Small 2-coloured path decompositions

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Abstract. Consider a complete graph of multiplicity 2, where between every pair of vertices there is one red and one blue edge. Can the edge set of such a graph be decomposed into isomorphic copies of a 2-coloured path of length $2k$ that contains k red and k blue edges? A necessary condition for this to be true is $n(n-1) \equiv 0 \bmod 2k$. We show that this is sufficient for $k \leqq 3$.

keywords: graph decompositions

1 Introduction

Henry Dudeney (1857-1930), creator of many mathematical puzzles, posed in [3] what became known as the "Nine Prisoners Problem." Given nine prisoners, the problem was to arrange a schedule so that for six days, the prisoners, chained in three distinct lines, each of three prisoners, may take their exercise in the prison yard so that on no two days is the same pair chained together. This problem is an example of what is known as a graph decomposition.

A graph G admits an H*-decomposition* for some subgraph H, if the edge set of G can be partitioned into edge-disjoint subgraphs each isomorphic to H. We say that G is H-decomposable, or decomposable by H.

To move the "Nine Prisoners Problem" into the language of graph theory, the following definitions must first be introduced. A *path* of length k is a sequence of $k+1$ distinct vertices, $v_1, v_2, \ldots, v_{k+1}$, joined by k edges, $v_1v_2, v_2v_3, \ldots, v_kv_{k+1}$. We denote it by P_k and will usually write it as $[v_1, v_2, v_3, \ldots, v_{k+1}]$. A *complete graph* on n vertices, denoted K_n, is the graph in which every pair of vertices is joined by a single edge. A complete graph on n vertices of *multiplicity* m, denoted mK_n, is a multigraph in which every pair of vertices is joined by m edges.

So, in graph theoretical terms, a solution to the "Nine Prisoners Problem" produces a P_2-decomposition of K_9. Of course, there is an added concern; the paths obtained by the decomposition must be such that they can be partitioned into 6 parts, where each vertex of K_9 occurs in each part exactly once. This condition is known as resolvability. It is mentioned here for completeness, but is not of concern to this paper.

The "Nine Prisoners Problem" can be considered more generally as: For what values of k, m, and n does a resolvable P_k-decomposition of mK_n exist? This question was brought to the fore in [5]. For $k = 2$, $m = 1$ and $n = 9$ (the conditions of Dudeney's original problem) it was shown in [2] that there are 332 non-isomorphic solutions. The more general case, only fixing $k = 2$, was solved in [6].

Theorem 1 *The graph mK_n has a resolvable P_2-decomposition if and only if $n \equiv 0 \mod 3$ and $m(n-1) \equiv 0 \mod 4$.*

The general solution was finally given in [1].

Theorem 2 *For $k \geqq 2$ the graph mK_n has a resolvable P_k-decomposition if and only if $n \equiv 0 \mod k+1$ and $m(k+1)(n-1) \equiv 0 \mod 2k$.*

However, the condition of resolvability may be dropped altogether, leaving the question: For what values of k, m, and n does a P_k-decomposition of mK_n exist? For $k \leqq 8$ this was completely solved in [7]. It was answered in [9], with the following theorem.

Theorem 3 *The complete multigraph mK_n is decomposable into paths of length k if and only if $n > k$ and $mn(n-1) \equiv 0 \mod 2k$.*

A thorough discussion of path decompositions, resolvable and otherwise, is available in [4]. The goal of this paper is to partially answer the following question posed by M. L. Yu in the same source.

> Colour the edges of P_{2k} red and blue so that there are k edges of each colour, and colour the edges of $2K_n$ red and blue so that there is both a red and a blue copy of K_n. If $n(n-1) \equiv 0 \mod 2k$, is there a decomposition of the coloured complete graph into copies of the coloured path?

The condition $n(n-1) \equiv 0 \mod 2k$ reflects that the number of edges of $2K_n$ should be divisible by the number of edges in each path. As well, Theorem 3 will see frequent use, though without providing a proof.

