On 3-restricted edge connectivity of Cartesian product graphs[1]

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Abstract This work presents explicit expressions of the 3-restricted edge connectivity of Cartesian product graphs, which yields some sufficient conditions for the product graphs to be maximally 3-restricted edge connected.
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1 Introduction

Since restricted edge connectivity of graphs plays an important role in analyzing reliability of communication network, its optimization problem draws a lot of attentions and many important observations are obtained as a result [1-7, 9-10, 12-13], for advances in this field the readers are suggested to refer to a survey [1]. This work pays attention to finding explicit expressions of restricted edge connectivity of Cartesian product graphs, these expressions lead to the solution of above optimization problem for Cartesian product networks.

The Cartesian product $G_1 \Box G_2$ of graphs G_1 and G_2 has vertex-set $V(G_1) \times V(G_2)$, where two vertices (x_1, y_1) and (x_2, y_2) are adjacent to each other if and only if either $y_1 = y_2$ and $x_1x_2 \in E(G_1)$ or $x_1 = x_2$ and $y_1y_2 \in E(G_2)$. Cartesian product graphs have many applications, for their properties we suggest the readers to refer to monograph [11].

For any subgraph F of graph G or subset of $V(G)$, let $\partial(F)$ denote the number of edges of G with exactly one end in F and $\xi_m(G) = \min\{\partial(F) : F$ is a connected vertex-induced subgraph of order m of $G\}$. Let $\lambda(G)$ denote the edge connectivity of graph G and $\lambda_3(G)$ its restricted edge connectivity. Let $p\xi_q(G_1) + q\xi_p(G_2) = \min\{l\xi_t(G_2) + t\xi_l(G_1) : 1 \le l \le |G_1| - 1, 1 \le t \le |G_2| - 1, m + 1 \le l + t \le (|G_1| + |G_2|)/2\}$. With these conventions, we present the expression on restricted edge connectivity of Cartesian product graphs in following theorem.

Theorem 2.2 If G_1 and G_2 are connected graphs of order at least 3, then $\lambda_m(G_1 \Box G_2) = \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), p\xi_q(G_1) + q\xi_p(G_2)\}$.

Theorem 2.3 Let G_1 and G_2 be two connected graphs of order at least three, and triangle free. If $\max\{\lambda(G_1), \lambda(G_2)\} \ge 3$ then $\lambda_3(G_1 \Box G_2) = \min\{\lambda(G_1)|G_2|, \lambda(G_2)|G_1|, \xi_3(G_1 \Box G_2)\}$.

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These observations will lead to sufficient and necessary conditions for Cartesian product graphs to be maximally restricted edge connected. Before proceeding, let us introduce some more symbols and terminologies. For any two subgraphs A and B of graph G or subsets of $V(G)$, let $\bar{A} = G - A$ or $V(G) - A$ and $[A, B]$ denote the set of edges with one end in A and the other in B. For any edge cut $S = [F, \bar{F}]$ of $G_1 \Box G_2$, we always assume that $|F| \leq |\bar{F}|$. Let G_2^x represents the subgraph of $G_1 \Box G_2$ induced by $\{x\} \Box G_2$. G_2^x is called separated by edge cut S if $G_2^x \cap F \neq \emptyset \neq G_2^x \cap \bar{F}$. For any vertex $x \in V(G_1)$, let $S_x = S \cap E(G_2^x)$ and similarly define S_y for any vertex $y \in V(G_2)$. Let$S_e = S \cap [G_2^{x_i}, G_2^{x_j}]$ for any edge $e = x_i x_j \in E(G_1)$. We follow [8] for other symbols and terminologies not specified herein.

2 Restricted edge connectivity

Lemma 2.1 If $G_1 \Box G_2$ contains 3-restricted edge cuts, then either it contains a minimum 3-restricted edge cut $S' = [F, \bar{F}]$ such that $F = X \Box Y$ for some connected subgraphs $X \subseteq G_1$ and $Y \subseteq G_2$, or it contains a component of order 3.

