A class of minimally spectrally arbitrary sign patterns*

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Abstract

A spectrally arbitrary pattern A is a sign pattern of order n such that every monic real polynomial of degree n can be achieved as the characteristic polynomial of a matrix with sign pattern A. A sign pattern A is minimally spectrally arbitrary if it is spectrally arbitrary but is not spectrally arbitrary if any nonzero entry (or entries) of A is replaced by zero. In this paper, we introduce some new sign patterns which are minimally spectrally arbitrary for all orders $n \geq 7$.

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1 Introduction

A *sign pattern matrix* (or sign pattern) is a matrix whose entries come from the set $\{+, -, 0\}$. If $A = [a_{ij}]$ is a sign pattern of order n, the *sign pattern class* (or qualitative class) of A, denoted by $Q(A)$, is the set of all real matrices $B = [b_{ij}]$ of order n with sign $b_{ij} = a_{ij}$ for all i and j.

A sign pattern $\breve{A} = [\breve{a}_{ij}]$ is a *superpattern* of a sign pattern $A = [a_{ij}]$ if $\breve{a}_{ij} = a_{ij}$ whenever $a_{ij} \neq 0$. Similarly, $\breve{A}$ is a *subpattern* of A if $\breve{a}_{ij} = 0$ whenever $a_{ij} = 0$. Note that each sign pattern is a superpattern and a subpattern of itself. If $\breve{A}$ is a subpattern of A and $\breve{A} \neq A$, then $\breve{A}$ is a *proper subpattern* of A.

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A *nilpotent realization* of a sign pattern A of order n is a real matrix $B \in Q(A)$, all of whose eigenvalues are zero. If A has a nilpotent realization, then we also say that A is *potentially nilpotent* (PN). A *spectrally arbitrary pattern* (SAP) is a sign pattern A of order n such that every monic real polynomial of degree n can be achieved as the characteristic polynomial of some matrix $B \in Q(A)$. A sign pattern A is a *minimally spectrally arbitrary pattern* (MSAP) if it is spectrally arbitrary but is not spectrally arbitrary if any nonzero entry (or entries) of A is replaced by zero. Note that each spectrally arbitrary sign pattern must be PN. For sign pattern A of order n, A is *sign nonsingular* if every matrix $B \in Q(A)$ is nonsingular, and A is *sign singular* if each matrix $B \in Q(A)$ is singular.

The question of the existence of a SAP arose in [1], where a general method (based on the Implicit Function Theorem) was given to prove that a sign pattern and all of its superpatterns are SAPs. The first SAP of order n for each $n \geq 2$ was provided in [2]. Later, some papers [3, 4, 5] introduce some sign patterns which are SAPs for all orders $n \geq 2$. In this paper, we introduce some new sign patterns which are MSAPs for all orders $n \geq 7$. This work is mainly motivated by the inertial and spectral problems considered in [1] (see section 3 for more details) and, more generally, by the inverse eigenvalue problem for matrices over the real field.

2 Some preliminaries

The following lemma describes the Nilpotent-Jacobian method (N-J method) for proving that a sign pattern and all of its superpattern are SAPs, which is stated as Observations 10 and 15 in [1] and is proved using the Implicit Function Theorem. Our work will rely on it largely.

Lemma 2.1 *([1]) Let A be a sign pattern of order n, and suppose that there exists some nilpotent realization $B \in Q(A)$ with at least n nonzero entries, say $b_{i_1j_1}, \ldots, b_{i_nj_n}$. Let X be the matrix obtained by replacing these entries in B by variables $x_1, \ldots, x_n$. If the Jacobian of the coefficients of the characteristic polynomial of X with respect to the variables $x_1, \ldots, x_n$ is nonzero at $(x_1, \ldots, x_n) = (b_{i_1j_1}, \ldots, b_{i_nj_n})$, then every superpattern of A is spectrally arbitrary.*

