting row $\sigma - 1$ with row 0. Figures 5, 11, 13 and 16 show each box contained in a dotted region.

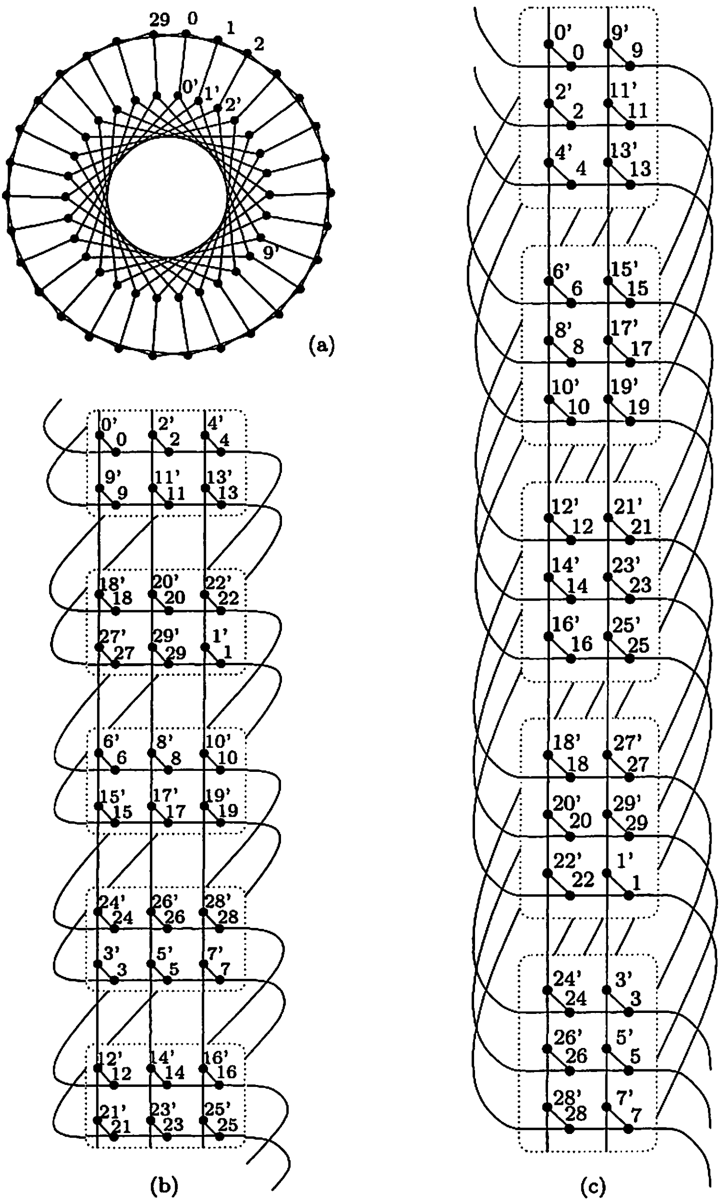


Figure 5: (a) SCay(Z_{30}, {2,9}), (b) $C_3 \circledast_4 C_{10}$, (c) $C_2 \circledast_9 C_{15}$

Lemma 2.2 *Given positive integers σ and τ, $C_\tau \circledast_0 C_\sigma \cong C_\sigma \circledast_0 C_\tau$. Further, if $k > 1$ and* $\gcd(\pi, k) = 1$, *then $C_\tau \circledast_{\pi\sigma} C_{k\sigma} \cong C_\sigma \circledast_{\phi\tau} C_{k\tau}$ for some ϕ satisfying* $\gcd(\phi, k) = 1$.

Proof.

By Lemma 2.1, $C_\tau \circledast_{\pi\sigma} C_{k\sigma} \cong \mathrm{SCay}(A, s, t) \cong \mathrm{SCay}(A, t, s) \cong C_\sigma \circledast_{\phi\tau} C_{k\tau}$ where $\pi = \tau s/(\sigma t) \pmod{k}$ and $\phi = \sigma t/(\tau s) \pmod{k}$. Since $\pi\phi = 1 \pmod{k}$ the result follows. □

Figure 5 illustrates the above lemmas. The spoked Cayley graph $\mathrm{Cay}(Z_{30}, 2, 9)$ in Figure 5(a) is isomorphic to the spoke product $C_3 \circledast_4 C_{10}$ in Figure 5(b) by Lemma 2.1. The two spoke products $C_3 \circledast_4 C_{10}$ and $C_2 \circledast_9 C_{15}$ in Figures 5(b) and (c) are isomorphic by Lemma 2.2.