Proof Let S be a minimum restricted edge cut of $G_1 \Box G_2$. If G_1^y is separated by S, then $G_1^y - S_y$ consists of two components. This observation is also true for any subgraph G_2^x. Let $|S_u| = \min\{|S_x| : G_2^x$ is separated by $S\}$, $|S_v| = \min\{|S_y| : G_1^y$ is separated by $S\}$, $r = |\{x \in V(G_1) : G_2^x$ is separated by $S\}|$ and $s = |\{y \in V(G_2) : G_1^y$ is separated by $S\}|$. Consider at first the case when $r \geq 1$ and $s \geq 1$. In this case, $S_u = [\{u\} \Box Y, \{u\} \Box \bar{Y}]$ and $S_v = [X \Box \{v\}, \bar{X} \Box \{v\}]$ for some connected subgraphs $X \subseteq G_1$ and $Y \subseteq G_2$. Assume without loss of generality that $|X| \leq |\bar{X}|$ and $|Y| \leq |\bar{Y}|$.

If $\max\{|X|, |Y|\} \geq 3$, or $|X| = |Y| = 2$, let $F = X \Box Y$ then $S' = [F, \bar{F}]$ is a 3-restricted edge cut such that

$$|S| \geq \sum_{x \in X} |S_x| + \sum_{y \in Y} |S_y| \geq |X||S_u| + |Y||S_v| = |S'|. \tag{1}$$

If $|X| = |Y| = 1$, then $X \Box \{v\}$ and $\{u\} \Box Y$ are isolated vertices in G_1^u and G_2^v respectively. Let $X = \{a\}$, $S_a = [\{a\} \Box Z, \{a\} \Box (V(G_2) - Z)]$, and $Y = \{b\}$, $S_b = [\{b\} \Box W, \{W\} \Box (V(G_1) - W)]$, Obviously, $max\{|Z|, |W|\} \geq 2$. If $max\{|Z|, |W|\} \geq 3$, suppose $|Z| \geq 3$. Let $S'_a = [\{a\} \Box Z, V(G_1 \Box G_2) - \{a\} \Box Z]$,Then

$$|S| \geq |S_a| + \sum_{y \in Z} |S_y| \geq |S_a| + |Z||S_v| = |S'_a| = |S'|.$$

If $|X| = |Y| = 2$, say $X = \{a, c\}$, and $S_c = [\{c\} \Box T, \{c\} \Box (V(G_2) - T)]$, and similar to the S'_a, we have S'_c. Then either $|F| = 3$, or $min\{|T|, |Z|\} \geq 2$. If the second case happens, and suppose $S'_a \geq S'_c$, then let $S' = X\Box T$

If $|X| = 1$, $|Y| = 2$, or $|X| = 2$, $|Y| = 1$, without loss of generality, suppose $|X| = |\{a, c\}| = 2$, $|Y| = 1$. If $min\{|T|, |Z|\} \geq 2$ and then similar to the above, we have done. Otherwise, suppose $|T| \leq |Z|$. If $|T| = 1$, and $|Z| \geq 3$, let $S' = [\{a\}\Box Z, V(G_1\Box G_2) - \{a\}\Box Z]$. If $|T| = 1$, and $|Z| = 2$, then either $|F| = 3$ or a has another neighbor in G_1, say d, such that $\{d, e\} \in G_2^d \cap F \neq \phi$. Then let $S' = X\Box K_2$, where K_2 is an induced subgraph of G_2 by $\{b, e\}$.

$$|S| \geq |S_a| + \sum_{y \in Z} |S_y| \geq |S_a| + |Z||S_v| = |S'|.$$

Since S is a restricted edge cut, it follows that Z induces a connected subgraph of order at least two. And so, S' is a restricted edge cut in either of these two subcases, the lemma follows in this case. When $r = 0$, we have $s = |V(G_1)|$. If let $Y = G_2$ and X be as before, then formula (1) is still true. This method also works when $s = 0$ and so the lemma follows. □

Theorem 2.2 If G_1 and G_2 are connected graphs of order at least two, then $\lambda_2(G_1\Box G_2) = \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), p\xi_q(G_1) + q\xi_p(G_2)\}$.

Proof Let S be a minimum restricted edge cut of $G_1\Box G_2$. By the lemma 2.1, we may assume that $S = [X\Box Y, \overline{X\Box Y}]$. Let r and s be as is defined in the proof of lemma 2.1. If $r = 0$, let $Y = G_2$ and $[X, \bar{X}]$ be a minimum edge cut of G_1, then by lemma 2.1 $[X\Box Y, \overline{X\Box Y}]$ is a minimum restricted edge cuts of $G_1\Box G_2$ with cardinality $|G_2|\lambda(G_1)$; if $r = |G_1|$, then $s = 0$. Let $X = G_1$ and $[Y, \bar{Y}]$ be a minimum edge cut of G_2. Then $[X\Box Y]$ is a minimum restricted edge cut of $G_1\Box G_2$ with cardinality $|G_1|\lambda(G_2)$.

