On Cliques and Forcing m-Convexity Numbers of Graphs

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ABSTRACT

This paper introduces the concepts of forcing m-convexity number and forcing clique number of a graph. We show that the forcing m-convexity numbers of some cartesian product and composition of graphs are related to the forcing clique numbers of the graphs. We also show that the forcing m-convexity number of the composition $G[K_n]$, where G is a connected graph with no extreme vertex, is equal to the forcing m-convexity number of G.

Key words: graph, m-convex, forcing m-subset, forcing m-convexity number, clique, forcing c-subset, forcing clique number, composition, cartesian product

1 Introduction

Given a connected graph G and two vertices u and v in $V(G)$, any chordless path connecting u and v is called a u-v m-path (monophonic or induced path). The set $J_G[u,v]$ denotes the closed interval consisting of u, v and all vertices lying on some u-v m-path in G. For a set S of vertices of G, the m-closure of S is the set $J_G[S] = \cup_{u,v \in S} J_G[u,v]$. A subset C of $V(G)$ is m-convex if $J_G[u,v] \subseteq C$ for every pair of vertices $u, v \in C$. The m-convexity number $con_m(G)$ of G is the maximum cardinality among the proper m-convex sets of G. An m-convex set S of G with $|S| = con_m(G)$

is called a maximum m-convex set. A subset T of a maximum m-convex set S of G is called an m-forcing subset of S if S is the unique maximum m-convex set containing T. The m-forcing convexity number $fcon_m(S)$ of a maximum m-convex set S of G is the minimum cardinality among the m-forcing subsets of S. The m-forcing convexity number of G is given by

$$fcon_m(G) = \min\{fcon_m(S) : S \text{ is a maximum } m\text{-convex set of } G\}.$$

The following remarks are easy.

Remark 1.1 *Let G be a connected graph. If S is a maximum m-convex set in G, then S is a forcing m-subset for itself. In particular, $fcon_m(G) \leq con_m(G)$.*

Remark 1.2 *Let G be a connected graph. If G has a unique maximum m-convex set S, then the empty set $\emptyset$ is an m-forcing subset for S. In this case, $fcon_m(G) = 0$.*

The next results are easy. These are the analogues of the ones obtained by Chartrand and Zhang in [1].

Theorem 1.3 *Let G be a connected graph. Then $fcon_m(G) = 0$ if and only if G has a unique maximum m-convex set S. Moreover, $fcon_m(G) \geq 1$ if and only if G does not have a unique maximum m-convex set and if there exists a vertex in G that is contained in a unique maximum m-convex set.*

Definition 1.4 *Let K_r be a maximum clique (complete subgraph) of a graph G. A set $S \subseteq V(K_r)$ is a forcing c-subset for K_r if K_r is the only maximum clique of G containing S. The forcing clique number of K_r is given by*

$$fcn(K_r) = min\ \{|S| : S \text{ is a forcing c-subset for } K_r\}.$$

The forcing clique number of G is given by

$$fcn(G) = min\ \{fcn(K_r) : K_r \text{ is a maximum clique of } G\}.$$

Remark 1.5 *Let G be a graph. Then $fcn(G) = 0$ if and only if G has a unique maximum clique.*

Remark 1.6 *Let G be a graph. If the maximum m-convex sets of G are cliques, then $fcon_m(K_r) = fcn(K_r)$ for every maximum clique K_r of G; hence, $fcon_m(G) = fcn(G)$.*

2 Forcing m-Convexity Number of the Cartesian Product $G \times H$

Let $C \subseteq V(G) \times V(H)$. The *G-projection* and *H-projection* of C are, respectively, the sets

$$C_G = \{u \in V(G) : (u, v) \in C \text{ for some } v \in V(H)\}$$

and

$$C_H = \{v \in V(H) : (u, v) \in C \text{ for some } u \in V(G)\}.$$

Our first result in this section characterizes the cliques in the cartesian product of graphs. Here, $\omega(G)$ denotes the clique number of G (the order of a maximum clique or complete subgraph of G).