Throughout this paper, whenever $2K_n$ is discussed, it is with the assumption that between every pair of vertices the two edges have different colours, nominally one red and one blue. A 2-coloured path of length $2k$ on the vertices $v_1, v_2, \ldots, v_{2k}, v_{2k+1}$ is said to be *colour–complementary* if for $1 \leq i \leq 2k$, the edges $[v_i, v_{i+1}]$ and $[v_{2k-i+1}, v_{2k-i+2}]$ are coloured differently.

Most of the constructions used in this paper involve labeling the vertices of the graph G being decomposed with the elements of a group. Then, given a permutation σ of the elements of the group, we can apply the permutation to the vertices of some subgraph H, to obtain a new subgraph $\sigma(H)$ as follows. The vertices of $\sigma(H)$ are $\sigma(V(H))$ and $[u, v]$ is an edge in H if and only if $[\sigma(u), \sigma(v)]$ is an edge of $\sigma(H)$. For example, applying the permutation $\sigma = (1 \quad 2 \quad 3)$ to the path $[1, 2, 3]$, we obtain the path $[2, 3, 1]$. Often, we will repeatedly apply a permutation to a subgraph. In this context, this refers to applying the permutation σ to obtain the subgraphs H, $\sigma(H)$, $\sigma^2(H), \ldots, \sigma^{t-1}(H)$ where $\sigma^t(H)$ is the identity. In the previous example, we may apply the permutation σ twice to obtain the path $\sigma^2([1,2,3]) = [3,1,2]$.

Sometimes a vertex of a graph is labeled ∞. This is a fixed point; applying a permutation does not alter it. So, if the permutation $(1\quad 2\quad 3)$ is applied to the path $[1, \infty, 2]$, the resulting path is $[2, \infty, 3]$.

Many decompositions consist of a subgraph and those that result from repeatedly applying a permutation to it. For example, label the vertices of K_4 by 0, 1, 2 and ∞. Repeatedly applying the permutation $\sigma = \quad (0\quad 1\quad 2)$ to the path $[0, 1, \infty]$ will produce the paths $[1, 2, \infty]$ and $[2, 0, \infty]$, the three of which form a P_2-decomposition of K_4.

Throughout this paper, coloured paths of a particular type will be written as $[v_1, v_2, v_3, \ldots, v_{2k}]$ with the colouring assumed to be written from left to right.

In the following sections, paths forming a decomposition are listed by length and type. By type we refer to the colour pattern of the edges. For instance, if the path $[a, b, c, d, e]$ is of type RBRB, the edges $[a, b]$ and $[c, d]$ are coloured red, while $[b, c]$ and $[d, e]$ are coloured blue.

As well, it should be noted that only non-isomorphic types are listed. So, for each type listed, it is not possible to obtain another type in the list by the process of "flipping" the path or by switching the colours. The former means that both RRBBBR and RBBBRR represent the same type, and the latter that both RRBBBR and BBRRRB are the same type.

2 Some Preliminary results

Both of the proofs of the following theorems describe constructions for decomposing particular complete graphs of multiplicity 2 into colour–complementary P_{2k}.

Theorem 4 *The graph $2K_{2kr+1}$ is decomposable into isomorphic copies of any fixed colour–complementary path of length $2k$, for all positive integers r and k.*

PROOF. Let the vertices of $2K_{2kr+1}$ be labeled with the elements of Z_{2kr+1}. We will define r paths to which the permutation $(0\quad 1\quad 2\quad \cdots\quad 2kr)$ will be successively applied. This construction will be dependent on the parity of k.

If k is even, the paths are
$[-\frac{k}{2}, -jk-1+\frac{k}{2}, -\frac{k}{2}+1, -jk-2+\frac{k}{2}, \ldots, -3, -jk+2, -2, -jk+1, -1, -jk, 0, jk, 1, jk-1, 2, jk-2, 3, \ldots, jk-\frac{k}{2}+2, \frac{k}{2}-1, jk-\frac{k}{2}+1, \frac{k}{2}]$, where $1 \le j \le r$.

