A proof to the odd-gracefulness of all lobsters[1]

Xiangqian ZHOU, Bing YAO[2], Xiang'en CHEN, and Haixia TAO

College of Mathematics and Information Science, Northwest Normal University, Lanzhou, Gansu 730070, China

Abstract

Gnanajothi conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. Since the conjecture is open now we present a proof to the odd-gracefulness of all lobsters and show a connection between set-ordered odd-graceful labellings and bipartite graceful labellings in a connected graph.

Mathematics Subject Classification 2010: 05C78

Keywords: graceful, set-ordered odd-graceful, trees, caterpillars, lobsters.

1 Introduction and concepts

Gnanajothi [5] defined that a graph G with q edges is odd-graceful if it has a proper labelling $f : V(G) \to \{0, 1, \dots, 2q-1\}$ such that $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 3, \dots, 2q-1\}$. She proved that the class of odd-graceful graphs lies between the class of α-graphs with α-labellings defined by Rosa [6] and the class of bipartite graphs; every α-graph is also odd-graceful, and the reverse case does not work. Furthermore, Gnanajothi conjectured: *All trees are odd-graceful.* This conjecture was verified for all trees with order up to 10 [5]. Barrientos ([2], [4]) has shown that all disjoint unions of caterpillars are odd-graceful and all trees of diameter 5 are odd-graceful. Eldergill [3] shows that a spider T with odd legs of the same length is odd-graceful. In [7], the author shown many odd-graceful graphs (including several classes of lobsters). A few classes of trees have been verified to be odd-graceful up to now [4].

In this note, we will prove that *every lobster is odd-graceful.* A connection between set-ordered odd-graceful labellings and bipartite graceful labellings in a connected graph will be shown.

We use standard terminology of graph theory. Graphs mentioned are simple, undirected, connected and finite. A (p, q)-graph G is one with p vertices and q edges. The shorthand symbol $[m, n]$ stands for an integer set $\{m, m+1, \dots, n\}$, where m and n are integers with $0 \le m < n$; and the notation $[s, t]^o$ indicates an odd-set $\{s, s+2, \dots, t\}$, where s and t are odd

[1] This work is supported by the National Science Fund of P.R. China (Grant No. 61163054 and No. 61163037).

[2] The corresponding author. E-mail: yybb918@163.com.

integers with $1 \leq s < t$. A graph G has a *proper* labelling $f : V(G) \to [0, k]$ if $f(u) \neq f(v)$ for distinct $u, v \in V(G)$. Correspondingly, each edge uv of G is assigned by the value $|f(u) - f(v)|$ (called the *edge label*), denoted by $f(uv) = |f(u) - f(v)|$. Write $f(V(G)) = \{f(u) : u \in V(G)\}$, $f(E(G)) = \{f(uv) : uv \in E(G)\}$, and $\max(f) = \max f(V(G))$ here. If a (p, q)-graph G admits a proper labelling $f : V(G) \to [0, q]$ such that $f(E(G)) = [1, q]$, then we say that G and f both are *graceful*.

Definition 1. [8] Let (X, Y) be the bipartition of a bipartite (p, q)-graph G. If G admits a graceful labelling f such that $\max\{f(x) : x \in X\} < \min\{f(y) : y \in Y\}$, then we call f a *bipartite graceful labelling*, and write this case as $f(X) < f(Y)$. Furthermore, if G has a perfect matching M and f is a (bipartite) graceful labelling such that $f(x) + f(y) = q$ for every edge $xy \in M$, so we call f a *strongly (bipartite) graceful labelling*.

Motivated by this definition we have the concept of a (strongly) set-ordered odd-graceful labelling as follows.

Definition 2. Let (V_1, V_2) be the bipartition of a bipartite (p, q)-graph G. If G admits an odd-graceful labelling f such that $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, then we call f a *set-ordered odd-graceful labelling*, and write this case as $f(V_1) < f(V_2)$. Furthermore, if G has a perfect matching M and f is a (set-ordered) odd-graceful labelling such that $f(u) + f(v) = 2q - 1$ for every edge $uv \in M$, we say that G and f both are *strongly (set-ordered) odd-graceful*.

A *leaf* is a vertex of degree one. A *caterpillar* is a tree such that the graph obtained by deleting all leaves from the tree is just a path. A *lobster* is a tree H such that the deletion of all leaves of H produces a caterpillar.