Let $F_0 = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma})$ be the subgraph of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ obtained by deleting the spokes. If F_0 is 2-regular, we call it the *natural 2-factor*. It consists of the τ vertical cycles and the σ horizontal cycles. To construct a Hamilton cycle in a spoke product we shall usually begin with the natural 2-factor, and *divert* it at certain locations. We now define this notion. Given a vertex (i, j) where $i \neq \tau - 1$, and some 2-factor F of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ which contains all the edges in $E_1(i,j) = \{\{(i,j),(i+1,j)\},\{(i,j+1),(i+1,j+1)\},\{(i,j)',(i,j+1)'\},\{(i+1,j)',(i+1,j+1)'\}\}$, but none of the edges (spokes) in $E_2(i,j) = \{\{(i,j),(i,j)'\},\{(i,j+1),(i,j+1)'\},\{i+1,j),(i+1,j)'\},\{(i+1,j+1),(i+1,j+1)'\}\}$, then the *diversion of F at* (i,j) is defined and denoted by $F \cdot \{(i,j)\} = (F \setminus E_1(i,j)) \cup E_2(i,j)$.

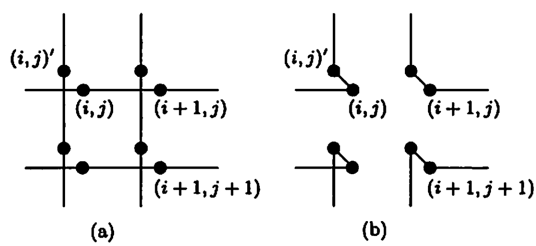


Figure 6: (a) 2-factor F and (b) diverted 2-factor $F \cdot \{(i,j)\}$

We say that the diversion occurs in column i and row j, see Figure 6. Note that if F is a 2-factor of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$, then $F \cdot \{(i,j)\}$ is also. If D is a set of permissible diversion locations for a 2-Factor F, we write $F \cdot D$ for the diversion of F at each element of D. If F_0 is the natural 2-factor of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ and $F_0 \cdot D$ is a Hamilton cycle, we say that D is a *joining set* for the graph $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$. Figure 7 shows joining sets for $C_\tau \circledast_0 C_\sigma$ for various σ and τ. In the figure, diversions are indicated by circles.