If k is odd, the paths are
$[-jk+\frac{k-1}{2}, -\frac{k-1}{2}, -jk-1+\frac{k-1}{2}, -\frac{k-1}{2}+1, \ldots, -jk+2, -2, -jk+1, -1, -jk, 0, jk, 1, jk-1, 2, jk-2, \ldots, \frac{k-1}{2}-1, jk-\frac{k-1}{2}+1, \frac{k-1}{2}, jk-\frac{k-1}{2}]$, where $1 \le j \le r$.

The basic idea behind this construction is that edges of all lengths are used exactly twice, and in particular, if e and f are two edges equidistant from the beginning and end of a path, then e and f have the same length. Since the paths are colour complementary, e and f are coloured differently. Therefore all lengths of both colours are represented. By applying the permutation, all edges of these lengths are used. ■

Theorem 5 *The graph $2K_{2kr}$, is decomposable into isomorphic copies of any fixed colour–complementary path of length $2k$ for all positive integers k, and $r > 1$.*

PROOF. Label $2kr-1$ of the vertices of $2K_{2kr}$ with the elements of Z_{2kr-1} and the remaining vertex ∞. Successively apply the permutation $(0 \quad 1 \quad 2 \quad \cdots \quad 2kr-2)$ to the given base paths. Once again, there will be two cases depending on the parity of k.

If k is even, the paths are $[-\frac{k}{2}, -jk+\frac{k}{2}, -\frac{k}{2}+1, -jk+\frac{k}{2}-1, \ldots, -3, -jk+3, -2, -jk+2, -1, -jk+1, 0, jk-1, 1, jk-2, 2, jk-3, 3, \ldots, jk-\frac{k}{2}+1, \frac{k}{2}-1, jk-\frac{k}{2}, \frac{k}{2}]$, for $1 < j \leq r$, and the path $[-\frac{k+2}{2}, -\frac{k}{2}, -\frac{k+2}{2}-1, -\frac{k}{2}+1, \ldots, -k+2, -3, -k+1, -2, -k, -1, \infty, 1, k, 2, k-1, 3, k-2, \ldots \frac{k}{2}-1, \frac{k}{2}+2, \frac{k}{2}, \frac{k}{2}+1]$.

If k is odd, the paths are $[-jk+\frac{k+1}{2}, -\frac{k-1}{2}, -jk+\frac{k+1}{2}-1, -\frac{k-1}{2}+1, \ldots, -2, -jk+2, -1, -jk+1, 0, jk-1, 1, jk-2, 2, \ldots, \frac{k-1}{2}-1, jk-\frac{k+1}{2}+1, \frac{k-1}{2}, jk-\frac{k+1}{2}]$, for $1 < j \leq r$, and the path $[-\frac{k+1}{2}, -\frac{k+3}{2}, -\frac{k+1}{2}+1, -\frac{k+3}{2}-1, \ldots, -k+2, -3, -k+1, -2, -k, -1, \infty, 1, k, 2, k-1, 3, k-2, \ldots, \frac{k+3}{2}+1, \frac{k+1}{2}-1, \frac{k+3}{2}, \frac{k+1}{2}]$.

Again, these paths are constructed so that edges equidistant from the endpoints of the path have the same length, and hence receive different colours. ■

We will use the following simple lemma, which relates path decompositions and colour–complementary path decompositions.

Lemma 6 Splitting Lemma
If G is a graph that admits a P_{2k}–decomposition, then the coloured multigraph $2G$ admits a colour–complementary P_{2k}-decomposition.

PROOF. For every path in the P_{2k}–decomposition of G, we obtain two colour–complementary paths in $2G$, in the following manner. If the uncoloured path is $[v_1, v_2, v_3, \ldots, v_{2k}, v_{2k+1}]$, form the first colour–complementary path by colouring the edges from left to right according to the given colour–complementary P_{2k}. The second colour–complementary path is formed by taking the same vertices in reverse order and again colouring from left to right. Thus, each edge is "split" into two edges, and each uncoloured path in the decomposition of G gives two colour-complementary paths in the decomposition of $2G$. ■

3 Paths of length 2 and 4

Up to isomorphism, there is only one 2-coloured path of length 2, and that is the path with one red edge and one blue edge. Clearly, $n(n-1) \equiv 0 \bmod 2$ is a necessary condition for the desired decomposition of $2K_n$. But this is equivalent to $n \equiv 0 \bmod 2$ or $n \equiv 1 \bmod 2$, or that all values of n must be considered. We denote this path by RB. Certainly, it is colour complementary, and so we may use Theorems 4 and 5, and we immediately obtain the following result.

