Graphs with complete minimal k-vertex separators

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Abstract

Dirac characterized chordal graphs by every minimal (2-)vertex separator inducing a complete subgraph. This generalizes to k-vertex separators and to a characterization of the class of $\{P_5, 2P_3\}$-free chordal graphs. The correspondence between minimal 2-vertex separators of chordal graphs and the edges of their clique trees parallels a correspondence between minimal k-vertex separators of $\{P_5, 2P_3\}$-free chordal graphs and certain $(k-1)$-edge substars of their clique trees.

1 k-vertex separators and chordal graphs

If G is a connected graph with an independent set $\mathcal{I}$ of two or more vertices, then $S \subset V(G)$ is an *$\mathcal{I}$-separator* of G if the vertices of $\mathcal{I}$ are in $|\mathcal{I}|$ separate components of the subgraph of G induced by $V(G) - S$; a *minimal $\mathcal{I}$-separator* is an inclusion-minimal $\mathcal{I}$-separator. When $|\mathcal{I}| = k$, these will also be called *(minimal) k-vertex separators.* Minimal 2-vertex separators are the well-studied 'minimal separators' (sometimes called 'minimal vertex separators' or 'minseps'); see [2, 6].

Chordal graphs can be defined by every cycle of length four or more having a *chord* (meaning an edge between two vertices of the cycle that are not consecutive along the cycle)—so, every cycle long enough to have a chord does have a chord. This is equivalent to a graph being chordal if and only if every induced subgraph contains a *simplicial vertex* (meaning a vertex whose neighborhood induces a complete subgraph). See [7] for history, proofs, and additional characterizations, including the following.

Dirac's Theorem ([4], 1961) *A graph is chordal if and only if every minimal 2-vertex separator induces a complete subgraph.*

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We extend Dirac's theorem to a new subclass $\mathcal{D}$ of chordal graphs—those graphs G in which, for all $k \geq 2$, every minimal k-vertex separator of G induces a complete subgraph of G. (The forbidden subgraph characterizations in Theorem 3 and Corollary 4 will give more memorable names for the class $\mathcal{D}$.) Figure 1 shows two graphs in the class $\mathcal{D}$. In each, for instance, $\{b, e\}$ is a minimal $\{a, d\}$-separator and $\{b, c, e\}$ is a minimal $\{a, d, f\}$-separator.

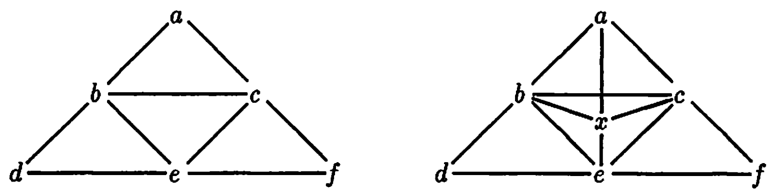

Figure 1: Two graphs in the class $\mathcal{D}$.

Figure 2 shows that the class $\mathcal{D}$ is properly contained in the class of chordal graphs. In each graph shown there, letting $\mathcal{I}$ consist of the 'hollow' vertices, the 'solid square' vertices form minimal $\mathcal{I}$-separators that do not induce complete subgraphs.

Figure 2: Two chordal graphs that are not in the class $\mathcal{D}$.

Let $v \sim w$ denote that vertices v and w are adjacent. For any $S \subset V(G)$, let $G - S$ denote the subgraph of G that is induced by $V(G) - S$. Let $d(v, w)$ denote the distance between v and w and $d(v, S)$ denote the minimum value of $d(v, w)$ over all $w \in S$. Lemma 1 consists of two immediate observations that will be used tacitly in later proofs.

Lemma 1 *If S is a minimal $\mathcal{I}$-separator of G, then each $v \in S$ has neighbors v' and v'' in two different components of $G - S$, and vertices w' and w'' in two different components of $G - S$ are not adjacent in G.* □

Theorem 2 gives a slightly refined characterization of the graphs in $\mathcal{D}$.