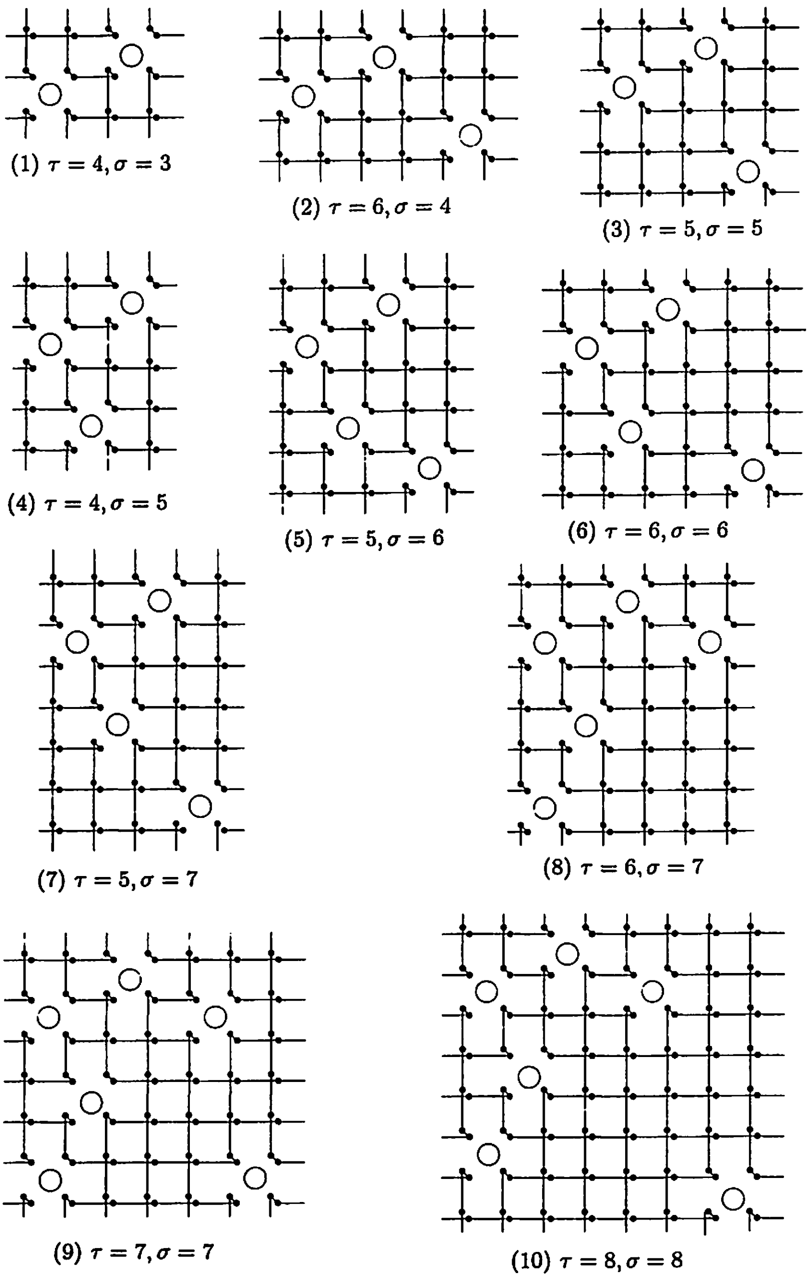


Figure 7: Joining sets for $C_\tau \circledast_0 C_\sigma$

In this paper we have no need to consider a diversion in column $\tau - 1$ of $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$. Further, we say a diversion set D is *proper* for a spoke product $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ if and only if none of the diversions in D are on horizontal cycle $\sigma - 1$. Thus the edges affected by a proper diversion set are in the boxes of the graph.

If A and B are subgraphs of some graph, we denote by $A - B$ the graph induced by the edges in A but not in B. Given an edge $\{v_1, v_2\}$ in a graph G, the graph G_1 obtained by deleting this edge and identifying vertices v_1 and v_2 is called an *edge contraction* of G. If a graph G_k can be obtained from G by a sequence of edge contractions, then G_k is said to be a *contraction* of G and we say that G is *contractible* to G_k. Clearly, graphs have the same number of components as their contractions.

3 Results

We prove our main theorem (Theorem 1.2) by first establishing which spoke products are Hamiltonian (Lemma 3.6). For spoke products $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ with large values of σ and τ, as in Lemma 3.5, joining sets can be found which lie properly inside a single box. For spoke products with small values of σ and τ, we sometimes can find joining sets which lie properly inside two or three boxes. Sometimes other constructions are more convenient. Lemmas 3.1 and 3.2 are technical lemmas which support inductive arguments on the values of τ and σ respectively. Lemmas 3.3 and 3.4 are also supporting lemmas concerning the number of boxes, k.

Lemma 3.1 *If D_1 is a joining set for $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ where $k\sigma > 2$, and (c,d) is the only element of D_1 in vertical cycle c, then $D_2 = \{(i,j) \in D_1 | i \leq c\} \cup \{(i+4,j) | (i,j) \in D_1, i \geq c\} \cup \{(c+2,d+1)\}$ is a joining set for $C_{\tau+4} \circledast_{\pi\sigma} C_{k\sigma}$. Also, (c,d) is the only element of D_2 in vertical cycle c of $C_{\tau+4} \circledast_{\pi\sigma} C_{k\sigma}$.*

Proof. Let $H_1 = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D_1$ and $H_2 = F_0(C_{\tau+4} \circledast_{\pi\sigma} C_{k\sigma}) \cdot D_2$. We show that H_1 is a contraction of H_2 (see Figure 8). For each row $y \neq d, d+1$ or $d+2$, the path $(c,y), (c+1,y), (c+2,y), (c+3,y), (c+4,y), (c+5,y)$ in H_2 contracts to the edge $\{(c,y), (c+1,y)\}$ in H_1. The path $(c,d+2), (c+1,d+2), \ldots, (c+4,d+2), (c+5,d+2)$ in H_2 contracts to the edge $\{(c,d+2), (c+1,d+2)\}$ in H_1. Thus both H_1 and H_2 have one component. Also H_2 is a 2-factor of $C_{\tau+4} \circledast_{\pi\sigma} C_{k\sigma}$, so it is a Hamilton cycle. □