Corollary 7 *The complete graph $2K_n$ is decomposable into 2-coloured paths of length 2 for all $n \geqq 3$.*

Concerning paths of length 4, we see that we have three types of paths to consider. We must investigate two different colour–complementary paths, which we denote by RRBB and RBRB, and the 2-coloured path RBBR. The necessary condition, $n(n-1) \equiv 0 \bmod 4$, results in the requirement that $n \equiv 0, 1 \bmod 4$. For the colour–complementary paths RRBB and RBRB, Theorems 4 and 5 take care of all possible cases, so we need only consider the path RBBR.

Lemma 8 *The graph $2K_{4r}$ is decomposable into paths of type RBBR for all $r \geqq 2$.*

PROOF. Label $4r-1$ of the vertices of $2K_{4r}$ with the elements of Z_{4r-1}, with the final vertex being labeled ∞. Then successively apply the permutation $(0 \quad 1 \quad 2 \quad \cdots \quad 4r-2)$ to the $r-1$ paths $[2-2j, 0, 2j-1, 1, 2j]$, for all $2 \leq j \leq r$, and the path $[0, \infty, 1, 2, 3]$. ■

Lemma 9 *The graph $2K_{4r+1}$ is decomposable into paths of type RBBR for all $r \geqq 1$.*

PROOF. Label the vertices of $2K_{4r+1}$ with the elements of Z_{4r+1}. Successively apply the permutation $(0 \quad 1 \quad 2 \quad \cdots \quad 4r)$ to the r paths $[1-2j, 0, 2j, 1, 2j+1]$, for all $1 \leq j \leq r$. ■

We obtain the following theorem.

Theorem 10 *The graph $2K_n$ is decomposable into a given 2-coloured path of length 4 if and only if $n \geqq 5$ and $n(n-1) \equiv 0 \bmod 4$.*

4 Paths of length 6.

Here, we have seven different path types to consider. They are RRRBBB, RBRBRB, RRBRBB, RBBRRB, RRBBBR, RBRBBR and RRBBRB. This time, the necessary condition $n(n-1) \equiv 0 \bmod 6$ implies that $n \equiv 0, 1, 3, 4 \bmod 6$. For the remaining constructions, we define the length of an edge $[a, b]$ in K_n whose vertices are the elements of Z_n is $|a-b|$. Thus edge lengths are $1, 2, \ldots,$ and $\left\lfloor \frac{n}{2} \right\rfloor$.

The colour–complementary paths RRRBBB and RBRBRB

Here, from Theorems 4 and 5, we immediately obtain the necessary decompositions of $2K_{6r}$ and $2K_{6r+1}$

Lemma 11 *The graph $2K_{6r+3}$ is decomposable into paths of type RRRBBB and paths of type RBRBRB for all $r \geqq 1$.*

PROOF. Label the $6r+3$ vertices of $2K_{6r+3}$ with the elements of Z_{6r+3}. Successively apply the permutation $(0 \quad 1 \quad 2 \quad \cdots \quad 6r+2)$ to the r paths $[-3j, -1, -3j-1, 0, 3j+1, 1, 3j]$, for $1 \leq j \leq r$. This will use all edges of $2K_{6r+3}$ except those of length 1.

To finish, successively apply the permutation $(0 \quad 3 \quad 6 \quad \cdots \quad 6r)(1 \quad 4 \quad 7 \quad \cdots \quad 6r+1)(2 \quad 5 \quad 8 \quad \cdots \quad 6r+2)$ to the path $[0, 1, 2, 3, 4, 5, 6]$. ■

Lemma 12 *The graph $2K_{6r+4}$ is decomposable into paths of type RRRBBB and paths of type RBRBRB for all $r \geqq 1$.*

PROOF. Label $6r+3$ vertices of $2K_{6r+4}$ with the elements of Z_{6r+3}; the remaining element will be labeled ∞. Then successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r+2)$ to the $r-1$ paths $[-3j, -1, -3j-1, 0, 3j+1, 1, 3j]$, for $2 \leq j \leq r$, and the path $[-2, 5, -1, \infty, 1, 5, 2]$.