Theorem 2 *If G is in the class $\mathcal{D}$ and S is a minimal k-vertex separator of G, then S induces a complete subgraph of* nonsimplicial *vertices of G, and S is a minimal $\mathcal{I}$-separator, $|\mathcal{I}| = k$, where $\mathcal{I}$ is an independent set of* simplicial *vertices of G.*

Proof. Suppose $G \in \mathcal{D}$ and S is a minimal $\mathcal{I}$-separator of G with $|\mathcal{I}| = k$.

Since each $v \in S$ has nonadjacent neighbors v' and v'' in two different components of $G - S$, vertex v cannot be a simplicial vertex of G.

For each component H_i ($1 \leq i \leq k$) of $G - S$, choose a vertex $v_i \in V(H_i)$ with maximum $d(v_i, S)$. Vertex v_i cannot be adjacent to vertices x and y with $x \not\sim y$, since that would force $d(x, S)$ or $d(y, S)$ to be greater than $d(v_i, S)$ (since Dirac's theorem implies that G is chordal and so the induced path x, v_i, y cannot be part of a cycle). Therefore, each v_i will be a simplicial vertex of G. The set S will be a minimal $\mathcal{I}'$-separator when $\mathcal{I}' = \{v_i : 1 \leq i \leq k\}$. □

A graph G is $\{H_1, \ldots, H_k\}$*-free* if G contains no induced subgraph that is isomorphic to any of $H_1, \ldots, H_k$. Recall that C_n denotes an induced cycle on $n \geq 3$ vertices, P_n denotes an induced path on $n \geq 1$ vertices, and $2P_3$ denotes a 2-component graph, each isomorphic to P_3. (In Figure 2, the graph on the left is a P_5, and deleting the vertex of degree six from the graph on the right leaves an induced $2P_3$ subgraph.)

Theorem 3 *The class $\mathcal{D}$ is the class of $\{P_5, 2P_3\}$-free chordal graphs.*

Proof. First suppose G is a connected chordal graph (so every minimal 2-vertex separator induces a complete subgraph by Dirac's Theorem). Suppose S is a minimal $\mathcal{I}$-separator with $|\mathcal{I}| \geq 3$ and H_1, H_2, and H_3 are among the distinct components of $G - S$. Suppose $G \notin \mathcal{D}$ with nonadjacent vertices $a, b \in S$ (arguing that G contains an induced P_5 or $2P_3$).

There cannot exist $y_i \in H_i$ and $y_j \in H_j$ with $i \neq j$ such that $a \sim y_i \sim b$ and $a \sim y_j \sim b$, (since they would form a chordless 4-cycle, contradicting that G is chordal). Yet each of a and b must be adjacent to vertices in at least two components of $G - S$. Hence, without loss of generality, there is a vertex $x_1 \in H_1$ with $a \sim x_1$ and $b \not\sim y$ for all $y \in V(H_1)$, and there is a vertex $x_2 \in H_2$ with $b \sim x_2$ and $a \not\sim y$ for all $y \in V(H_2)$. So $\{a, b, x_1, x_2\}$ induces a $2P_2$ subgraph with edge set $\{ax_1, bx_2\}$. Vertex a [respectively, b] also has a neighbor $x_a \in H_i$ [$x_b \in H_j$] with $\{1, 2\} \cap \{i, j\} = \emptyset$; hence, neither a nor b is a simplicial vertex of G. Let $E' = \{x_1a,\ ax_a,\ x_2b,\ bx_b\} \subset E(G)$. If $i = j$ and π_i is an induced a-to-b-path in H_i, then will G will contain an induced P_k with $k \geq 5$ that consists of edges in $E' \cup E(\pi_i)$; in particular, G will contain an induced P_5. If $i \neq j$, then the subgraph H of G induced by $\{x_1, a, x_a, x_2, b, x_b\}$ will form an induced $2P_3$.