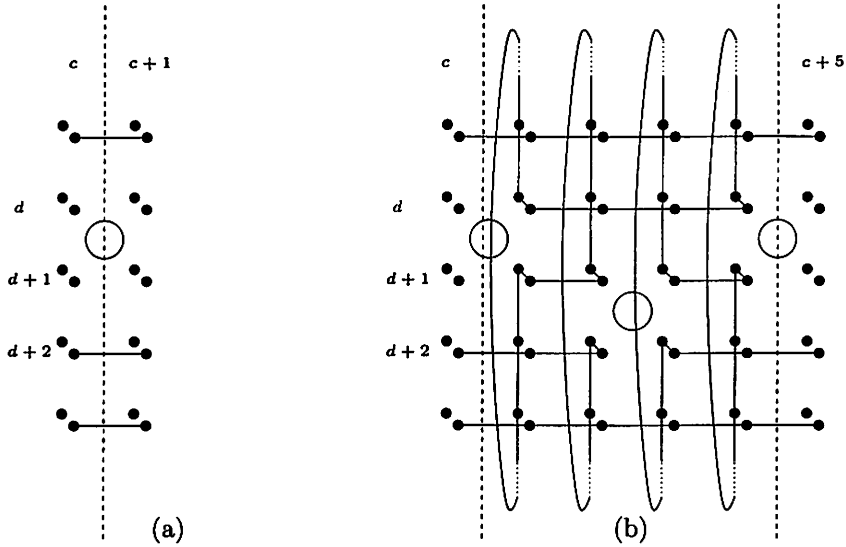


Figure 8: (a) $H_1 = F_0(C_\tau *_{\pi\sigma} C_{k\sigma}) \cdot D_1$ and (b) $H_2 = F_0(C_{\tau+4} *_{\pi\sigma} C_{k\sigma}) \cdot D_2$

Lemma 3.2 *If D_1 is a joining set for $C_\tau *_{\pi\sigma} C_{k\sigma}$ where $\tau > 2$, and (c, d) is the only element of D_1 in horizontal cycle $d \pmod{\sigma}$, then, $D_2 = \{(i, j) \in D_1 | j \leq d\} \cup \{(i, j+4) | (i, j) \in D_1, j \geq d\} \cup \{(c+1, d+2)\}$ is a joining set for $C_\tau *_{\pi(\sigma+4)} C_{k(\sigma+4)}$. Also, (c, d) is the only element of D_2 in horizontal cycle $d \pmod{\sigma}$ of $C_\tau *_{\pi(\sigma+4)} C_{k(\sigma+4)}$.*

Proof. This follows immediately from Lemma 3.1 and Lemma 2.2. □

Lemma 3.3 *If D is a proper joining set for either $C_\tau *_0 C_\sigma$ or $C_\tau *_\sigma C_{2\sigma}$, then D is a joining set for $C_\tau *_{\pi\sigma} C_{k\sigma}$ for any $k > 1$ and π coprime to k.*

Proof. The proof splits into two cases.

Case(1) D is a proper joining set for $C_\tau *_0 C_\sigma$.

Let $R = F_0(C_\tau *_{\pi\sigma} C_{k\sigma}) \cdot D - B_0 = F_0(C_\tau *_{\pi\sigma} C_{k\sigma}) - B_0$. So R consists of $\sigma + \tau$ disjoint paths: a subpath of each vertical cycle and of each horizontal cycle. Contracting each path of R in $F_0(C_\tau *_{\pi\sigma} C_{k\sigma}) \cdot D$ to a single edge gives $F_0(C_\tau *_0 C_\sigma) \cdot D$ which is a Hamilton cycle. Thus the 2-factor $F_0(C_\tau *_{\pi\sigma} C_{k\sigma}) \cdot D$ also has one component, and hence it is a Hamilton cycle.

Case(2) D is a proper joining set for $C_\tau *_\sigma C_{2\sigma}$.

Let $R = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D - B_0 - B_1 = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) - B_0 - B_1$. So R consists of $2(\sigma + \tau)$ disjoint paths: two subpaths of each vertical cycle and each horizontal cycle. Contracting each path of R in $F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D$ to a single edge gives $F_0(C_\tau \circledast_\sigma C_{2\sigma}) \cdot D$ which is a Hamilton cycle. Thus the 2-factor $F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D$ also has one component, and hence it is a Hamilton cycle. □

Lemma 3.4 *For any integers $\sigma > 1, \tau > 1, k \geq 3$ and π coprime to k, if D_1 is a proper joining set for $C_\tau \circledast_\sigma C_{3\sigma}$ and D_2 is a proper joining set for $C_\tau \circledast_{2\sigma} C_{3\sigma}$, then either D_1 or D_2 is a joining set for $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$.*

Proof. Let D be a proper diversion set for $C_\tau \circledast_{\pi\sigma} C_{3\sigma}$ where $\pi = 1$ or 2. We show that $F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D$ is contractible to either $F_0(C_\tau \circledast_\sigma C_{3\sigma}) \cdot D$ or $F_0(C_\tau \circledast_{2\sigma} C_{3\sigma}) \cdot D$. To see this, consider $R = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D - B_0 - B_1 - B_2 = F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) - B_0 - B_1 - B_2$. Note that R consists of $3(\sigma + \tau)$ disjoint paths of two kinds:

(i) 3 subpaths of vertical cycle i for each $0 \leq i < \tau$, and

(ii) 3 subpaths of horizontal cycle j for each $0 \leq j < \sigma$.