The edges remaining will be those of lengths 1 and 2. These are used by successively applying the permutation $(0\ \ 3\ \ 6\ \ \cdots\ \ 6r)(1\ \ 4\ \ 7\ \ \cdots\ \ 6r+1)$ $(2\ \ 5\ \ 8\ \ \cdots\ \ 6r+2)$ to the two paths [0, 1, 2, 3, 4, 5, 6] and [0, 2, 4, 6, 8, 10, 12]. ■

The colour–complementary path RRBRBB

For decompositions into paths of type RRBRBB, we can use Theorems 4 and 5 for graphs $2K_{6r}$ and $2K_{6r+1}$.

Lemma 13 *The graph $2K_{6r+3}$ is decomposable into paths of type RRBRBB for all $r \geqq 1$.*

PROOF. For $2K_9$, we need only examine the underlying graph K_9. By Theorem 3, K_9 is decomposable into paths of length 6. By Lemma 6, the result follows.

For $r \geqq 2$, label the vertices of $2K_{6r+3}$ with the elements of Z_{6r+3}. Then successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r+2)$ to the $r-1$ paths $[3-3j, -1, 2-3j, 0, 3j-2, 1, 3j-3]$, for $2 \leq j \leq r$.

To obtain the remaining edges of $2K_{6r+3}$, successively apply the permutation $(0\ \ 3\ \ 6\ \ \cdots\ \ 6r)\ (1\ \ 4\ \ 7\ \ \cdots\ \ 6r+1)\ (2\ \ 5\ \ 8\ \ \cdots\ \ 6r+2)$ to the four paths $[-3, -3r-2, -1, 0, 3r, 1, 1-3r]$, $[-2, -3r-1, 0, 1, 3r+1, 2, 1-3r]$, $[-1, -3r, 1, 2, 3r+2, 3, 2-3r]$ and $[0, 1, 2, 3r+2, 3r+3, 3, 3r+4]$. ■

Lemma 14 *The graph $2K_{6r+4}$ is decomposable into paths of type RRBRBB for all $r \geqq 1$.*

PROOF. For $2K_{10}$, label 9 of the vertices with the elements of Z_9, and the remaining vertex ∞. Then successively apply the permutation $(0\ \ 3\ \ 6)(1\ \ 4\ \ 7)(2\ \ 5\ \ 8)$ to the paths $[0, 1, 2, \infty, 5, 4, 3]$, $[1, 5, 0, \infty, 3, 8, 4]$ and $[2, 0, 7, \infty, 4, 6, 8]$. The following paths use the remaining edges of $2K_{10}$ [6, 0, 8, 2, 4, 7, 3], [3, 7, 4, 2, 8, 0, 6], [3, 6, 5, 8, 1, 4, 0], [0, 4, 1, 8, 5, 6, 3], [0, 3, 2, 5, 7, 1, 6] and [6, 1, 7, 5, 2, 3, 0].

For $2K_{6r+4}$, with $r \geqq 2$, label $6r+3$ of the vertices with the elements of Z_{6r+3}, and the remaining vertex ∞. Successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r+2)$ to the paths $[2-3j, -1, 1-3j, 0, 3j-1, 1, 3j-2]$, for $2 \leq j \leq r$, and $[-2, -3r-2, -1, \infty, 1, 3r+2, 2]$, and successively apply the permutation $(0\ \ 3\ \ 6\ \ \cdots\ \ 6r)$ $(1\ \ 4\ \ 7\ \ \cdots\ \ 6r+1)(2\ \ 5\ \ 8\ \ \cdots\ \ 6r+2)$ to the paths [0, 1, 2, 3, 5, 7, 9] and [9, 7, 5, 3, 2, 1, 0]. ■

The colour–complementary path RBBRRB

Once again, we may use the constructions of Theorems 4 and 5 when considering $2K_{6r}$ and $2K_{6r+1}$. We may also use the construction of Lemma 14 when considering $2K_{6r+4}$. This leaves the case $2K_{6r+3}$.