In this paper, we shall study the sign patterns of order $n \geq 7$ defined as

$$A = \begin{bmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \eta \\ \beta_1 & 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \beta_2 & 0 & 0 & + & \ddots & & & & & \vdots \\ \beta_3 & \vdots & & \ddots & + & \ddots & & & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\ \beta_{n-5} & 0 & \cdots & \cdots & \cdots & 0 & + & \ddots & & \vdots \\ \beta_{n-4} & \beta_{n-2} & 0 & 0 & \cdots & 0 & 0 & + & \ddots & \vdots \\ 0 & \beta_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & + & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \alpha & + \\ \gamma & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}, \qquad (2.1)$$

where the entries $\alpha, \gamma, \eta \in \{+,-\}$ and $\beta_i \in \{+,-\}$ for $i = 1, 2, \ldots, n-2$. We shall demonstrate three patterns of form (2.1) are MSAPs, and the other sign patterns of form (2.1) are not SAPs. For convenience, suppose that $B = [b_{ij}] \in Q(A)$ has been scaled such that $b_{11} = -1$, $b_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ (otherwise they can be adjusted to be 1 by suitable similarities), and has the following form.

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & b \\ d_1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ d_2 & 0 & 0 & 1 & \ddots & & & & & \vdots \\ d_3 & \vdots & & \ddots & 1 & \ddots & & & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\ d_{n-5} & 0 & \cdots & \cdots & \cdots & 0 & 1 & \ddots & & \vdots \\ d_{n-4} & d_{n-2} & 0 & 0 & \cdots & 0 & 0 & 1 & \ddots & \vdots \\ 0 & d_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 1 \\ c & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}. \qquad (2.2)$$

Lemma 2.2 *Let $f_B(\lambda) = \det(\lambda I - B)$. Then*

(1) $f_B(\lambda) = \lambda^n + f_1\lambda^{n-1} + f_2\lambda^{n-2} + \cdots + f_{n-1}\lambda + f_n$, *where*

$f_1 = 1 - a$,

$f_2 = -a - d_1 - bc$,

$f_3 = abc + ad_1 - d_2$ *(If $n = 7$, then $f_3 = abc + ad_1 - d_2 - d_5$),*

$f_i = ad_{i-2} - d_{i-1}$, *for $i = 4, 5, \ldots, n-5$ $(n \geq 9)$,*

$f_{n-4} = ad_{n-6} - d_{n-2} - d_{n-5}$ $(n \geq 8)$,

$f_{n-3} = -d_{n-2} + ad_{n-2} - d_{n-3} - d_{n-4} + ad_{n-5}$,
$f_{n-2} = bcd_{n-2} - d_{n-3} + ad_{n-3} + ad_{n-2} + ad_{n-4}$,
$f_{n-1} = bcd_{n-3} - abcd_{n-2} + ad_{n-3}$,
$f_n = -c - abcd_{n-3}$.

(2) *For arbitrary given* d_1,

$$\frac{\partial(f_1, f_2, f_3, \ldots, f_{n-1}, f_n)}{\partial(a, b, c, d_2, \ldots, d_{n-2})} = b^2c^3 \neq 0.$$

Proof (1)

$$f_B(\lambda) = \begin{vmatrix} \lambda+1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & -b \\ -d_1 & \lambda & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -d_2 & 0 & \lambda & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ -d_{n-5} & 0 & \cdots & 0 & \lambda & -1 & \ddots & & \vdots \\ -d_{n-4} & -d_{n-2} & 0 & \cdots & 0 & \lambda & -1 & \ddots & \vdots \\ 0 & -d_{n-3} & 0 & \cdots & 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda-a & -1 \\ -c & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \lambda \end{vmatrix}_n .$$

By adding λ times of the ith row to the $(i+1)$th row, for $i = 1, 2, \ldots, n-3$, and expanding along the third column in order, we have

$$f_B(\lambda) = \begin{vmatrix} \lambda+1 & -1 & 0 & -b \\ \lambda^{n-3}(\lambda+1) - \sum_{i=1}^{n-4} d_i\lambda^{n-3-i} & -d_{n-3} - d_{n-2}\lambda & -1 & -b\lambda^{n-3} \\ 0 & 0 & \lambda-a & -1 \\ -c & 0 & 0 & \lambda \end{vmatrix}$$

