Vertex-magic total labeling of the union of suns

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Abstract. Let G be a graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$ and let $e = |E(G)|$ and $v = |V(G)|$. A one-to-one map λ from $V \cup E$ onto the integers $\{1, 2, ..., v+e\}$ is called *vertex-magic total labeling* if there is a constant k so that for every vertex x,

$$\lambda(x) + \sum \lambda(xy) = k$$

where the sum is over all edges xy, where y is adjacent to x. Let us call the sum of labels at vertex x the *weight* $w_\lambda(x)$ of the vertex under labeling λ; we require $w_\lambda(x) = k$ for all x. The constant k is called the *magic constant* for λ.
A sun S_n is a cycle on n vertices C_n, for $n \geq 3$, with an edge terminating in a vertex of degree 1 attached to each vertex.
In this paper, we present the vertex-magic total labeling of the union of suns, including the union of m non-isomorphic suns for any positive integer $m \geq 3$, proving the conjecture given in [6].

1 Introduction

In this paper all graphs are finite, simple and undirected. The graph G has vertex-set $V(G)$ and edge-set $E(G)$ and we let $e = |E(G)|$ and $v = |V(G)|$. A general reference for graph-theoretic ideas is in [8].

MacDougall *et al.* [4] introduced the notion of a *vertex-magic total labeling*. This is an assignment of the integers from 1 to $v + e$ to the vertices and edges of G so that at each vertex the vertex label and the labels on the edges incident at that vertex add to a constant. More formally, a one-to-one map λ from $V \cup E$ onto the integers $\{1, 2, \cdots, v+e\}$ is a *vertex-magic total labeling* if there is a constant k so that for every vertex x,

$$\lambda(x) + \sum \lambda(xy) = k$$

where the sum is over all edges xy, where y is adjacent to x. Let us call the sum of labels at vertex x the *weight* $\omega_\lambda(x)$ of the vertex x; we require $w_\lambda(x) = k$ for all x. The constant k is called the *magic constant* for λ.

If a regular graph G possesses a vertex-magic total labeling λ, we can create a new labeling λ' from λ by setting

$$\lambda'(x) = v + e + 1 - \lambda(x)$$

for every vertex x, and

$$\lambda'(xy) = v + e + 1 - \lambda(xy)$$

for every edge xy. Clearly the new labelingλ' is also a one-to-one map from the set $V \cup E$ to $\{1, 2, \cdots, v+e\}$, and we call this new labeling as the dual of the previous labeling. If r is the degree of each vertex of G, then

$$k' = (r+1)(v+e+1) - k$$

is the new magic constant.

Since the introduction of this notion, there have been several results on vertex-magic total labeling of particular classes of graphs. For example, MacDougall *et al.* [4] proved that cycle C_n for $n \geq 3$,path P_n for $n \geq 3$, complete graph K_n for odd n and complete bipartite graph $K_{n,n}$ for $n > 1$, have vertex-magic total labeling. Baca, Miller and Slamin [1] proved that for $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n,m)$ has vertex-magic total labeling with magic constants $k = 9n+2,\ \ k = 10n+2,\ \ and\ \ k = 11n+2$. A complete survey on the vertex-magic total labeling of graphs can be found in [2].

Most of the known results are concerning the vertex-magic total labeling of connected graphs. For the case of disconnected graphs, Wallis [7] proved the following theorem.

Theorem 1. *Suppose G is a regular graph of degree r which has a vertex-magic total labeling.*

(i) If r is even, then nG has a vertex-magic total labeling when n is odd.
(ii) If r is odd, then nG has a vertex-magic total labeling for every positive integer n.

The result above concerns the vertex-magic total labeling of disconnected graphs whose components are regular and isomorphic graphs. For the case of disconnected graphs whose components are not regular graphs, Gray *et al.* [3] proved that the n copies of the star on 3 vertices $nK_{1,2}$ has a vertex-magic total labeling. Slamin *et al.* [6] proved that for $t \geq 3$ and $n \geq 1$, the n copies of sun nS_t has a vertex-magic total labeling with the magic constant $k = 6nt + 1$. These two results are concerning on the vertex-magic total labeling of n copies of graphs where all components are isomorphic.