Lemma 15 *The graph $2K_{6r+3}$ is decomposable into paths of type RBBRRB for all $r \geqq 1$.*

PROOF. We know that K_9 admits a P_6–decomposition (by Theorem 3). Using Lemma 6, we have the desired decomposition of $2K_9$.

Consider $2K_{6r+3}$, with $r \geq 2$, and label the vertices with the elements of Z_{6r+3}. Then successively apply the permutation $(0\quad 1\quad 2\quad \cdots\quad 6r+2)$ to the $r-1$ paths $[-3j, -1, -3j-1, 0, 3j+1, 1, 3j]$, for $2 \leq j \leq r$, and successively apply the permutation $(0\quad 3\quad 6\quad \cdots\quad 6r)(1\quad 4\quad 7\quad \cdots\quad 6r+1)(2\quad 5\quad 8\quad \cdots\quad 6r+2)$ to the four paths $[0, 1, 2, 4, 5, 6, 7]$, $[-3, -1, -4, 0, 4, 1, 3]$, $[-2, 0, -3, 1, 5, 2, 3]$ and $[-1, 1, -2, 2, 6, 3, 5]$. ■

The path RRBBBR

Lemma 16 *The graph $2K_{6r}$ is decomposable into paths of type RRBBBR for all $r \geqq 2$.*

PROOF. Label $6r-1$ of the vertices of $2K_{6r}$ with the elements of Z_{6r-1}. Label the remaining vertex ∞. Successively apply the permutation $(0\quad 1\quad 2\quad \cdots\quad 6r-2)$ to the $r-1$ paths $[-1, 2-3j, 0, 3j-1, 2, 3j, 1]$, for $2 \leq j \leq r$ and to the path $[-1, -2, \infty, 2, 1, 3, 5]$. ■

Lemma 17 *The graph $2K_{6r+1}$ is decomposable into paths of type RRBBBR for all $r \geqq 1$.*

PROOF. Label the $6r+1$ vertices of $2K_{6r+1}$ with the elements of Z_{6r+1}. Successively apply the permutation $(0\quad 1\quad 2\quad \cdots\quad 6r)$ to the r paths $[-1, -3j+1, 0, 3j, 2, 3j+1, 1]$, for $1 \leq j \leq r$. ■

Lemma 18 *The graph $2K_{6r+3}$ is decomposable into paths of type RRBBBR for all $r \geqq 1$.*

PROOF. Let the vertices of $2K_{6r+3}$ be labeled with the elements of Z_{6r+3}. Then successively apply the permutation $(0\quad 1\quad 2\quad \cdots\quad 6r+2)$ to the r paths $[-1, -3j, 0, 3j+1, 2, 3j+2, 1]$, for $1 \leq j \leq r$, and the permutation $(0\quad 3\quad 6\quad \cdots\quad 6r)$ $(1\quad 4\quad 7\quad \cdots\quad 6r+1)$ $(2\quad 5\quad 8\quad \cdots\quad 6r+2)$ to the path $[0, 1, 2, 3, 4, 5, 6]$. ■

Lemma 19 *The graph $2K_{6r+4}$ is decomposable into paths of type RRBBBR for all $r \geqq 1$.*

PROOF. Let $6r+3$ of the vertices of $2K_{6r+4}$ be labeled with the elements of Z_{6r+3} – the final vertex being labeled ∞. Then, successively apply the permutation $(0\quad 1\quad 2\quad \cdots\quad 6r+2)$ to the $r-1$ paths $[-1, -3j+2, 0, 3j-1, 2, 3j, 1]$, for $2 \leq j \leq r$ and the path $[-3r, 0, \infty, 1, 3r+2, 2, 1-3r]$. Also, successively apply the permutation $(0\quad 3\quad 6\quad \cdots\quad 6r)$ $(1\quad 4\quad 7\quad \cdots\quad 6r+1)$ $(2\quad 5\quad 8\quad \cdots\quad 6r+2)$ to the paths $[0, 1, 2, 3, 4, 5, 6]$ and $[0, 2, 4, 6, 8, 10, 12]$. ■