Conversely, suppose $G \in \mathcal{D}$. If G were not chordal, then G would have a minimal 2-vertex separator S with $\{x, y\} \subseteq S$ and $x \not\sim y$ (by Dirac's theorem). If G contained an induced P_5 path a, b, c, d, e, then $\{b, d\}$ would be in a minimal $\{a, c, e\}$-separator S of G with $b \not\sim d$. In the remaining case, suppose G had an induced $2P_3$ subgraph consisting of the paths x_1, a, x_2

and x_3, b, x_4. Let $\mathcal{I}$ be the independent set $\{x_1, x_2, x_3, x_4\}$ and S be a minimal $\mathcal{I}$-separator. Then a and b would have to be in S with $a \not\sim b$. Hence, in every case, S would not induce a complete subgraph. □

Corollary 4 *The class $\mathcal{D}$ is the class of $\{C_4, C_5, P_5, 2P_3\}$-free graphs.*

Proof. This follows directly from Theorem 3 and the observation that every induced C_k with $k \geq 6$ contains an induced P_5. □

The *starlike graphs* (meaning the intersection graphs of substars of stars) from [5] are characterized in [3] as being the $\{C_4, C_5, P_5, 2P_3\}$-free graphs that, in addition, contain neither of the graphs shown in Figure 3 as induced subgraphs. Hence, the starlike graphs form a subclass of $\mathcal{D}$, and this is a proper subclass since the graphs in Figure 3 are in $\mathcal{D}$. (In the graph on the left, notice that the two 'solid square' vertices form a minimal $\mathcal{I}$-separator where $\mathcal{I}$ consists of the three 'hollow' vertices *or* the three degree-2 vertices *or* just the two hollow' degree-2 vertices.)

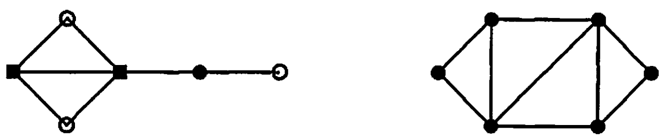

Figure 3: Two $\{P_5, 2P_3\}$-free chordal graphs that are not starlike graphs.

Lemma 5 will be needed for the proofs in section 2.

Lemma 5 *Suppose G is a $\{P_5, 2P_3\}$-free chordal graph, $S \subset V(G)$ induces a complete subgraph of G, and $\mathcal{I} \subset V(G)$ is an independent set with $|\mathcal{I}| \geq 2$ and $S \cap \mathcal{I} = \emptyset$. Then S is a minimal $\mathcal{I}$-separator of G if and only if both the following hold:*

(5.1) *$S = S_1 \cup \cdots \cup S_{|\mathcal{I}|-1}$, where each S_i is a minimal 2-vertex separator and, for every $v \neq w$ in $\mathcal{I}$, some S_i is a minimal $\{v, w\}$-separator.*

(5.2) *Each vertex in S is adjacent to vertices in at least two components of $G - S$.*

Proof. Suppose G is a $\{P_5, 2P_3\}$-free chordal graph, $S \subset V(G)$ induces a complete subgraph of G, $\mathcal{I} = \{v_1, \ldots, v_k\} \subset V(G)$ is an independent set, $k \geq 2$, and $S \cap \mathcal{I} = \emptyset$.

First suppose S is a minimal $\mathcal{I}$-separator of G, where each v_i is in a component H_i of $G - S$, with $i \neq j \Rightarrow H_i \neq H_j$. When $1 \leq i \leq k$, let $S_i = \{s \in S : N(s) \cap V(H_i) \neq \emptyset\}$. When $1 \leq i < j \leq k$, each v_i-to-v_j path will intersect both S_i and S_j, so each of S_i and S_j is a (possibly nonminimal) $\{v_i, v_j\}$-separator. If $S_i \subseteq S_j$, then every $s \in S_i$ is on a v_i-to-v_j

path that only contains s from S, and so S_i is a minimal $\{v_i, v_j\}$-separator. If $S_i \not\subseteq S_j$ and $s^* \in S_i - S_j$, then every $s \in S_j$ is adjacent to s^* (since S induces a complete subgraph) and s is on a v_i-to-v_j path that contains exactly s and s^* from S, so S_j is a minimal $\{v_i, v_j\}$-separator.