Depending on the values of π and k, the (directed) paths of type (ii) will connect the boxes B_0, B_1 and B_2 in either the cyclic order (012) or (021). Thus, contracting each path of R in $F_0(C_\tau \circledast_{\pi\sigma} C_{k\sigma}) \cdot D$ to a single edge gives either $F_0(C_\tau \circledast_\sigma C_{3\sigma}) \cdot D$ or $F_0(C_\tau \circledast_{2\sigma} C_{3\sigma}) \cdot D$. In the first case let $D = D_1$. In the second let $D = D_2$. Then either D_1 or D_2 is a joining set for $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$.

□

Lemma 3.5 *The spoke product $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$, where either π is coprime to k, or both $k = 1$ and $\pi = 0$, has a joining set for all integers σ, τ, π and k satisfying*

(i) $\sigma > 4$ *and* $\tau > 4$, *or*
(ii) $\{\sigma, \tau\} = \{4, x\}$ *where* $x \geq 3$ *and* $x \not\equiv 0 \pmod 4$, *or*
(iii) $\{\sigma, \tau\} = \{3, x\}$ *where* $x \equiv 0 \pmod 4$.

Proof. If the values of τ and σ satisfy one of the three conditions of this lemma, then τ and σ can be expressed in terms of positive integers a and b, by exactly one of the sixteen cases listed below:

(1) $\tau = 4a,\ \sigma = 4b - 1$
(2) $\tau = 4a + 2,\ \sigma = 4b$
(3) $\tau = 4a + 1,\ \sigma = 4b + 1$
(4) $\tau = 4a,\ \sigma = 4b + 1$
(5) $\tau = 4a + 1,\ \sigma = 4b + 2$
(6) $\tau = 4a + 2,\ \sigma = 4b + 2$
(7) $\tau = 4a + 1,\ \sigma = 4b + 3$
(8) $\tau = 4a + 2,\ \sigma = 4b + 3$
(9) $\tau = 4a + 3,\ \sigma = 4b + 3$
(10) $\tau = 4a + 4, \sigma = 4b + 4$
(11) $\tau = 4a - 1, \sigma = 4b$
(12) $\tau = 4a,\ \sigma = 4b + 2$
(13) $\tau = 4a + 1, \sigma = 4b$
(14) $\tau = 4a + 2, \sigma = 4b + 1$
(15) $\tau = 4a + 3, \sigma = 4b + 1$
(16) $\tau = 4a + 3, \sigma = 4b + 2$

Case (1) $\tau = 4a,\ \sigma = 4b - 1$:

We show that $D_{(1)} = \{(0,1),(2,0),(4,1),(6,0),\ldots,(4a-2,0),(1,3),(0,5),(1,7),(0,9),\ldots,(0,4b-3)\}$ is a joining set for $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$. Figure 7(1) illustrates that $\{(0,1),(2,0)\}$ is a joining set for $C_4 \circledast_0 C_3$. This is easily checked by hand. Applying Lemma 3.2 to $C_4 \circledast_0 C_3$ with $(c,d) = (0,1)$ as many times as necessary, we obtain that $\{(0,1),(2,0)\} \cup \{(1,3),(0,5),(1,7),(0,9),\ldots,(0,4b-3)\}$ is a joining set for $C_4 \circledast_0 C_\sigma$. Applying Lemma 3.1 to $C_4 \circledast_0 C_\sigma$ with $(c,d) = (2,0)$ as many times as necessary, establishes that $D_{(1)}$ is a joining set for $C_\tau \circledast_0 C_\sigma$. Note that $D_{(1)}$ is a proper joining set for $C_\tau \circledast_0 C_\sigma$. Lemma 3.3 establishes that $D_{(1)}$ is also a joining set for $C_\tau \circledast_{\pi\sigma} C_{k\sigma}$ for any $k \geq 1$ and any π coprime to k.

Cases (2) - (10) are proven similarly and we omit the proofs. The joining set for each case is listed below, and illustrated in Figure 7 for the values $a = b = k = 1$.