For the case of disconnected graphs whose components are not regular and are not isomorphic, Slamin *et al.* [6] posed a conjecture that there is a vertex-magic total labeling of the disjoint union of n non-isomorphic suns, for any positive integer $n \geq 3$.

In this paper we prove the conjecture as described in the following section.

2 Main Result

Before presenting the main result of this paper, we give the definition of sun as follows.

A sun S_n is a graph with a cycle C_n having an edge terminating in a vertex of degree 1 attached to each vertex of the cycle. The sun S_n consists of the vertex set $V(S_n) = \{v_i | 1 \leq i \leq n\} \bigcup \{a_i | 1 \leq i \leq n\}$ and edge set $E(S_n) = \{v_i v_{i+1} | 1 \leq i \leq n\} \bigcup \{v_i a_i | 1 \leq i \leq n\}$, where $i+1$ is taken modulo n.

Theorem 2. *If $t_i \geq 3$ for every $i = 1, 2, \cdots, n$ and $n \geq 1$ the n disjoint copies of suns $S_{t_1} \bigcup S_{t_2} \bigcup \cdots \bigcup S_{t_n}$ has a vertex-magic total labeling with magic constant $6 \sum_{k=1}^{n} t_k + 1$.*

Proof. We label the vertices and edges of the graph in the following way:

$$\lambda(v_i^{t_j}) = 2 \sum_{k=1}^{j-1} t_k + 2i \;\; ; \; i = 1, 2, \cdots t_j \; \textit{and} \; j = 1, 2, \cdots, n$$

$$\lambda(a_i^{t_j}) = \begin{cases} 4 \sum\limits_{k=1}^{n} t_k - 2 \sum\limits_{k=1}^{j} t_k + 2 & \text{for } i = 1 \\ 2 \sum\limits_{k=1}^{n} t_k + 2 \sum\limits_{k=j}^{n} t_k - 2(i-2) & \text{for } i = 2, 3, \cdots, t_j \end{cases}$$

$$\lambda(v_i^{t_j}v_{i+1}^{t_j}) = \begin{cases} 2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j} t_k + 1 & \text{for } i = t_j \\ 2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j-1} t_k - 2i + 1 & \text{for } i = 1, 2, \cdots, t_j - 1 \end{cases}$$

$$\lambda(v_i^{t_j}a_i^{t_j}) = \begin{cases} 2\sum\limits_{k=1}^{n} t_k + 2\sum\limits_{k=1}^{j} t_k - 1 & \text{for } i = 1 \\ 2\sum\limits_{k=1}^{n} t_k + 2\sum\limits_{k=1}^{j-1} t_k + 2i - 3 & \text{for } i = 2, 3, \cdots, t_j \end{cases}$$

It is easy to verify that the labeling λ is a bijection from the set $V(S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_n}) \cup E(S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_n})$ onto the set
$\{1, 2, \cdots, 4\sum\limits_{k=1}^{n} t_k\}$.

Let us denote the weights of the vertices $v_i^{t_j}$ of S_{t_i} under the labeling λ by

$$\omega_\lambda(v_i^{t_j}) = \lambda(v_i^{t_j}) + \lambda(v_i^{t_j}v_{i+1}^{t_j}) + \lambda(v_i^{t_j}a_i^{t_j}) + \lambda(v_{i-1}^{t_j}v_i^{t_j})$$

and the weights of the vertices $a_i^{t_j}$ by

$$\omega_\lambda(a_i^{t_j}) = \lambda(a_i^{t_j}) + \lambda(v_i^{t_j}a_i^{t_j})$$

Then for all $j = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, t_j$ the weights of the vertices $v_i^{t_j}$ can be determined as follows:

- for $i = 1$ we have
  $\omega_\lambda(v_1^{t_j}) = \lambda(v_1^{t_j}) + \lambda(v_1^{t_j}v_2^{t_j}) + \lambda(v_1^{t_j}a_1^{t_j}) + \lambda(v_{t_j}^{t_j}v_1^{t_j})$
  $= 2\sum\limits_{k=1}^{j-1} t_k + 2 + 2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j-1} t_k - 2 + 1 + 2\sum\limits_{k=1}^{n} t_k + 2\sum\limits_{k=1}^{j} t_k - 1 +$
  $2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j} t_k + 1$
  $= 6\sum\limits_{k=1}^{n} t_k + 1;$
- for $i = 2, 3, ..., t_j - 1$ we have
  $\omega_\lambda(v_i^{t_j}) = \lambda(v_i^{t_j}) + \lambda(v_{i-1}^{t_j}v_i^{t_j}) + \lambda(v_i^{t_j}v_{i+1}^{t_j}) + \lambda(v_i^{t_j}a_i^{t_j})$
  $= 2\sum\limits_{k=1}^{j-1} t_k + 2i + 2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j-1} t_k - 2i + 1 + 2\sum\limits_{k=1}^{n} t_k - 2\sum\limits_{k=1}^{j-1} t_k - 2(i -$

$1) + 1 + 2\sum_{k=1}^{n} t_k + 2\sum_{k=1}^{j-1} t_k + 2i - 3$
$= 6\sum_{k=1}^{n} t_k + 1;$

- for $i = t_j$ we have
$\omega_\lambda(v_i^{t_j}) = 2\sum_{k=1}^{j-1} t_k + 2t_j + 2\sum_{k=1}^{n} t_k - 2\sum_{k=1}^{j} t_k + 1 + 2\sum_{k=1}^{n} t_k - 2\sum_{k=1}^{j-1} t_k -$
$2(t_j - 1) + 1 + 2\sum_{k=1}^{n} t_k + 2\sum_{k=1}^{j-1} t_k + 2t_j - 3$
$= 6\sum_{k=1}^{n} t_k + 1;$

- for $i = 1$ the weights of the vertices $a_1^{t_j}$ are given by
$\omega_\lambda(a_1^{t_j}) = \lambda(a_1^{t_j}) + \lambda(v_1^{t_j} a_1^{t_j})$
$= 4\sum_{k=1}^{n} t_k - 2\sum_{k=1}^{j} t_j + 2 + 2\sum_{k=1}^{n} t_k + 2\sum_{k=1}^{j} t_k - 1$
$= 6\sum_{k=1}^{n} t_k + 1;$

- for $i = 2, 3, \cdots, t_j$ the weights of the vertices $a_i^{t_j}$ are given by
$\omega_\lambda(a_i^{t_j}) = \lambda(a_i^{t_j}) + \lambda(v_i^{t_j} a_i^{t_j})$
$= 2\sum_{k=1}^{n} t_k + 2\sum_{k=j}^{n} t_k - 2i + 4 + 2\sum_{k=1}^{n} t_k + 2\sum_{k=1}^{j-1} t_k + 2i - 3$
$= 6\sum_{k=1}^{n} t_k + 1.$

Example: In the following figure vertex-magic total labeling of $S_4 \cup S_5 \cup S_6$ is given with our formula.

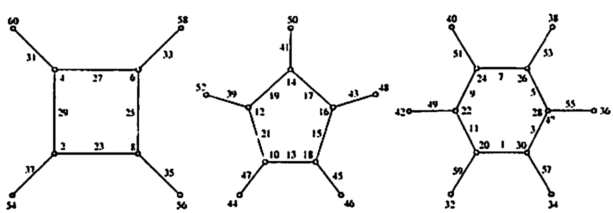

Fig. 1. Vertex-magic total labeling of $S_4 \cup S_5 \cup S_6$ with magic constant $k = 91$

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