Therefore, $i \neq j$ implies that at least one of S_i and S_j is a minimal $\{v_i, v_j\}$-separator, and so, in particular, at least $k-1$ of the sets $S_1, \ldots, S_k$ are minimal 2-vertex separators. Reorder these sets if necessary so that $S_1, \ldots, S_{k-1}$ are minimal 2-vertex separators. Since each $s \in S$ is adjacent to vertices in at least two H_i components, each such s is in at least two S_i sets, $1 \leq i \leq k$, so S is the union of the minimal 2-vertex separators $S_1, \ldots, S_{k-1}$ and these include a minimal $\{v_i, v_j\}$-separator for every vertex pair $v_i, v_j \in \mathcal{I}$. Thus (5.1) holds. The minimality of S then ensures that (5.2) holds.

Conversely, suppose that both (5.1) and (5.2) hold. Then (5.1) ensures that S is an $\mathcal{I}$-separator of G, and (5.2) ensures the minimality of S. □

2 k-vertex separators and clique trees

The clique tree approach to chordal graphs, as detailed in [7], is one of the traditional uses of Dirac's theorem and one of the important occurrences of minimal separators. Briefly, for any chordal graph G, a *clique tree* T for G is a tree that has *nodes*—called that to lessen confusion with the vertices of G—that are the *maxcliques* (meaning the inclusion-maximal complete subgraphs) of G such that, for each $v \in V(G)$, the subgraph T_v of T that is induced by the nodes containing v is connected: in other words, every T_v is a subtree of T. A graph G is chordal if and only if it has such a clique tree T, clique trees can be elegantly constructed from the intersection graph of the maxcliques of G, and G is the intersection graph of the subtrees T_v of T. Each edge QQ' of T corresponds to the minimal separator $Q \cap Q'$ of G, and every minimal separator similarly corresponds to an edge of T. This is in spite of the possibility that G has more than one clique tree and the possibility that the same minimal separator corresponds to more than one edge of T; again, see [7] for history, proofs, and details.

As an example, Figure 4 shows a graph G from Figure 1 with a clique tree T for G (T happens to be unique in this case). The nodes of T are the maxcliques of G; the edges of T correspond to the minimal 2-vertex separators $\{a,b,c,x\} \cap \{b,c,e,x\} = \{b,c,x\}$, $\{b,d,e\} \cap \{b,c,e,x\} = \{b,e\}$, and $\{c,e,f\} \cap \{b,c,e,x\} = \{c,e\}$.

Theorem 6 will generalize the correspondence of minimal 2-vertex separators with the edges of cliques trees of chordal graphs to the correspondence of minimal k-vertex separators with certain substars of edges of clique trees of $\{P_5, 2P_3\}$-free chordal graphs. A *substar* of a clique

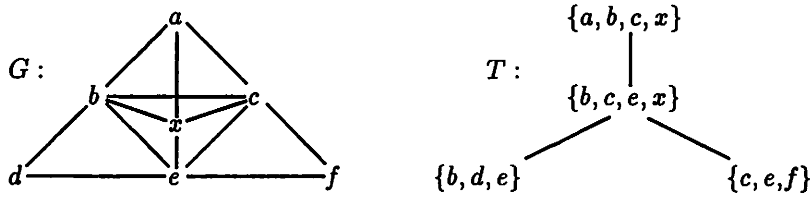

Figure 4: A $\{P_5, 2P_3\}$-free chordal graph G with its clique tree T.

tree T is any subgraph that consists of a node Q and one or more of its neighbor nodes $Q_1, \ldots, Q_d$; that substar will *correspond* to the set $\bigcup\{Q \cap Q_i : 1 \leq i \leq d\} \subset V(G)$.

Theorem 6 *Suppose G is a $\{P_5, 2P_3\}$-free chordal graph that has a clique tree T and suppose $k \geq 2$. Then $S \subset V(G)$ is a minimal k-vertex separator of G if and only if both the following hold*:

(6.1) *S corresponds to a substar of $k-1$ edges of T.*

(6.2) *For every $v \in S$, subtree T_v has at least two nodes not equal to S.*

Proof. Suppose G, T, and k are as in the statement of the theorem.