$$= c[-1 + b(\lambda-a)(d_{n-3} + d_{n-2}\lambda) - b\lambda^{n-3}(\lambda-a)] + \lambda(\lambda-a)[(\lambda +1)(-d_{n-3} - d_{n-2}\lambda) + lambda^{n-3}(\lambda+1) - \textstyle\sum_{i=1}^{n-4} d_i\lambda^{n-3-i}]$$
$$= \lambda^n + (1-a)\lambda^{n-1} - (a + bc + d_1)\lambda^{n-2} + abc\lambda^{n-3} - \textstyle\sum_{i=2}^{n-5} d_i\lambda^{n-1-i} + a\sum_{i=1}^{n-6} d_i\lambda^{n-2-i} - d_{n-2}\lambda^4 + (ad_{n-2} - d_{n-2} - d_{n-3} - d_{n-4} + ad_{n-5})\lambda^3 + (bcd_{n-2} + ad_{n-2} - d_{n-3} + ad_{n-3} + ad_{n-4})\lambda^2 + (bcd_{n-3} - abcd_{n-2} + ad_{n-3})\lambda - c - abcd_{n-3}.$$

So result (1) is right.

(2) For arbitrary given d_1,

$$\frac{\partial(f_1, f_2, f_3, \ldots, f_{n-1}, f_n)}{\partial(a, b, c, d_2, \ldots, d_{n-2})} =$$

$$
\left|\begin{array}{ccc}
-1 & 0 & 0 \\
-1 & -c & -b \\
bc+d_1 & ac & ab \\
d_2 & 0 & 0 \\
\vdots & \vdots & \vdots \\
d_{n-7} & \vdots & \vdots \\
d_{n-6} & \vdots & \vdots \\
d_{n-5}+d_{n-2} & 0 & 0 \\
d_{n-4}+d_{n-3}+d_{n-2} & cd_{n-2} & bd_{n-2} \\
d_{n-3}-bcd_{n-2} & cd_{n-3}-acd_{n-2} & bd_{n-3}-abd_{n-2} \\
-bcd_{n-3} & -acd_{n-3} & -1-abd_{n-3}
\end{array}\right.
$$

$$
\left.\begin{array}{cccccccc}
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & & & & & \vdots & \vdots & \vdots \\
-1 & \ddots & & & & \vdots & \vdots & \vdots \\
a & -1 & \ddots & & & \vdots & \vdots & \vdots \\
\ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
 & \ddots & a & -1 & \ddots & \vdots & \vdots & 0 \\
 & & \ddots & a & -1 & 0 & 0 & -1 \\
\cdots & \cdots & \cdots & 0 & a & -1 & -1 & a-1 \\
0 & 0 & \cdots & 0 & 0 & a & a-1 & a+bc \\
0 & 0 & \cdots & 0 & 0 & 0 & bc+a & -abc \\
0 & 0 & \cdots & 0 & 0 & 0 & -abc & 0
\end{array}\right|.
$$

By adding a times of the ith row to the $(i+1)$th row, for $i = 2, 3, \ldots, n-1$, and expanding along the first row, we have

$$
\frac{\partial(f_1, f_2, f_3, \ldots, f_{n-1}, f_n)}{\partial(a, b, c, d_2, \ldots, d_{n-2})} = -\left|\begin{array}{cccc}
-c & -b & 0 & 0 \\
cd_{n-2} & bd_{n-2} & -1 & bc \\
cd_{n-3} & bd_{n-3} & bc & 0 \\
0 & -1 & 0 & 0
\end{array}\right| = b^2c^3.
$$

Thus result (2) follows. □

Lemma 2.3 *([3]) For $n \geq 2$, an irreducible spectrally arbitrary sign pattern of order n has at least $2n-1$ nonzero entries.*

Lemma 2.4 *Let A be a sign pattern of order $n \geq 7$ having the form* (2.1). *If A is a SAP, then A is a MSAP.*

Proof Suppose $T = [t_{ij}]$ is a subpattern of A, and T is a SAP. Then

(1) $t_{n-1,n-1} \neq 0$. Otherwise, the trace of T is negative.

(2) $t_{n,1} \neq 0$ and $t_{i,i+1} \neq 0$ for $i = 2, 3, \ldots, n-3$. Otherwise, sign pattern T is sign singular.

(3) $t_{1,n} \neq 0$, $t_{n-2,2} \neq 0$, $t_{1,2} \neq 0$, $t_{n-2,n-1} \neq 0$, and $t_{n-1,n} \neq 0$. Otherwise, sign pattern T is sign nonsingular or sign singular.