2 Every lobster is odd-graceful

Lemma 1. *If a graph G admits an odd-graceful labelling, then G is a bipartite graph.*

Proof. Let f be an odd-graceful labelling of G. We define $V_1 = \{v : f(v)$ is odd, $v \in V(G)\}$, $V_2 = \{v : f(v)$ is even, $v \in V(G)\}$. Clearly, $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Since G is odd-graceful, then V_1 and V_2 both are independent. □

The addition of m vertices $x_1, x_2, \ldots, x_m$ to the vertex set of a graph G, each of them joined with a vertex of G, is called "adding leaves to G."

Theorem 2. *For any connected set-ordered odd-graceful graph H, adding leaves to H produces an odd-graceful graph.*

Proof. Let H be a connected graph with q edges and a set-ordered odd-graceful labelling f. By Lemma 1, $V(T) = V_1 \cup V_2$, where $V_1 = \{u_i : i \in [1,s]\}$ and $V_2 = \{v_j : j \in [1,t]\}$ with $s+t = |T|$. Since f is set-ordered odd-graceful, thus, $f(u_i) < f(u_{i+1})$ for $i \in [1, s-1]$ and $f(v_j) < f(v_{j+1})$ for $j \in [1, t-1]$, and $f(V_1) < f(V_2)$. Notice that $f(u_1) = 0$, $f(u_s v_1) = 1$ and $f(v_t) = 2q-1$. Clearly, $f(u_i)$ is even, $i \in [1,s]$; and $f(v_j)$ is odd, $j \in [1,t]$.

The sets of leaves joining with u_i and v_j are denoted as $\{u_{i,1}, u_{i,2}, \ldots, u_{i,\ell_i}\}$ and $\{v_{j,1}, v_{j,2}, \ldots, v_{j,k_j}\}$, respectively, where $\ell_i \geq 0$ for $i \in [1,s]$ and $k_j \geq 0$ for $j \in [1,t]$. The resulting tree is denoted as G^*. Let $M(s) = \sum_{i=1}^{s} \ell_i$ and $M(t) = \sum_{j=1}^{t} k_j$. Hence, $|V(G^*)| = |V(H)| + M(s) + M(t)$, $|E(G^*)| = q + M(s) + M(t)$. We define a labelling h of G^* as follows.

(i) Let $h(u_i) = f(u_i)$ for $i \in [1,s]$, $h(v_j) = f(v_j) + 2(M(s)+M(t))$ for $j \in [1,t]$.

(ii) Let $h(u_1 u_{1,1}) = 1$, $h(u_{1,1}) = h(u_1) + h(u_1 u_{1,1}) = 1$ since $h(u_1) = 0$. We define edge labels $h(u_1 u_{1,j}) = 2j-1$ for $j \in [1, \ell_1]$, and then define vertex labels as $h(u_{1,j}) = h(u_1) + h(u_1 u_{1,j}) = 2j-1$ for $j \in [1,\ell_1]$. In general, we define edge labels $h(u_i u_{i,j}) = h(u_i u_{i,1}) + 2(j-1) = 1 + 2(\sum_{q=1}^{i-1} \ell_q) + 2(j-1) = 2(j + \sum_{q=1}^{i-1} \ell_q) - 1$ and vertex labels $h(u_{i,j}) = h(u_i) + h(u_i u_{i,j})$ for $j \in [1, \ell_i]$ and $i \in [2,s]$.

(iii) Let $h(v_1 v_{1,1}) = 2(M(s)+M(t)) - 1$, $h(v_{1,1}) = h(v_1) - h(v_1 v_{1,1})$, and $h(v_1 v_{1,l}) = h(v_1 v_{1,1}) - 2(l-1) = 2(M(s)+M(t)) - 2l + 1$ for $l \in [1, k_1]$; and $h(v_{1,l}) = h(v_1) - h(v_1 v_{1,l})$ for $l \in [1,k_1]$. In general, we have $h(v_i v_{i,j}) = h(v_i v_{i,1}) - 2(j-1) = h(v_1 v_{1,1}) - 2\sum_{q=1}^{i-1} k_q - 2(j-1) = 2(M(s)+M(t) - \sum_{q=1}^{i-1} k_q) - 2j + 1$ and $h(v_{i,j}) = h(v_i) - h(v_i v_{i,j})$ for $j \in [1,k_i]$ and $i \in [2,t]$.

We, now, verify that h is an odd-graceful labelling of G^*.