First suppose that S is a minimal $\mathcal{I}$-separator of G with $|\mathcal{I}| = k \geq 2$. Conditions (5.1) and (5.2) of Lemma 5 and the correspondence of minimal 2-vertex separators of G with the edges of T imply that the minimal 2-vertex separators $S_1, \ldots, S_{k-1}$ from (5.1) correspond to $k-1$ edges of T whose removal from T would leave the vertices of $\mathcal{I}$ in nodes of k different components of the resulting forest. (Some of the S_i sets may be identical, but their multiplicity will equal the number of edges QQ' of T with equal $Q \cap Q'$; see [7].) Moreover, those $k-1$ edges are all incident with a node of T that corresponds to a maxclique of G that contains the complete subgraph induced by S. Thus (6.1) holds. Condition (5.2) directly implies (6.2).

Conversely, suppose (6.1) and (6.2) hold. Then (6.1) implies (5.1) by constructing $\mathcal{I}$ to consist of one simplicial vertex from each component of the forest that results from deleting the $k-1$ edges of the substar from T and by taking the sets S_i to correspond to the edges of the substar. Condition (6.2) directly implies (5.2). □

In Figure 4 for example, the minimal 2-vertex separators $\{b, e\}$ and $\{c, e\}$ correspond to two adjacent edges of T, and their union $S = \{b, c, e\}$ is a minimal 3-vertex separator of G (T_b, T_c, and T_e each has three nodes). On the other hand, the minimal 2-vertex separators $\{b, c, x\}$ and $\{b, e\}$ correspond to two adjacent edges of T, but T_x has only one node, $\{a, b, c, x\}$, not equal to S, and their union $\{b, c, e, x\} = S$ is not a *minimal* 3-vertex separator of G.

Condition (6.2) holds automatically in the $k = 2$ case, since v being in a minimal 2-vertex separator S implies that v is in both nodes of the edge of T that corresponds to S, and S will be a proper subset of both of the maxcliques that correspond to those two nodes.

Theorem 7 will resemble Theorem 2 for the $k \geq 3$ cases when S is a maxclique of G (rather than just a complete subgraph). In it, a vertex v is *bisimplicial* [1] if $N(v)$ is the union of two complete subgraphs; equivalently, if v is in at most two maxcliques of G (or, in the case when G is chordal, if v is in at most two nodes of a clique tree). Every simplicial vertex is also bisimplicial.

Theorem 7 *If G is a $\{P_5, 2P_3\}$-free chordal graph and S is a maxclique that is a minimal k-vertex separator of G with $k \geq 3$, then S induces a complete subgraph of non-bisimplicial vertices of G, and S is a minimal $\mathcal{I}$-separator, $|\mathcal{I}| = k$, where $\mathcal{I}$ is a set of vertices that are pairwise distance two apart in G.*

Proof. Suppose G is a $\{P_5, 2P_3\}$-free chordal graph with clique tree T and S is a maxclique of G that is also a minimal k-vertex separator of G. Then $k \neq 2$; otherwise, Lemma 5 would imply that S corresponds to both an edge and a node of T (contradicting that maxcliques are incomparable subsets of $V(G)$). Thus $k \geq 3$ and Theorem 6 implies that S corresponds to a substar of $k - 1 \geq 2$ edges $Q_0Q_1, \ldots, Q_0Q_{k-1}$ of T where $S \subseteq Q_0$ and each $v \in S$ is in at least three Q_i nodes of T. So no $v \in S$ is bisimplicial. Let $\mathcal{I}$ consist of one vertex v_i from each set $Q_i - Q_0$. Then $\mathcal{I}$ is an independent set, and S is a minimal $\mathcal{I}$-separator. Lemma 5 implies that every two vertices $v_i, v_j \in \mathcal{I}$ have a minimal vertex separator $S_i \subseteq S$ as in (5.1), making v_i and v_j both adjacent to some vertex in S. Hence, every v_i and v_j are distance two apart in G. □

To illustrate the role of bisimplicial vertices, observe that, in the example of Figure 4, the maxclique $S = \{b, c, e, x\}$ is the center vertex of a 2-edge substar with leaf nodes $\{b, d, e\}$ and $\{c, e, f\}$, but $x \in S$ is bisimplicial and S is not a *minimal* 3-vertex separator.

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