Theorem 2.1 *Let G and H be connected graphs. A subset C of $V(G \times H)$ induces a complete graph (clique) in $G \times H$ if and only if $C = S \times \{b\}$ or $C = \{a\} \times R$, where $a \in V(G)$, $b \in V(H)$, and $\langle S\rangle$ and $\langle R\rangle$ are cliques in G and H, respectively. In particular, $\omega(G \times H) = max\{\omega(G), \omega(H)\}$.*

Proof. Suppose $\langle C\rangle$ is complete and let $p = (x, y)$ and $q = (a, b)$ be elements of C. Then either $xa \in E(G)$ and $y = b$ or $x = a$ and $yb \in E(H)$. Suppose $xa \in E(G)$ and $y = b$. Pick any $r = (z, w) \in C$ different from p and q (if such element exists). Suppose $w \neq y$. Since $pr \in E(G \times H)$, it follows that $x = z$ and $wy \in E(H)$. Also, since $qr \in E(G \times H)$ and $w \neq b$, $a = z$ and $wb \in E(H)$. This implies that $a = x$, contrary to our assumption that $xa \in E(G)$. Therefore, $w = y = b$. Hence, by the adjacency in $G \times H$, we have $xw, aw \in E(G)$. This implies that $C = S \times \{b\}$, where $\langle S\rangle$ is a clique of G.

If we assume that $x = a$ and $yb \in E(H)$, then a similar argument can be used to show that $C = \{a\} \times R$, where $\langle R\rangle$ is a clique of H.

The converse is obvious. □

The next result is due to Paluga and Canoy.

Theorem 2.2 [3] *Let G and H be connected noncomplete graphs. Then a proper subset C of $V(G \times H)$ is m-convex in $G \times H$ if and only if $C = S \times R$, where $S \subseteq V(G)$, $R \subseteq V(H)$, and one of $\langle S\rangle$ and $\langle R\rangle$ is complete and the other is K_1.*

By Theorem 2.1, the following remark is immediate.

Remark 2.3 *Let G and H be connected graphs. Then a proper subset C of $V(G \times H)$ is m-convex in $G \times H$ if and only if $\langle C\rangle$ is complete.*

The next result gives the forcing m-convexity number of the cartesian product of two connected non-complete graphs.

Theorem 2.4 *Let G and H be connected non-complete graphs with $\omega(G) \neq \omega(H)$. Then*

$$fcon_m(G \times H) = \begin{cases} fcn(G), & \text{if } \omega(G) > \omega(H) \\ fcn(H), & \text{if } \omega(H) > \omega(G). \end{cases}$$

Proof. Suppose $\omega(G) > \omega(H)$ and C is a maximum m-convex set of $G \times H$ with $fcon_m(G \times H) = fcon_m(C)$. Then $\langle C \rangle$ is complete by Theorem 2.2; hence $fcon_m(G \times H) = fcon_m(C) = fcn(\langle C \rangle)$. Also, $C = S \times \{b\}$, where $\langle S \rangle$ is a maximum clique in G and $b \in V(H)$, by Theorem 2.2. Now, let R be a forcing c-subset of $\langle C \rangle$ such that $fcon_m(C) = |R|$. Then $R_G = \{x \in S : (x, b) \in R\}$ is a forcing c-subset of $\langle S \rangle$. It follows that

$$fcon_m(G \times H) = |R| = |R_G| \geq fcn(\langle S \rangle) \geq fcn(G).$$

Next, let $\langle S' \rangle$ be a maximum clique of G and P a forcing c-subset of $\langle S' \rangle$ such that $fcn(G) = fcn(\langle S' \rangle) = |P|$. Then $C' = S' \times \{b\}$ induces a maximum clique of $G \times H$ by Theorem 2.1. Thus, C' is a maximum m-comvex set of $G \times H$ by Theorem 2.2. Moreover, $Q = P \times \{b\}$ is a forcing c-subset of C'. Consequently, $fcon_m(G \times H) \leq fcon_m(C') \leq |Q| = |P| = fcn(G)$.