$D_{(2)} = D_{(1)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(3)} = D_{(1)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(4)} = D_{(1)} \cup \{(1, 4b - 1)\}$
$D_{(5)} = D_{(4)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(6)} = D_{(4)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(7)} = D_{(4)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(8)} = D_{(4)} \cup \{(4a, 1), (0, 4b + 1)\}$
$D_{(9)} = D_{(8)} \cup \{(\tau - 2, \sigma - 2)\}$
$D_{(10)} = D_{(8)} \cup \{(\tau - 2, \sigma - 2)\}$

Cases (11) - (16) These spoke products are isomorphic (by Lemma 2.2) to the spoke products of cases (1),(2),(4),(5),(7) and (8) respectively.

□

Lemma 3.6 *Every spoke product contains a Hamilton cycle, except for the following, which do not:*

(0) $C_1 \circledast_0 C_n \cong C_n \circledast_0 C_1$ *where* $n \geq 1$,

(1) $C_1 \circledast_r C_n$ *where* $n \equiv 5 \pmod 6$ *and* $r = \pm 2, (n \pm 1)/2$,

(2) $C_n \circledast_1 C_2 \cong C_1 \circledast_n C_{2n}$ *where* n *is even and* $n \geq 4$, *and*

(3) $C_n \circledast_0 C_2 \cong C_2 \circledast_0 C_n$ *where* n *is odd.*

Proof. Without loss of generality, let our spoke product be $C_\tau *_{\pi\sigma} C_{k\sigma}$ where π is coprime to k, or $\pi = 0$ and $k = 1$. By Lemma 2.2 it suffices to examine spoke products where $\sigma \leq \tau$. We divide the proof into cases on the value of σ.

Case (i) $\sigma = 1, \tau \geq 1$: We divide this into two subcases.

(a) If $k = 1$, our graph is $C_\tau *_0 C_1$. All the primed vertices of this graph have degree one, so this graph is not Hamiltonian. Hence the class (0) of exceptions listed above.

(b) If $k > 1$, by Lemma 2.1, $C_\tau *_\pi C_k \cong \mathrm{SCay}(Z_{k\tau}, 1, t) \cong \mathrm{GP}(k\tau, t)$ for some $t \neq 0$. By Theorem 1.1 these are all Hamiltonian except for classes (1) and (2) listed above.

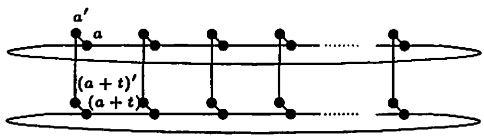


Figure 9:

Case (ii) $\sigma = 2, \tau \geq 2$: We divide this into three subcases.

(a) If τ is odd and $k = 1$, each of the primed vertices of $C_\tau *_0 C_2$ has degree 2. See Figure 9. Suppose this graph contains a Hamilton cycle. Then the Hamilton cycle contains the paths $(i, 0), (i, 0)', (i, 1)', (i, 1)$ for each $0 \leq i < \tau$. Since the number of such paths is odd, this is impossible. We see that $C_\tau *_0 C_2 \cong C_2 *_0 C_\tau$ is class (3) of non-Hamiltonian graphs.

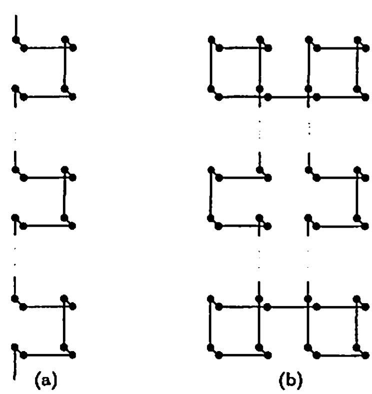


Figure 10:

(b) If τ is odd and $k = 2$, or if τ is even, then by Lemma 2.2 the spoke product is isomorphic to $C_2 \circledast_r C_{2n}$ for some r and n. Figure 10(a) illustrates a Hamilton cycle in this graph.

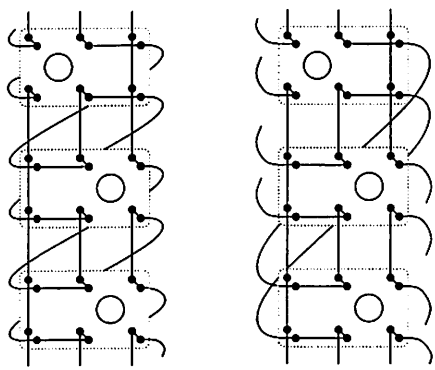


Figure 11:

(c) If τ is odd and $k \geq 3$, the spoke product is shown to be Hamiltonian as follows. Firstly, if $k = 3$, then the diversion set $D_\tau = \{(0,0),(1,3),(2,0),(3,3),\ldots,(\tau-3,0),(\tau-2,3),(\tau-2,5)\}$ is a joining set for the spoke product $C_\tau \circledast_{\pi 2} C_6$, for both values $\pi = 1$ and $\pi = 2$. This can be shown by induction on τ: Figure 11 illustrates the basis case(s) when $\tau = 3$.