The paths RBRBBR and RRBBRB

Lemma 20 *The graph $2K_{6r}$ is decomposable into paths of type RBRBBR and paths of type RRBBRB for all $r \geqq 2$.*

PROOF. Label $6r-1$ vertices of $2K_{6r}$ with the elements of Z_{6r-1}. Label the remaining vertex ∞, a fixed point. Successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r-2)$ to the $r-1$ paths $[-1, 1-r-2j, 0, r+2j-1, 1, j, 2j-1]$, for $2 \le j \le r$ and the path $[-1, -r-1, 0, r+1, 1, \infty, 2]$. ■

Lemma 21 *The graph $2K_{6r+1}$ is decomposable into paths of type RBRBBR and paths of type RRBBRB for all $r \geqq 1$.*

PROOF. For $2K_7$, label the vertices with the elements of Z_7 and successively apply the permutation $(0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$ to the path $[6, 0, 4, 2, 3, 1, 5]$.

For $r \geqq 2$, label the $6r+1$ vertices of $2K_{6r+1}$ with the elements of Z_{6r+1}. Successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r)$ to the r paths $[-1, -2j-r, 0, 2j+r, 1, j+1, 2j+1]$, for $1 \le j \le r$. ■

Lemma 22 *The graph $2K_{6r+3}$ is decomposable into paths of types RBRBBR and paths of type RRBBRB for all $r \geqq 1$.*

PROOF. For $2K_9$, label the vertices with the elements of Z_9. Then the following 2-coloured paths make up a decomposition. $[8, 0, 7, 1, 6, 2, 5]$, $[5, 3, 4, 8, 1, 0, 2]$, $[2, 7, 3, 6, 4, 5, 8]$, $[1, 8, 4, 3, 5, 2, 6]$, $[4, 6, 3, 7, 2, 0, 1]$, $[6, 1, 7, 0, 8, 5, 4]$, $[8, 2, 1, 3, 0, 4, 7]$, $[7, 5, 6, 8, 3, 2, 4]$, $[4, 1, 5, 0, 6, 7, 8]$, $[3, 8, 6, 5, 7, 4, 0]$, $[6, 0, 5, 1, 4, 2, 3]$, $[0, 3, 1, 2, 8, 7, 6]$.

For $r \geqq 2$, label the vertices of $2K_{6r+3}$ with the elements of Z_{6r+3}. Successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 6r+2)$ to the paths $[-1, -3r+3j-4, 0, 3r-3j+4, 1, 3r-3j+3, 6r-6j+5]$, for $1 \le j \le r-1$.

In addition, to use all the edges of lengths 1 and 2, we need the $2r+1$ paths: $[0, 1, 2, 3, 4, 5, 6]$, $[6, 7, 8, 9, 10, 11, 12]$, ..., $[6r-6, 6r-5, 6r-4, 6r-3, 6r-2, 6r-1, 6r]$, $[6r, 6r+1, 6r+2, 0, 2, 4, 6]$, $[6, 8, 10, 12, 14, 16, 18]$,..., $[6r-9, 6r-7, 6r-5, 6r-3, 6r-1, 6r+1, 0]$, and $[4, 3, 2, 1, 0, 6r+1, 6r-1]$, $[6r-1, 6r-3, 6r-5, 6r-7, 6r-9, 6r-11, 6r-13]$, ..., $[14, 12, 10, 8, 6, 4, 2]$, $[2, 0, 6r+2, 6r+1, 6r, 6r-1, 6r-2]$, ..., and $[10, 9, 8, 7, 6, 5, 4]$.