(4) Since T is a SAP, there exists a real matrix $B \in Q(T)$ which is nilpotent. Suppose B has the form (2.2). Then from $f_1 = f_2 = \ldots = f_n = 0$ as in Lemma 2.2, we can conclude that

$$\begin{aligned}
&a = 1,\\
&bc = -d_1 - 1,\\
&d_i = -1, \text{ for } i = 2, 3, \ldots, n-6,\\
&f_{n-4} = -1 - d_{n-5} - d_{n-2} = 0,\\
&f_{n-3} = d_{n-5} - d_{n-4} - d_{n-3} = 0,\\
&f_{n-2} = d_{n-4} - d_1 d_{n-2} = 0,\\
&f_{n-1} = d_1 d_{n-2} + d_{n-2} - d_1 d_{n-3} = 0,\\
&f_n = -c - bcd_{n-3} = 0.
\end{aligned}$$

(4*a*) $d_i \neq 0$, for $i = 2, 3, \ldots, n-6$, since $d_i = -1$.

(4*b*) $d_1 \neq 0$. Otherwise, $f_{n-2} = d_{n-4} = 0$ and $f_{n-1} = d_{n-2} = 0$, and so $d_{n-4} = 0$ and $d_{n-2} = 0$. Then the number of nonzero entries of T is less than $2n - 1$, and we know T is not a SAP by Lemma 2.3.

(4*c*) $d_{n-4} \neq 0$. Otherwise, $f_{n-2} = -d_1 d_{n-2} = 0$, and $d_{n-2} = 0$ by Case (4b). Then $f_{n-1} = -d_1 d_{n-3} = 0$, which is contrary to $t_{n-2,2} \neq 0$ in Case 3, or $d_1 \neq 0$ in Case (4b).

(4*d*) $d_{n-2} \neq 0$. Otherwise, $f_{n-1} = -d_1 d_{n-3} = 0$, which is contrary to $t_{n-2,2} \neq 0$ in Case 3, or $d_1 \neq 0$ in Case (4b).

(4*e*) $d_{n-5} \neq 0$. Otherwise, $f_{n-4} = -1 - d_{n-2} = 0$, and so $d_{n-2} = -1$. Then from $f_{n-2} = d_{n-4} - d_1 d_{n-2} = 0$, we have $d_{n-4} = -d_1$. From $f_{n-3} = -d_{n-4} - d_{n-3} = 0$, we have $d_{n-3} = d_1$. So $f_{n-1} = d_1 d_{n-2} + d_{n-2} - d_1 d_{n-3} = -(d_1^2 + d_1 + 1) = 0$, and which contradicts the fact that d_1 is real.

Thus, there are no proper subpatterns of A which are SAPs. □

3 Main results

Let A_1, A_2 and A_3 be sign patterns of order n ($n \geq 7$) as follows.

$$A_1 = \begin{bmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & - \\ - & 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ - & 0 & 0 & + & \ddots & & & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & & & \vdots \\ - & \vdots & & & \ddots & + & \ddots & & & \vdots \\ - & 0 & \cdots & \cdots & \cdots & 0 & + & \ddots & & \vdots \\ - & + & 0 & 0 & \cdots & 0 & 0 & + & \ddots & \vdots \\ 0 & + & 0 & 0 & \cdots & 0 & 0 & 0 & + & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & + \\ - & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & + \\ - & 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ - & 0 & 0 & + & \ddots & & & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & & & \vdots \\ - & \vdots & & & \ddots & + & \ddots & & & \vdots \\ - & 0 & \cdots & \cdots & \cdots & 0 & + & \ddots & & \vdots \\ - & + & 0 & 0 & \cdots & 0 & 0 & + & \ddots & \vdots \\ 0 & - & 0 & 0 & \cdots & 0 & 0 & 0 & + & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & + \\ - & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + \\ + & 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ - & 0 & 0 & + & \ddots & & & & & & \vdots \\ - & \vdots & & \ddots & + & \ddots & & & & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & & & \vdots \\ - & \vdots & & & & \ddots & + & \ddots & & & \vdots \\ - & 0 & \cdots & \cdots & \cdots & \cdots & 0 & + & \ddots & & \vdots \\ - & - & 0 & 0 & \cdots & \cdots & 0 & 0 & + & \ddots & \vdots \\ 0 & - & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & + & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & + \\ - & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Then A_i ($i = 1, 2, 3$) has the form (2.1). We shall show that sign patterns A_1, A_2 and A_3 are MSAPs, and the other sign patterns of form (2.1) are not SAPs.