Therefore, $fcon_m(G \times H) = fcn(G)$. Using a similar argument we can show that $fcon_m(G \times H) = fcn(H)$ if $\omega(H) > \omega(G)$. □

Theorem 2.5 *Let G and H be connected noncomplete graphs with $\omega(G) = \omega(H)$. Then $fcon_m(G \times H) = 1$ if and only if one of the following conditions hold:*

(a) *There exists $x \in V(G)$ that is not in any maximum clique of G and $fcn(H) \leq 1$.*

(b) *There exists $y \in V(H)$ that is not in any maximum clique of H and $fcn(G) \leq 1$.*

Proof. Suppose $fcon_m(G \times H) = 1$. Let C be a maximum m-convex set with $fcon_m(G \times H) = fcon_m(C) = fcn(\langle C \rangle) = 1$. Then there exists $(x, y) \in C$ that is not in any other maximum clique of $G \times H$. By Theorem 2.2, $C = \{x\} \times R$ or $C = S \times \{y\}$, where $\langle R \rangle$ and $\langle S \rangle$ are maximum cliques of H and G, respectively. If $C = \{x\} \times R$, then $y \in R$. Since $\{(x, y)\}$, is a forcing c-subset of C, $\{y\}$ is a forcing c-subset of $\langle R \rangle$. Hence, $fcn(\langle R \rangle) = 1$

or $fcn(\langle R\rangle) = 0$. This implies that $fcon_m(H) \leq 1$. Suppose now that $x \in S'$ for some maximum clique $\langle S'\rangle$ of G. Then $(x, y) \in S' \times \{y\}$, contrary to our assumption that C is the only maximum m-convex set containing (x, y). It follows that x is not in any maximum clique of G. Therefore, condition (a) holds. A similar argument will show that (b) holds if $C = S \times \{y\}$.

Conversely, suppose that condition (a) holds. Let $\langle R\rangle$ be a maximum clique of H such that $fcn(H) = fcn(\langle R\rangle)$. Since $fcn(H) = 0$ or $fcn(H) = 1$, it follows that there exists $y \in R$ such that y is not in any other maximum clique of H. Hence, $C = \{x\} \times R$ is the only maximum clique of $G \times H$ containing (x, y). It follows that $fcn(\langle C\rangle) = 1$. Therefore, $fcon_m(G \times H) = 1$, by Remark 2.3. Similarly, the assumption that condition (b) holds will yield the same conclusion. □

3 Forcing m-Convexity Number of the Composition of Graphs

Recall that a vertex v of G is an *extreme vertex* if the set $N_G(v)$ (consisting of the neighbors of v or vertices of G adjacent to v) induces a complete subgraph of G. Throughout this section, $Ext(G)$ denotes the set of extreme vertices of G.

To achieve one of our goals in this section, we need the following result.

Lemma 3.1 *Let G be a connected graph and $a \in V(G)$. Then $\langle N_G(a)\rangle$ is complete if and only if $\langle N_{G[K_n]}((a, x))\rangle$ is complete for every $x \in V(K_n)$*

Proof. The assertion is clear for $|V(G)| \leq 2$ and $n \leq 2$. So assume that $|V(G)| \geq 3$ and $n \geq 3$. Suppose $\langle N_G(a)\rangle$ is complete in G and let $x \in V(K_n)$. Let $(b, y), (c, z) \in \langle N_{G[K_n]}((a, x))\rangle$. Then $(a, x)(b, y), (a, x)(c, z) \in E(G[K_n])$. Hence, by definition, $ab \in E(G)$ or $a = b$ and $ac \in E(G)$ or $a = c$. Now, if $a = b$ or $a = c$, then $d(b, c) \leq d(b, a) + d(a, c) \leq 1$, i.e., either $d(b, c) = 0$ or $d(b, c) = 1$. In either case, (b, y) and (c, z) are adjacent in $G[K_n]$. On the other hand, if $ab, ac \in E(G)$ $(b \neq c)$, the completeness of $N_G(a)$ implies that $bc \in E(G)$. It follows that (b, y) and (c, z) are adjacent in $G[K_n]$. Accordingly, $\langle N_{G[K_n]}((a, x))\rangle$ is complete.