The inductive step $\tau \Rightarrow \tau + 2$, is illustrated in Figure 12.

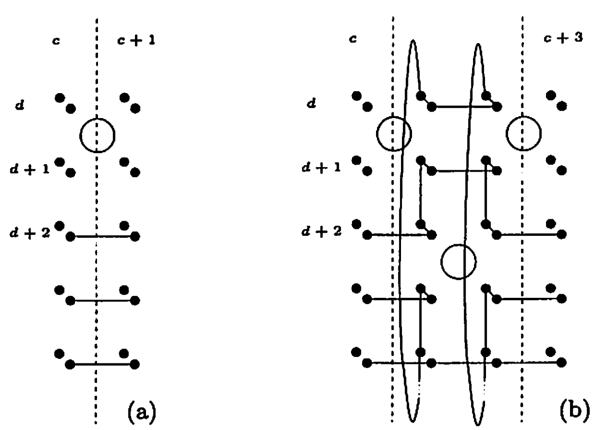
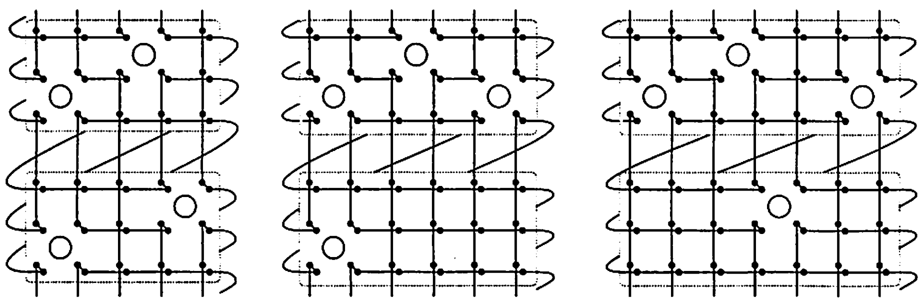


Figure 12:

The proof of this inductive step is similar to that of Lemma 3.1 and we omit it. Finally, Lemma 3.4 establishes that the diversion set D_τ is also a joining set for $C_\tau \circledast_{\pi 2} C_{k2}$ for any $k \geq 3$ and any π coprime to k.

Case (iii) $\sigma = 3, \tau \geq 3$: We divide this into seven subcases.

(a) If $\tau \equiv 0 \pmod 4$, then the proof is given in Lemma 3.5 case (1).



(a) (b) (c)

Figure 13:

(b) If $\tau \equiv 1, 2$ or $3 \pmod 4$, $\tau \geq 5$ and $k > 1$, then the proof, similar to that of Lemma 3.5, uses an inductive argument on τ based on joining sets given in Figures 13(a), 13(b) and 13(c) respectively. Lemmas 3.1 and 3.3 then imply the result.

(c) If $\tau \equiv 1 \pmod 4$ and $k = 1$, then a Hamilton cycle is given in Figure 14.

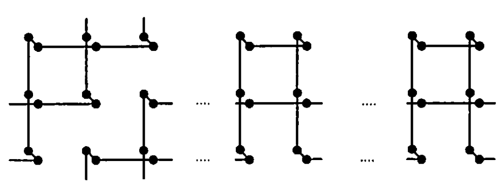


Figure 14:

(d) If $\tau \equiv 2 \pmod 4$ and $k = 1$, then $D = \{(0,0), (2,1), (4,2)\}$ is a joining set for the basis case $\tau = 6$. Applying Lemma 3.1 to this joining set with $(c, d) = (0, 0)$ proves the result.

(e) If $\tau \equiv 3 \pmod 4$ and $k = 1$, then Figure 14 shows a Hamilton cycle.

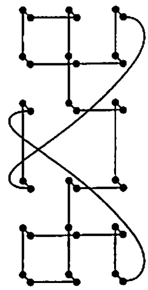


Figure 15:

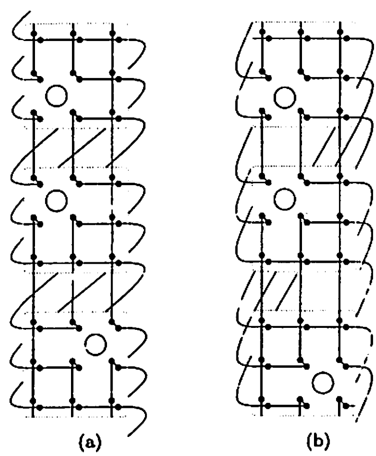


Figure 16:

(f) If $\tau = 3$ and $k = 2$, then Figure 15 shows a Hamilton cycle.