Finally, if $1 \leq r \leq |G_1| - 1$, by lemma 2.1, $G_1\Box G_2$ contains a minimum restricted edge cut $S = [X\Box Y, \overline{X\Box Y}]$. If either $|X| > |G_1|/2$ or $|Y| > |G_2|/2$, then $[\bar{X}\Box Y, \overline{\bar{X}\Box Y}]$ or $[X\Box \bar{Y}, \overline{X\Box \bar{Y}}]$ is a restricted edge cut with less cardinality than S. The theorem follows from these observations. □

Considering that the expression of theorem 2.2 is somewhat complicated, in the last of this section we turn to determine the restricted edge connectivity of Cartesian product of triangle-free graphs.

Theorem 2.3 Let G_1 and G_2 be two triangle-free connected graphs of order at least three. If $\max\{\lambda(G_1), \lambda(G_2)\} \geq 2$ then $\lambda_2(G_1\Box G_2) = \min\{\lambda(G_1)|G_2|, \lambda(G_2)|G_1|, \xi_2(G_1\Box G_2)\}$.

Proof Let $[X, \overline{X}]$ and $[Y, \overline{Y}]$ be minimum edge cuts of G_1 and G_2 respectively, uv be an edge of $G_1\Box G_2$ with minimum edge degree. Then $[X\Box G_2, \overline{X}\Box G_2]$,

$[G_1\Box Y, G_1\Box\overline{Y}]$ and $[\{u,v\}, V(G_1\Box G_2)-\{u,v\}]$ contains a restricted edge cut of $G_1\Box G_2$ respectively. It follows that $\lambda_2(G_1\Box G_2) \le \min\{\lambda(G_1)|G_2|,\ \lambda(G_2)|G_1|,\ \xi_2(G_1\Box G_2)\}$.

To show that $\lambda_2(G_1\Box G_2) \ge \min\{\lambda(G_1)|G_2|, \lambda(G_2)|G_1|, \xi_2(G_1\Box G_2)\}$, let S be a minimum restricted edge cut of $G_1\Box G_2$. By lemma 2.1, we may assume that $S = [X\Box Y, \overline{X\Box Y}]$. Let r and s be as are defined in the proof of lemma 2.1.

If $r = 0$, then $s = n_2$, $Y = G_2$ and $|S| = |[X\Box Y, \overline{X\Box Y}]| \ge \lambda(G_1)|G_2|$; if $r = n_1$, then $s = 0$ and similarly we have $|S| \ge |G_1|\lambda(G_2)$. And so, assume in what follows that $1 \le r \le n_1 - 1$ and $1 \le s \le n_2 - 1$. Since $\max\{\lambda(G_1), \lambda(G_2)\} \ge 2$, by the symmetry of G_1 and G_2 in $G_1\Box G_2$ it suffices to consider the case when $\lambda(G_1) \ge 2$.

Case 1. X contains at least two vertices.

For any given edge x_1x_2 of X, since G_1 is triangle-free it follows that $N_{G_1}(x_1)\cap N_{G_1}(x_2) = \emptyset$.

$$\begin{aligned}
|S| &\ge \sum_{e\in[X,\overline{X}]} |S_e| + \sum_{x\in N_X(x_1)} |S_x| + \sum_{x\in N_X(x_2)} |S_x| \\
&\ge \lambda(G_1)|Y| + (d_{G_1}(x_1) - |[x_1,\overline{X}]| + d_{G_1}(x_2) - |[x_2,\overline{X}]|)|[Y,\overline{Y}]| \\
&= 2|Y| + (\lambda(G_1)-2)|Y| + 2|[Y,\overline{Y}]| + \\
&\quad +(d_{G_1}(x_1) + d_{G_2}(x_2) - 2 - |[x_1,\overline{X}]| - |[x_2,\overline{X}]|)|[Y,\overline{Y}]| \\
&\ge 2(|Y| + |[Y,\overline{Y}]|) + \lambda(G_1) + d_{G_1}(x_1) + d_{G_2}(x_2) - 4 - |[x_1,\overline{X}]| - |[x_2,\overline{X}]| \\
&\ge 2(|Y| + |[Y,\overline{Y}]|) + d_{G_1}(x_1) + d_{G_2}(x_2) - 4.
\end{aligned}$$