Now assume that $\langle N_{G[K_n]}((a, x))\rangle$ is complete in $G[K_n]$ for every $x \in V(K_n)$. Let $b, c \in N_G(a)$, $b \neq c$, and $y, z \in V(K_n)$. It follows that $(b, y), (c, z) \in V(G[K_n])$. Since $ab, ac \in E(G)$, $(a, x)(b, y)$ and $(a, x)(c, z)$ are edges in $G[K_n]$ for $x \in V(K_n)$ and hence, $(b, y), (c, z) \in N_{G[K_n]}((a, x))$. Moreover, since $\langle N_{G[K_n]}((a, x))\rangle$ is complete, $(b, y)(c, z) \in E(G[K_n])$. Thus, either $bc \in E(G)$ or $b = c$. By choice, $b \neq c$, hence $bc \in E(G)$. It follows that $\langle N_G(a)\rangle$ is complete. □

The next result is immediate from the above lemma.

Corollary 3.2 *If G is a connected graph, then*

$$Ext(G[K_n]) = Ext(G) \times V(K_n).$$

The authors earlier obtained the following result.

Theorem 3.3 [2] *Let G be a connected graph with k extreme vertices. Then $fcon_m(G) = k - 1$.*

Theorem 3.4 *If G is a connected graph containing an extreme vertex, then*

$$fcon_m(G[K_n]) = n|Ext(G)| - 1.$$

Proof. By Corollary 3.2, the graph $G[K_n]$ has $|Ext(G[K_n])| = n|Ext(G)|$ extreme vertices. It follows from Theorem 3.3 that

$$\begin{aligned} fcon_m(G[K_n]) &= |Ext(G[K_n])| - 1 \\ &= n|Ext(G)| - 1. \end{aligned}$$

□

We now consider the case $Ext(G) = \varnothing$. The following result from [3] comes in handy.

Theorem 3.5 [3] *Let G be a connected graph of order $p \geq 4$. If G has no extreme vertex, then $con_m(G[K_n]) = ncon_m(G)$. In particular, a subset C of $V(G[K_n])$ is a maximum m-convex set of $G[K_n]$ if and only if $C = S \times V(K_n)$ for some maximum m-convex set S of G.*

We also need the following results.

Lemma 3.6 *Let G be a connected graph of order $p \geq 4$ with no extreme vertex, and C a maximum m-convex set of $G[K_n]$. Then $Q \subseteq C$ is a forcing m-subset of C if and only if Q_G is a forcing m-subset of C_G.*

Proof. Suppose $Q \subseteq C$ is a forcing m-subset of C. By Theorem 3.5, $C = C_G \times V(K_n)$ and C_G is a maximum m-convex set of G. Suppose further that Q_G is not a forcing m-subset of C_G. Then there exists a maximum m-convex set $S \neq C_G$ such that $Q_G \subseteq S$. It follows that Q is contained in the maximum m-convex set $S \times V(K_n)$. This contradicts the assumption that Q is a forcing m-subset of C. Therefore, Q_G is a forcing m-subset of C_G.

Assume that Q_G is a forcing m-subset of C_G. Then there exists no maximum m-convex set $S \neq C_G$ such that $Q_G \subseteq S$. Hence, $C = C_G \times V(K_n)$ is the only maximum m-convex set in $G[K_n]$ containing Q. Therefore, Q is a forcing m-subset of C. □

Lemma 3.7 *Let G be a connected graph of order $p \geq 4$ with no extreme vertices, and C a maximum m-convex set of $G[K_n]$. Then $Q \subseteq C$ is a minimum forcing m-subset of C if and only if Q_G is a minimum forcing m-subset of C_G and $|Q| = |Q_G|$.*

Proof. Suppose $Q \subseteq C$ is a minimum forcing m-subset of C. By Lemma 3.6, Q_G is a forcing m-subset of C_G. It follows that if $w \in V(K_n)$, $Q^* = Q_G \times \{w\}$ is a forcing m-subset of $C = C_G \times V(K_n)$. Clearly, $|Q^*| = |Q_G| \leq |Q|$. Since Q is a minimum forcing m-subset of C, $|Q^*| = |Q_G| = |Q|$.

Now, suppose that Q_G is not a minimum forcing m-subset of C_G. Then there exists $R \subseteq C_G$ such that R is a forcing m-subset of C_G and $|R| < |Q_G|$. Let $x \in V(K_n)$. Then $R \times \{x\}$ is a forcing m-subset of $C_G \times V(K_n) = C$. Moreover, $|R \times \{x\}| = |R| < |Q_G| = |Q|$. This contradicts the minimality of Q. Therefore, Q_G is a minimum forcing m-subset of C_G.