It is easy to see $h(u_i)$ is even for $i \in [1,s]$, $h(v_j)$ is odd for $j \in [1,t]$; $h(u_{i,j})$ is odd for $j \in [1,\ell_i]$ and $i \in [1,s]$, and $h(v_{l,r})$ is even for $r \in [1,k_l]$ and $l \in [1,t]$. Clearly, $h(u_{i,j}) \neq h(v_{l,r})$. Notice that $h(u_i) < h(u_{i+1})$ for $i \in [1,s-1]$, $h(v_j) < h(v_{j+1})$ for $i \in [1,t-1]$ and $h(u_s) < h(v_1)$; and $h(u_1) = 0$, $h(u_s v_1) = 1 + 2(M(s)+M(t))$, $h(v_t) = 2q - 1 + 2(M(s)+M(t))$.

We have $h(u_i u_{i,j}) < h(u_i u_{i,j+1})$ for $j \in [1, \ell_i - 1]$ and $i \in [1,s]$; and $h(u_i u_{i,\ell_i}) < h(u_{i+1} u_{i+1,1})$ for $i \in [1,s-1]$. Similarly, $h(v_i v_{i,j}) > h(v_i v_{i,j+1})$ for $j \in [1,k_i - 1]$ and $i \in [1,t]$; and $h(v_i v_{i,k_i}) > h(v_{i+1} v_{i+1,1})$ for $i \in [1,t-1]$. Furthermore, $h(u_{i,j}) < h(u_{i,j+1})$ for $j \in [1,\ell_i - 1]$ and $i \in [1,s]$; and $h(u_{i,\ell_i}) < h(u_{i+1,1})$ for $i \in [1,s-1]$. Analogously, $h(v_{i,j}) < h(v_{i,j+1})$ for $j \in [1,k_i - 1]$ and $i \in [1,t]$; and $h(v_{i,k_i}) < h(v_{i+1,1})$ for $i \in [1,t-1]$.

Since $h(u_s v_1) = h(v_1) - h(u_s) = 1 + 2(M(s)+M(t))$, $h(v_1 v_{1,1}) = h(v_1) - h(v_{1,1}) = 2(M(s)+M(t)) - 1$, $h(u_s u_{s,\ell_s}) = h(u_{s,\ell_s}) - h(u_s) = 2M(s) - 1$, then $h(u_{s,\ell_s}) = h(u_s) + 2M(s) - 1 < f(v_1) + 2(M(s)+M(t)) = h(v_1)$, and $h(v_{1,1}) = h(v_1) - 2(M(s)+M(t)) + 1 = f(v_1) + 1 > f(u_s) = h(u_s)$.

Therefore, $h(x) \neq h(y)$ for distinct $x, y \in V(G^*)$.

The set $h(E(G^*))$ of edge labels of G^* consists of $\{h(xy) : xy \in E(G^*) \setminus E(H)\} = [1, 2(M(s) + M(t)) - 1]^o$ and $\{h(xy) : xy \in E(H) \subseteq E(G^*)\} = [1 + 2(M(s) + M(t)), 2q - 1 + 2(M(s) + M(t))]^o$, that is, $h(E(G^*)) = [1, 2q - 1 + 2(M(s) + M(t))]^o = [1, 2|E(G^*)| - 1]^o$.

The proof of the theorem is completed. □

Figures 1, 2 and 3 are used for illustrating the proof of Theorem 2.

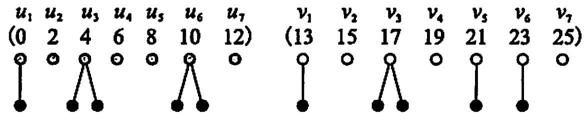

Figure 1: Adding ten leaves (black vertices) to the set-ordered odd-graceful tree T shown in Figure 3(a).

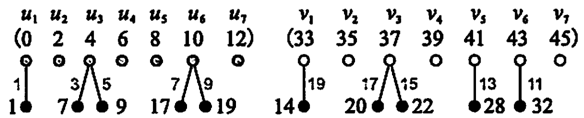

Figure 2: A procedure of relabelling each vertex of the tree shown in Figure 3(b).

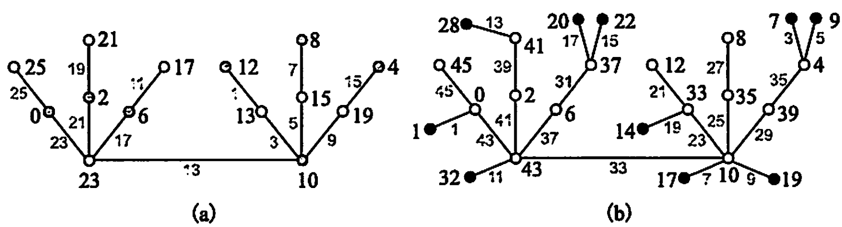

Figure 3: (a) A set-ordered odd-graceful tree T; (b) adding leaves (black vertices) to T yields a tree with an odd-graceful labelling.