(g) If $\tau = 3$ and $k \geq 3$, Figure 16(a) shows a joining set D_1 for $C_3 \circledast_3 C_9$, and Figure 16(b) shows a joining set D_2 for $C_3 \circledast_6 C_9$. Lemma 3.4 establishes that either D_1 or D_2 is a joining set for $C_3 \circledast_{3\pi} C_{3k}$ for any $k \geq 3$ and any π coprime to k.

Case (iv) $\sigma \geq 4, \tau \geq 4$,

Lemma 3.5 shows that these spoke products are Hamiltonian except possibly when $\sigma = 4$ and $\tau \equiv 0 \pmod 4$, or when $\tau = 4$ and $\sigma \equiv 0 \pmod 4$. By Lemma 2.2 it suffices to show the latter are Hamiltonian. Figure 10(b) illustrates a solution.

□

We can now prove our main result.

Proof. **(of Theorem 1.2)** By Lemma 2.1, each spoked Cayley graph is isomorphic to some spoke product. Lemma 3.6 lists four classes of non-Hamiltonian spoke products. Of these four, classes (1) - (3) of Lemma 3.6 are isomorphic to classes (1) - (3) of Theorem 1.2 respectively. The isomorphisms are given by Lemmas 2.1, 2.2 and group automorphisms. However, no spoked Cayley graphs are isomorphic to the spoke products in class (0) of Theorem 3.6, as this would imply that either s or t is zero. Hence the result follows by Lemma 3.6.

□

4 Epilogue

We present two consequences of Theorem 1.2 and ask a question.

Corollary 4.1 *Every regular spoked Cayley graph on an Abelian group is 1-factorable, except those which consist of disjoint copies of the Petersen graph.*

Proof. If the spoked Cayley graph $\mathrm{SCay}(A, s, t)$ is not connected, then it consists of disjoint copies of a connected spoked Cayley graph. A regular graph is 1-factorable if and only if each of its components is also 1-factorable. Hence we need only consider connected graphs. Except those graphs listed in Class (1) of Theorem 1.2, every connected regular spoked Cayley graph on an Abelian group is Hamiltonian. A Hamilton cycle in a 3-regular graph immediately gives a 1-factorization. The graphs in Class (1) of Theorem 1.2 are all 3-regular generalized Petersen graphs. Castagna and Prins [4] showed that the Petersen graph is the only 3-regular generalized Petersen graph which is not 1-factorable. Finally, any 2-regular spoked Cayley graph can be 1-factored by taking the spokes as a 1-factor. So, the Petersen graph is the only connected regular spoked Cayley graph on an Abelian group which is not 1-factorable. □

Thomassen has conjectured that there are only finitely many connected vertex-transitive non-Hamiltonian graphs [2]. Only five have been published in the literature. The four non-trivial ones are the Petersen graph, the Coxeter graph C_{28} and the graphs obtained from these two by replacing each vertex with a triangle (see [8] or [9]).

Classes (2) and (3) of Theorem 1.2, are not vertex-transitive since they are not regular. Class (1) graphs of Theorem 1.2 are the generalized Petersen graphs in Class (1) of Theorem 1.1. Frucht, Graver and Watkins [11] have shown that the only vertex-transitive generalized Petersen graphs are GP(n,k) where $k^2 \equiv \pm 1 \pmod{n}$ or $(n,k) = (10,2)$. These conditions, together with the conditions $n \equiv 5 \pmod{6}$, and $k = \pm 2$ or $(n \pm 1)/2$ imply the following result.

Corollary 4.2 *Excepting the Petersen graph, every connected vertex-transitive spoked Cayley graph on an Abelian group, has a Hamilton cycle.*

Another class of graphs, similar to the spoked Cayley graphs may also be of interest. Given an Abelian group A and three nonzero elements $s,t,u \in A$, we define the *clawed Cayley graph* CCay(A,s,t,u) to have vertices a, a', a'', a''' for each $a \in A$ and edges $\{a,a'\}, \{a,a''\}, \{a,a'''\}, \{a',(a+s)'\}, \{a'',(a+t)''\}, \{a''',(a+u)'''\}$ for each $a \in A$. Clawed Cayley graphs on a cyclic group are called *Y-graphs* in [10]. The Coxeter graph C_{28} is CCay$(Z_7,1,2,4)$ [3]. Thus, P is *spoked* K_5, and C_{28} is *clawed* K_7. Which clawed Cayley graphs are Hamiltonian?

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