Lemma 3.1 *Let sign pattern A of order $n \geq 7$ have form (2.1). Then A is potentially nilpotent if and only if A is one of the sign patterns A_1, A_2 and A_3.*

Proof Necessity. Suppose sign pattern A of form (2.1) is potentially nilpotent. Then there exists a real matrix $B \in Q(A)$ which is nilpotent. We may suppose that B has the form (2.2). From $f_1 = f_2 = \cdots = f_n = 0$ as in Lemma 2.2, since there are n equations and $n+1$ unknowns, we can express the other n unknowns by d_1. So we can conclude that $a = 1 > 0$, $d_i = -1 < 0$ for $i = 2, 3, \ldots, n-6$, $d_{n-2} = -\frac{d_1}{(d_1+1)^2}\begin{cases} < 0 & \text{if } d_1 > 0 \\ > 0 & \text{if } d_1 < 0 \end{cases}$, $d_{n-3} = -\frac{1}{d_1+1}\begin{cases} < 0 & \text{if } d_1 > -1 \\ > 0 & \text{if } d_1 < -1 \end{cases}$, $d_{n-4} = -\frac{d_1^2}{(d_1+1)^2} < 0$, $d_{n-5} = -\frac{d_1^2+d_1+1}{(d_1+1)^2} < 0$, $c = -1 < 0$, and $b = d_1 + 1\begin{cases} > 0 & \text{if } d_1 > -1 \\ < 0 & \text{if } d_1 < -1 \end{cases}$. From the signs of $a, b, c, d_2, \ldots, d_{n-2}$ corresponding to $d_1 < -1$, $-1 < d_1 < 0$ and $d_1 > 0$, we know that A must be one of the sign patterns A_1, A_2 and A_3.

Sufficiency. Let $B \in Q(A)$ have form (2.2). If $(a, b, c, d_1, d_2, \ldots, d_{n-2}) = (1, -1, -1, -2, d_2, \ldots, d_{n-6}, -3, -4, 1, 2)$ with $d_i = -1$ for $i = 2, \ldots, n-6$, then $B \in Q(A_1)$ is nilpotent. If $(a, b, c, d_1, d_2, \ldots, d_{n-2}) = (1, 1/2, -1, -1/2, d_2, \ldots, d_{n-6}, -3, -1, -2, 2)$ with $d_i = -1$ for $i = 2, \ldots, n-6$, then $B \in Q(A_2)$ is nilpotent. If $(a, b, c, d_1, d_2, \ldots, d_{n-2}) = (1, 2, -1, 1, d_2, \ldots, d_{n-6}, -3/4, -1/4, -1/2, -1/4)$ with $d_i = -1$ for $i = 2, \ldots, n-6$, then $B \in Q(A_3)$ is nilpotent. □

Lemma 3.2 *Sign patterns A_1, A_2 and A_3 are MSAPs, and every superpattern of them is a SAP.*

Proof By Lemmas 2.2(2), 2.1 and 2.4, the result is clear. □

Combining Lemmas 3.1 and 3.2, we have the following.

Theorem 3.3 *Let sign pattern A of order $n \geq 7$ have form (2.1). Then A is a SAP if and only if A is one of the sign patterns A_1, A_2 and A_3.*

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References

[1] J.H. Drew, C.R. Johnson, D.D. Olesky, and P. van den Driessche, Spectrally arbitrary patterns, *Linear Algebra Appl.*, 308(2000), 121–137.

[2] J.J. McDonald, D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche, On the spectra of striped sign patterns, *Linear Multilinear Algebra*, 51:1(2003), 39–48.

[3] M.S. Cavers, and K.N. Vander Meulen, Spectrally and inertially arbitrary sign patterns, *Linear Algebra Appl.*, 394(2005), 53–72.

[4] T. Britz, J.J. McDonald, D.D. Olesky, and P. van den Driessche, Minimal spectrally arbitrary sign patterns, *SIAM J. Matrix Anal. Appl.*, 26:1(2004), 257–271.

[5] M.S. Cavers, I.-J. Kim, B.L. Shader, and K.N. Vander Meulen, On determining minimal spectrally arbitrary patterns, *Elec. J. Linear Algebra*, 13(2005), 240–248.