Theorem 3. *Every lobster is odd-graceful.*

Proof. First, we will show that every caterpillar is set-ordered odd-graceful. By induction on the orders of the caterpillars. Since the deletion of all leaves from a caterpillar produces a path, so we can describe a caterpillar T in the following way. T contains a path $P = u_1 u_2 \cdots u_m$, and each set of leaves $u_{i,j}$ adjacent to a vertex u_i is denoted as $\mathcal{L}(u_i) = \{u_{i,j} : j \in [1, \alpha_i]\}$ with integers $\alpha_i \geq 0$ for $i \in [1, m]$. It is allowed that some $\alpha_i = 0$ if no leaf is adjacent to u_i in T.

Clearly, each graph $T_i = T - V_i$, where $V_i = (\bigcup_{j=i+1}^{m} \mathcal{L}(u_j)) \cup \{u_j : j \in [i+2, m]\}$, is a caterpillar too. T_1 is a star with the vertex set $\{u_1, u_2\} \cup \mathcal{L}(u_1) = \{u_1, u_2, u_{1,j} : j \in [1, \alpha_1]\}$. We label the vertices of T_1 as: $\pi_1(u_1) = 0$, $\pi_1(u_{1,j}) = 2j - 1$ for $j \in [1, \alpha_1]$, and $\pi_1(u_2) = 2\alpha_1 + 1$. Obviously, π_1 is a set-ordered odd-graceful labelling of T_1. Next, we define the *complementary labelling* π'_1 of π_1 as $\pi'_1(x) = \max(\pi_1) - \pi_1(x)$ for $x \in V(T_1)$. Clearly, π'_1, also, is a set-ordered odd-graceful labelling of T_1. Notice that $\pi'_1(u_1) = 2\alpha_1 + 1$ and $\pi'_1(u_2) = 0$. Joining each $x \in \mathcal{L}(u_2) \cup \{u_3\}$ with the vertex u_2 of T_1 results T_2. We define a proper labelling π_2 of T_2 as: $\pi_2(x) = \pi'_1(x)$ for $x \in V(T_1) \subset V(T_2)$, $\pi_2(u_{2,j}) = 2\alpha_1 + 1 + 2j = 2(\alpha_1 + j) + 1$ for $j \in [1, \alpha_2]$, and $\pi_2(u_3) = 2(\alpha_1 + \alpha_2 + 1) + 1$. Since $\pi_2(T_2) = [1, 2\alpha_1 + 1]^o \cup [2\alpha_1 + 3, 2(\alpha_1 + \alpha_2 + 1) + 1]^o$, so π_2 is set-ordered odd-graceful. Similarly, the complementary labelling π'_2 of π_2 is defined as $\pi'_2(x) = \max(\pi_2) - \pi_2(x)$ for $x \in V(T_2)$; and it is set-ordered odd-graceful such that $\pi'_2(u_2) = 2(\alpha_1 + \alpha_2 + 1) + 1$ and $\pi'_2(u_3) = 0$. In general, each caterpillar T_i admits a set-ordered odd-graceful labelling π_i such that its complementary labelling π'_i, defined as $\pi'_i(x) = \max(\pi_i) - \pi_i(x)$ for $x \in V(T_i)$, is set-ordered odd-graceful with $\pi'_i(u_i) = 2(i - 1 + \sum_{j=1}^{i} \alpha_j) + 1$ and $\pi'_i(u_{i+1}) = 0$. Therefore, every caterpillar is set-ordered odd-graceful by the principle of induction.

Notice that deleting all leaves from a lobster remains a caterpillar. This theorem follows by Theorem 2. □

Observe that the proof of Theorem 2 needs the help of set-ordered odd-graceful graphs. So we wish *to characterize set-ordered odd-graceful graphs.*

Theorem 4. *A connected graph H has a (strongly) set-ordered odd-graceful labelling if and only if H admits a (strongly) bipartite graceful labelling.*

Proof. Let H be a connected bipartite graph with q edges. So $V(T) = V_1 \cup V_2$, where $V_1 = \{u_i : i \in [1, s]\}$ and $V_2 = \{v_i : i \in [1, t]\}$ with $s + t = |T|$. The proof about "*A connected graph H has a strongly set-ordered odd-graceful labelling π if and only if H admits a strongly bipartite graceful labelling*" is very similar with the following proof.