Conversely, suppose Q_G is a minimum forcing m-subset of C_G and $|Q| = |Q_G|$. By Lemma 3.6, Q is a forcing m-subset of $C = C_G \times V(K_n)$. Suppose Q is not a minimum forcing m-subset of C. Then there exists $Q^* \subseteq C$ such that $fcon_m(C) = |Q^*| < |Q|$. Hence, Q^*_G is a forcing m-subset of C_G and $|Q^*| = |Q^*_G|$. It follows that $|Q^*_G| < |Q_G|$, contrary to our assumption that Q_G is a minimum forcing m-subset of C_G. Therefore Q is a minimum forcing m-subset of C. □

Theorem 3.8 *Let G be a connected graph with no extreme vertex. Then*

$$fcon_m(G[K_n]) = fcon_m(G).$$

Proof. Let C be a maximum m-convex set of $G[K_n]$ such that

$$fcon_m(G[K_n]) = fcon_m(C)$$

and let Q be a forcing m-subset of C such that $fcon_m(C) = |Q|$. Then Q_G is a minimum forcing m-subset of C_G and, by Theorem 3.7,

$$fcon_m(G[K_n]) = |Q| = |Q_G| = fcon_m(C_G) \geq fcon_m(G).$$

On the other hand, if S is a maximum m-convex set of $G[K_n]$ such that $fcon_m(G) = fcon_m(S)$ and R is a minimum forcing m-subset of S such that $fcon_m(S) = |R|$, then $Q' = R \times \{x\}$ is a minimum forcing m-subset of $S \times V(K_n) = C$ by Theorem 3.7. It follows that

$$fcon_m(G[K_n]) \leq fcon_m(C) = |Q'| = |R| = fcon_m(S) = fcon_m(G).$$

Therefore, $fcon_m\,(G[K_n]) = fcon_m(G)$. □

Our next goal is to determine $fcon_m(G[H])$, where G is a connected graph and H is a connected non-complete graph. The following results will be useful.

Theorem 3.9 [3] *Let G be a connected graph of order $n > 1$ and H a connected non-complete graph. Then C is a proper m-convex set of $G[H]$ if and only if C induces a complete subgraph of $G[H]$.*

Theorem 3.10 [3] *Let G be a connected graph of order $n > 1$ and H a connected non-complete graph. Then $con_m(G[H]) = \omega(G[H])$.*

The preceding two results, however, do not describe nor give the possible forms of the maximum cliques of $G[H]$. The next results describe the cliques of the composition of two graphs.

Theorem 3.11 *Let G and H be graphs. A subset C of $V(G[H])$ induces a clique of $G[H]$ if and only if $C = \cup_{s\in S}(\{s\} \times T_s)$, where $\langle S\rangle$ is a clique of G and $\langle T_s\rangle$ is a clique of H for every $s \in S$.*

Proof. Suppose C induces a clique of $G[H]$. Clearly, $C = \cup_{s\in S}(\{s\} \times T_s)$, where $S \subseteq V(G)$ and $T_s \subseteq V(H)$ for every $s \in S$. If S is a singleton, then $\langle S\rangle$ is a clique of G. Suppose S is not a singleton. Let $u, v \in S$ with $u \neq v$ and $a, b \in V(H)$ such that $(u, a), (v, b) \in C$. Since $\langle C\rangle$ is a clique, $(u, a)(v, b) \in E(G[H])$. By the adjacency in $G[H]$, it follows that $uv \in E(G)$. Thus, $\langle S\rangle$ is a clique of G.

Next, let $s \in S$. If T_s is a singleton set, then it induces a clique of H. Suppose it is not a singleton. Let $t_1, t_2 \in T_s$. Then $(s, t_1), (s, t_2) \in C$; hence $(s, t_1)(s, t_2) \in E(G[H])$. By the adjacency in $G[H]$, it follows that $t_1t_2 \in E(H)$. Thus, $\langle T_s\rangle$ is a clique of G.