Noticing that for any vertex $y \in V(G_2)$ we have $|Y| + |[Y,\overline{Y}]| \ge d_{G_2}(y) + 1$, the above formula implies that

$$\begin{aligned}
|S| &\ge 2(d_{G_2}(y)+1) + d_{G_1}(x_1) + d_{G_2}(x_2) - 4 \\
&= d(x_1,y) + d(x_2,y) - 2 \ge \xi_2(G_1\Box G_2).
\end{aligned}$$

Case 2. $X = \{x\}$ for some vertex $x \in V(G_1)$.

Since S is a restricted edge cut, it follows that Y contains at least two vertices. If $|[Y,\overline{Y}]| \ge 2$ in this case, then $|S| \ge \xi_2(G_1\Box G_2)$ by a similar reasoning as is employed in the proof of case 1. Hence, we assume in what follows that $|[Y,\overline{Y}]| = 1$, which implies $\lambda(G_2) = 1$. With this assumption we have

$$|S| = \sum_{e\in[x,G_1-x]} |S_e| + |S_x| \ge d_{G_1}(x)|Y| + 1.$$

Since $|[Y,\overline{Y}]| = 1$, every vertex in Y has degree at most $|Y|$ in graph G_2 and at most one vertex, say y, has degree $|Y|$. It is obviously that $|S| \ge \xi_2(G_1\Box G_2)$ when $|Y| = 2$. In the case when $|Y| = 3$, for any vertex $y' \in N_Y(y)$ we have

$$d_{G_1}(x)|Y| + 1 \ \ge\ 3d_{G_1}(x) + 1 \ge 2d_{G_1}(x) + 3$$

$$
\begin{aligned}
&= (d_{G_1}(x)+|Y|)+(d_{G_1}(x)+|Y|-1)-2\\
&\geq (d_{G_1}(x)+d_{G_2}(y))+(d_{G_1}(x)+d_{G_2}(y'))-2\\
&= d((x,y))+d((x,y'))-2\geq \xi_2(G_1\Box G_2).
\end{aligned}
$$

Consider the final case when $|Y|\geq 4$. Noticing that $d_{G_1}(x)\geq \lambda(G_1)\geq 2$, if $d_{G_1}(x)=2$ then

$$
\begin{aligned}
d_{G_1}(x)|Y|+1 &= 2|Y|+1\\
&= (d_{G_1}(x)+|Y|)+(d_{G_1}(x)+|Y|-1)-2\\
&\geq d((x,y))+d((x,y'))-2\geq \xi_2(G_1\Box G_2);
\end{aligned}
$$

if $d_{G_1}(x)=3$ then

$$
\begin{aligned}
d_1(x_1,y)|Y|+1 &\geq 3|Y|+1\geq 2|Y|+5\\
&= (d_{G_1}(x)+|Y|)+(d_{G_1}(x)+|Y|-1)\\
&> d((x,y))+d((x,y'))\geq \xi_2(G_1\Box G_2);
\end{aligned}
$$

if $d_{G_1}(x)\geq 4$ then

$$
\begin{aligned}
d_{G_1}(x)|Y|+1 &\geq 4\max\{d_{G_1}(x),|Y|\}+1\\
&\geq 2(d_{G_1}(x)+|Y|)+1> d((x,y))+d((x,y'))\\
&\geq \xi_2(G_1\Box G_2).
\end{aligned}
$$

The theorem follows from these discussions. □

3 Optimization of restricted edge connectivity

The following corollary 3.1 and 3.2 follow directly from theorem 2.2 and 2.3 respectively.

Corollary 3.1 For two connected graphs G_1 and G_2 of order at least two, their Cartesian product is is maximum restricted edge connected if and only if $\min\{|G_1|\lambda(G_2),|G_2|\lambda(G_1),p\xi_q(G_1)+q\xi_p(G_2)\}\geq \xi_2(G_1\Box G_2)$. □

Corollary 3.2 Let G_1 and G_2 be two triangle-free connected graphs of order at least three. If $\max\{\lambda(G_1),\lambda(G_2)\}\geq 2$ and $\min\{\lambda(G_1)|G_2|,\ \lambda(G_2)|G_1|\}\geq \xi_2(G_1\Box G_2)\}$, then $G_1\Box G_2$ is maximally restricted edge connected. □

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