We consider the proof of "if". Let f be a set-ordered odd-graceful labelling of H such that $f(u_i) < f(u_{i+1})$ for $i \in [1, s-1]$ and $f(v_j) < f(v_{j+1})$ for $j \in [1, t-1]$, and $f(V_1) < f(V_2)$. Notice that $f(u_1) = 0$, $f(u_s v_1) = 1$ and $f(v_t) = 2q - 1$; and $f(u_i)$ is even for $i \in [1, s]$, $f(v_j)$ is odd for $j \in [1, t]$. We define a proper labelling θ of H by setting $\theta(u_i) = \frac{1}{2} f(u_i)$ for $u_i \in V_1$, and $\theta(v_i) = \frac{1}{2}(f(v_i) + 1)$ for $v_i \in V_2$. Hence,

$$0 = \theta(u_1) < \theta(u_2) < \cdots < \theta(u_s) < \theta(v_1) < \theta(v_2) < \cdots < \theta(v_t) = q.$$

Since $|f(u_i) - f(v_j)| = f(v_j) - f(u_i) = 2k - 1 \in [1, 2q-1]^o$ for $k \in [1, q]$, thus, $|\theta(u_i) - \theta(v_j)| = \frac{1}{2}(f(v_j) + 1) - \frac{1}{2}f(u_i) = k$, which means that θ is bipartite graceful from $\theta(V_1) < \theta(V_2)$.

To show the proof of "only if", we let g be a bipartite graceful labelling of H such that $g(u_i) < g(u_{i+1})$ for $i \in [1, s-1]$ and $g(v_j) < g(v_{j+1})$ for $j \in [1, t-1]$, and $g(V_1) < g(V_2)$. Notice that $g(u_1) = 0$, $g(u_s v_1) = 1$ and $f(v_t) = q$. Next, we extend the labelling g to a labelling φ of H by setting $\varphi(u_i) = 2g(u_i)$ for $u_i \in V_1$, $\varphi(v_i) = 2g(v_i) - 1$ for $v_i \in V_2$. Thereby,
$0 = \varphi(u_1) < \varphi(u_2) < \cdots < \varphi(u_s) < \varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_t) = 2q - 1$.

Notice that $|g(u_i) - g(v_j)| = g(v_j) - g(u_i) = k \in [1, q]$. Hence, $|\varphi(u_i) - \varphi(v_j)| = 2g(v_j) - 1 - 2g(u_i) = 2k - 1 \in [1, 2q-1]^o$, which implies that φ is set-ordered odd-graceful since $\varphi(V_1) < \varphi(V_2)$.

This theorem is covered. □

Since the Graceful Tree Conjecture [4] is open now, Theorem 4 shows that it is not easy to settle Gnanajothi's conjecture. Furthermore, "*each tree having a perfect matching is strongly odd-graceful*" does not work since there are trees with perfect matchings and diameter four that are not strongly odd-graceful.

Acknowledgment. The authors would like to thank the referee for his valuable and thoughtful suggestions, which greatly improve the present note. This research is supported by the National Science Fund of China (No. 61163054 and No. 61163037).

References

[1] R. Balakrishnan and R. Sampathkumar. Decompositions of Regular Graphs into $K_n^c \bigvee 2K_2$. *Discrete Mathematics*, Vol. 156 (1996), 19-28.

[2] C. Barrientos. Odd-Graceful Labelings of Trees of Diameter 5. *AKCE J. Graphs Combin* 6 (2009), 307-313.

[3] P. Eldergill. Decomposition of the Complete Graph with an Even Number of Vertices. M.Sc. Thesis, McMaster University, 1997.

[4] Joseph A. Gallian. A Dynamic Survey of Graph Labelling. *The Electronic Journal of Combinatorics*, 14 (2009), #DS6.

[5] R.B. Gnanajothi. Topics in Graph Theory. Ph. D. Thesis, *Madurai Kamaraj University*, 1991.

[6] Alexander Rosa. On certain valuations of the vertices of a graph. *Theory of Graphs* (International Symposium in Rome in July 1966) pp. 349-355.

[7] C. Sekar. Studies in Graph Theory. Ph. D. Thesis, *Madurai Kamaraj University*, 2002.

[8] Bing Yao, Hui Cheng, Ming Yao and Meimei Zhao. A Note on Strongly Graceful Trees. *Ars Combinatoria* 92 (2009), 155-169.