For the converse, suppose $C = \cup_{s\in S}(\{s\} \times T_s)$, where $\langle S\rangle$ is a clique of G and $\langle T_s\rangle$ is a clique of H for every $s \in S$. Let (a, b) and (x, y) be distinct elements of C. If $a = x$, then $b, y \in T_a$. By assumption, $by \in E(H)$. Thus, by the adjacency in $G[H]$, it follows that $(a, b)(x, y) \in E(G[H])$. If $a \neq x$, then $ax \in E(G)$ by assumption that $\langle S\rangle$ is a clique of G. Hence, $(a, b)(x, y) \in E(G[H])$. Therefore, $\langle C\rangle$ is a clique of $G[H]$. □

Theorem 3.12 *Let G and H be graphs. A subset C of $V(G[H])$ is a maximum clique of $G[H]$ if and only if $C = \cup_{s\in S}(\{s\} \times T_s)$, where $\langle S\rangle$ is a maximum clique of G and $\langle T_s\rangle$ is a maximum clique of H for every $s \in S$. In particular, $\omega(G[H]) = \omega(G)\omega(H)$.*

Proof. Let C be a maximum clique of $G[H]$. Then $C = \cup_{s\in S}(\{s\} \times T_s)$, where $\langle S\rangle$ is a clique of G and $\langle T_s\rangle$ is a clique of H for every $s \in S$, by Theorem 3.11. Let s_1 and s' be two distinct elements of S. Suppose $|T_{s_1}| \neq |T_{s'}|$, say $|T_{s_1}| > |T_{s'}|$. Consider the set $C^* = \cup_{s\neq s'}(\{s\} \times T_s) \cup (\{s'\} \times T_{s_1})$. Then C^* induces a clique of $G[H]$ by Theorem 3.11. This, however, contradicts the maximality of C because $|C^*| > |C|$. Therefore, $|T_{s_1}| = |T_{s'}|$. Hence, for $s \in S$, we have $|C| = |S||T_s| \leq \omega(G)\omega(H)$.

Now, let S' and T' induce maximum cliques of G and H, respectively. Then clearly, $S' \times T'$ induces a clique of $G[H]$. Since $\langle C\rangle$ is a maximum clique of $G[H]$, it follows that $|C| \geq |S' \times T'| = \omega(G)\omega(H)$. Therefore, $\omega(G[H]) = |C| = \omega(G)\omega(H)$. Consequently, S induces a maximum clique of G and T_s induces a maximum clique of H for each $s \in S$.

The converse also follows from Theorem 3.11. □

The following is a quick consequence of Theorem 3.10 and Theorem 3.12.

Corollary 3.13 *Let G be a connected graph of order $n > 1$ and H a connected non-complete graph. Then $con_m(G[H]) = \omega(G)\omega(H)$.*

Theorem 3.14 *Let G be a connected graph of order $n > 1$ and H a connected non-complete graph with $fcn(H) = 0$. Then $fcon_m(G[H]) = fcn(G[H]) = fcn(G)$.*

Proof. Let C be a maximum m-convex set of $G[H]$ such that $fcon_m(G[H]) = fcon_m(C)$. Then $C = \cup_{s\in S}(\{s\} \times T_s)$, where $\langle S\rangle$ is a maximum clique of G and $\langle T_s\rangle$ is a maximum clique of H for every $s \in S$, by Theorem 3.12. Since $fcn(H) = 0$, H has a unique maximum clique, say T. It follows that $T = T_s$ for all $s \in S$ and $C = S \times T$. Let Q be a forcing m-subset of C such that $fcon_m(C) = |Q|$. Then, clearly, Q_G is a forcing c-subset of S. Hence, $Q^* = Q_G \times \{t\}$, where $t \in T$, is a forcing m-subset of C, and $|Q^*| = |Q_G| \leq |Q|$. By our assumption of Q, we must have $|Q| = |Q_G|$. Thus $fcon_m(G[H]) = |Q_G| \geq fcn(G)$.