Finally, to use the edges of lengths 3 and 4, we use the $2r+1$ paths $[0, 4, 1, 5, 2, 6, 3]$, $[3, 7, 4, 8, 5, 9, 6]$, ..., $[6r-3, 6r+1, 6r-2, 6r+2, 6r-1, 0, 6r]$, $[6r, 1, 6r+1, 2, 6r+2, 3, 0]$, and $[5, 1, 4, 0, 3, 6r+2, 2]$, $[2, 6r+1, 1, 6r, 0, 6r-1, 6r+2]$, $[6r+2, 6r-2, 6r+1, 6r-3, 6r, 6r-4, 6r-1]$, ..., $[11, 7, 10, 6, 9, 5, 8]$, and $[8, 4, 7, 3, 6, 2, 5]$. ■

Lemma 23 *The graph $2K_{6r+4}$ is decomposable into paths of type RBRBBR and paths of type RRBBRB for all $r \geqq 1$.*

PROOF. For $2K_{10}$, label nine of the vertices with the elements of Z_9 and the remaining vertex ∞. Successively apply the permutation $(0\ \ 1\ \ 2\ \ \cdots\ \ 8)$ to the 2-coloured path $[5, 0, \infty, 2, 6, 8, 1]$. Also, take the paths $[0, 3, 2, 5, 4, 7, 6]$, $[6, 0, 8, 2, 1, 4, 3]$, $[3, 6, 5, 8, 7, 1, 0]$, $[4, 5, 2, 3, 0, 1, 7]$, $[7, 8, 5, 6, 3, 4, 1]$ and $[1, 2, 8, 0, 6, 7, 4]$.

For $r \geqq 2$, label $6r+3$ of the vertices of $2K_{6r+4}$ with the elements of Z_{6r+3}, and the remaining vertex ∞. Successively apply the permutation $(0 \quad 1 \quad 2 \quad \cdots \quad 6r+2)$ to the $r-1$ paths $[-1, -3r+3j-4, 0, 3r-3j+4, 1, -3r+3j-1, -6r+6j-3]$, for $1 \leq j \leq r-1$ and the path $[-4, 0, \infty, 2, 6, 9, 12]$.

Finally, we also need the $2r+1$ paths: $[0,1,2,3,4,5,6]$, $[6,7,8,9,10,11,12]$,..., $[6r-6, 6r-5, 6r-4, 6r-3, 6r-2, 6r-1, 6r]$, $[6r, 6r+1, 6r+2, 0, 2, 4, 6]$, $[6, 8, 10, 12, 14, 16, 18]$,..., $[6r-9, 6r-7, 6r-5, 6r-3, 6r-1, 6r+1, 0]$ and $[4, 3, 2, 1, 0, 6r+1, 6r-1]$, $[6r-1, 6r-3, 6r-5, 6r-7, 6r-9, 6r-11, 6r-13]$, ..., $[14, 12, 10, 8, 6, 4, 2]$, $[2, 0, 6r+2, 6r+1, 6r, 6r-1, 6r-2]$,..., and $[10, 9, 8, 7, 6, 5, 4]$. ■

Combining the results of this section, we obtain the following theorem.

Theorem 24 *The complete graph $2K_n$ is decomposable into 2-coloured paths of length 6 if and only if $n > 6$ and $n(n-1) \equiv 0 \bmod 6$.*

5 Conclusions

Combining the results of Sections 2, 3, and 4, we obtain the following theorem.

Theorem 25 *The complete graph $2K_n$ is decomposable into 2-coloured paths of length $2k$, $k \leqq 3$, if and only if $n > 2k$ and $n(n-1) \equiv 0 \bmod 2k$.*

Theorem 25 seems to imply that 2-coloured path decompositions will exist for all possible k. It is not surprising that this would be true for large k, as in [8], a theorem is given proving the asymptotic existence of many graph decompositions, and this result is also applicable in this situation. However, as the number of non-isomorphic types of path of length 8 is 23, and only 8 of these 23 is colour–complementary, extending Theorem 25 in this manner does not seem reasonable.

Further, when $k = 3$, it is clear from the previous lemmas that decompositions were much easier to formulate when the paths were colour–complementary. It would seem that focusing on colour-complementary paths would allow extensions of these results to larger k.

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