On the other hand, if S is maximum clique of G and R is a forcing c-subset of S such that $fcn(G) = fcn(\langle S\rangle) = |R|$, then $W = R \times \{t\}$, where $t \in T$, is a forcing m-subset of $C = S \times T$. Hence, $fcn(G) = |R| = |W| \geq fcon_m(C) = fcon_m(G[H])$. Therefore, $fcon_m(G[H]) = fcn(G)$. By Remark 1.6, we have the desired result. □

Theorem 3.15 *Let G be a connected graph of order $n > 1$ and H a connected non-complete graph with $fcn(H) \neq 0$. Then*

$$fcon_m(G[H]) = fcn(G[H]) \leq \omega(G)fcn(H).$$

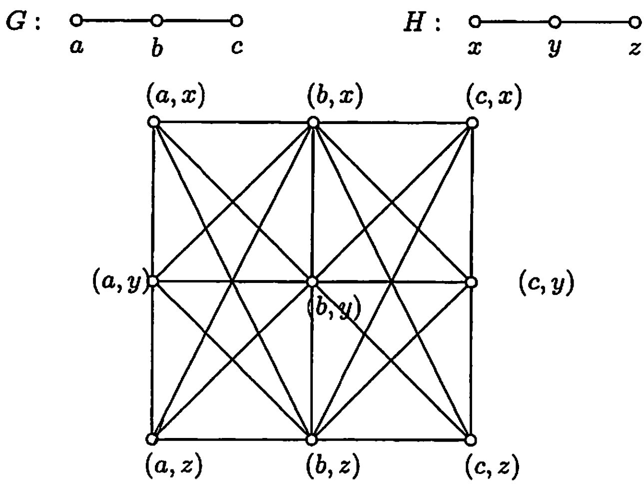
Proof. Let $\langle S\rangle$ and $\langle T\rangle$ be maximum cliques of G and H, respectively, such that $fcn(H) = fcn(\langle T\rangle)$. For each $s \in S$, define $T_s = T$. Then $C = \cup_{s\in S}(\{s\} \times T_s)$ is a maximum clique (hence, a maximum m-convex set) of $G[H]$ by Theorem 3.12.

Next, let V be a forcing c-subset of T such that $fcn(\langle T\rangle) = |V|$. By assumption, $V \neq \emptyset$. Consider the set $Q = S \times V$. Then $Q_G = S$, $Q_H = V$ and $Q \subseteq C$. Suppose $Q \subseteq C^*$, where C^* is a maximum clique of $G[H]$ different from C. By Theorem 3.12, $C^* = \cup_{p\in S'}(\{p\} \times T'_p)$, where S' and T'_p ($p \in S'$) induce maximum cliques of G and H, respectively. Hence, $Q_G = S = S'$ and $V \subseteq W = \cup_{p\in S'}T'_p$. Since $C^* \neq C$, there exists an $x \in S$ such that $T'_x \neq T$. Suppose $V \nsubseteq T'_x$. Then there exists $y \in V$ such that $y \notin T'_x$. Hence, $(x, y) \notin \{x\} \times T'_x$. Since $(x, y) \notin \{p\} \times T'_p$ for all $p \in S' \setminus \{x\}$, it follows that $(x, y) \notin C^*$, contrary to the fact that $Q \subseteq C^*$. Therefore, $V \subseteq T'_x$. This is not possible because V is a minimum c-forcing subset of T. Hence, C is the only maximum clique in $G[H]$ containing Q , i.e., Q is a forcing subset of C. Therefore,

$$fcon_m(G[H]) \leq fcon(C) \leq |Q| = |S||V| = \omega(G)fcn(H).$$

This completes the proof. □

Example 3.16 Consider the composition $G[H]$, where G and H are paths with $V(G) = \{a, b, c\}$ and $V(H) = \{x, y, z\}$, respectively.



$G[H]$

The set $A = \{(a,x),(b,x),(a,y),(b,y)\}$ is a maximum m-convex set and the set $B = \{(a,x),(a,y)\}$ is a forcing m-subset of A. Observe that no singleton is a forcing m-subset of any maximum m-convex set since each singleton is contained in at least two maximum m-convex sets. Thus, $fcon_m(G[H]) = 2$. On the other hand, $\omega(G)fcn(H) = 2(1) = 2$. This shows that the upper bound given in Theorem 3.15 is sharp. We leave to interested readers to verify whether or not equality in Theorem 3.15 does hold.

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