Equitable Coloring and Equitable Choosability of Planar Graphs without 6- and 7-Cycles *

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Abstract

For any given k-uniform list assignment L, a graph G is equitably k-choosable, if and only if G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. A graph G is equitable k-colorable if G has a proper vertex coloring with k colors such that the size of the color classes differ by at most 1. In this paper, we prove that every planar graph G without 6- and 7-cycles is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 6\}$.

Keywords: Equitable choosability; Planar graph; Degenerate
MSC(2000): 05C15

1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (face) x in G. A vertex (face) x is called a k-*vertex* (k-*face*), k^+-*vertex* (k^+-*face*), k^--vertex or k^{--}-vertex, if $d(x) = k$, $d(x) \geq k$, $2 \leq d(x) \leq k$ or $1 \leq d(x) \leq k$. We use $(d_1, d_2, \cdots, d_n)$ to denote a face f if $d_1, d_2, \cdots, d_n$ are the degree of vertices incident to the face f. Let $\delta(f)$ denote the minimal degree of vertices incident to f. A graph G is k-*degenerate* if every subgraph has a vertex of degree at most k.

A graph $G = (V, E)$ is said to be equitably k-colorable if the vertex set $V(G)$ can be partitioned into k independent subsets V_1, V_2, $\cdots$, V_k such that $||V_i| - |V_j|| \leq 1$ $(1 \leq i, j \leq k)$. The equitable chromatic number of

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G, denoted by $\chi_e(G)$, is the smallest integer k such that G is equitably k-colorable. The equitable chromatic threshold of G, denoted by $\chi_e^*(G)$, is the smallest integer k such that G is equitably l-colorable ($l \geq k$). It is obvious that $\chi_e(G) \leq \chi_e^*(G)$ for any graph G. They might not be equal. For example, if $K_{2n+1,2n+1}$ (n is a positive integer) is a complete bipartite graph, then $\chi_e(K_{2n+1,2n+1}) = 2$, $\chi_e^*(K_{2n+1,2n+1}) = 2n+2$.

In many application of graph coloring, it is desirable that the color classes are not too large. For example, when using a coloring model to find an optimal final exam schedule, one would like to have approximately equal number of final exams in each time slot because the whole exam period should be as short as possible and the number of classrooms available is limited. Recently, Pemmaraju [13] and Janson and Ruciński [6] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence. In all of these applications, the fewer colors we use, the better the deviation bound is. Equitable coloring has a well-known property that restricts the size of each color class by its definition.

In 1970, Hajnál and Szemerédi proved that $\chi_e^*(G) \leq \Delta(G) + 1$ for any graph G [5]. This bound is sharp as shows the example of $K_{2n+1,2n+1}$. In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture [11]:

Conjecture 1 *If G is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.*

In 1994, Chen et al. put forth the following conjecture [2]:

Conjecture 2 *For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.*

Chen et al. proved the conjecture for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq \frac{|V(G)|}{2}$ or a tree [2, 3]. Yap and Zhang proved that the conjecture holds for outer planar graphs and planar graphs with $\Delta(G) \geq 13$ [17, 18]. Lih and Wu verified $\chi_e^*(G) \leq \Delta(G)$ for bipartite graphs other than a complete bipartite graph [9]. Wang et al. proved the conjecture for line graphs [16]. It follows from [8] that the conjecture hold for d-degenerate graphs with maximum degree $\Delta(G) \geq 14d + 1$.

For a graph G and a list assignment L assigned to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, a L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$. A graph G is equitably k-choosble if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the equitable list coloring of graphs. They proposed the following conjecture[7].

Conjecture 3 *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 4 *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It has been proved that Conjecture 3 holds for graphs with $\Delta(G) \leq 3$ independently in [12, 14]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in the later case) and $k \geq \max\{\Delta, \frac{|V(G)|}{2}\}$, or G is a connected inteval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ [7]. Pelsmajer proved that every graph is equitably k-choosable for any $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$ [12]. There are several results for planar graphs without short cycles [10, 19].

In this paper, we show that every planar graph G without 6- and 7-cycles is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 6\}$.

2 Planar graphs without 6- and 7-cycles

First let us introduce some important lemma.

Lemma 2.1 *([4]) Every planar graph without 6-cycles is 3-degenerate.*

Corollary 2.2 *If G is a planar graph without 6- and 7-cycles, then $\delta(G) \leq 3$.*

Lemma 2.3 *([5]) Every graph has an equitable k-coloring whenever $k \geq \Delta(G) + 1$.*

Lemma 2.4 *([19]) Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of k different vertices in G such that $G - S$ has an equitable k-coloring, if $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

Lemma 2.5 *([12, 14]) Every graph G with maximum degree $\Delta(G) \leq 3$ is equitably k-choosable whenever $k \geq \Delta(G) + 1$.*

Lemma 2.6 *([7]) Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \cdots, v_k\}$, where $\{v_1, v_2, \cdots, v_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring and $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

Lemma 2.7 *Let G be a connected planar graph with order at least 5 and without 6- and 7-cycles, then G has at least one of the following structures in Figure 1.*

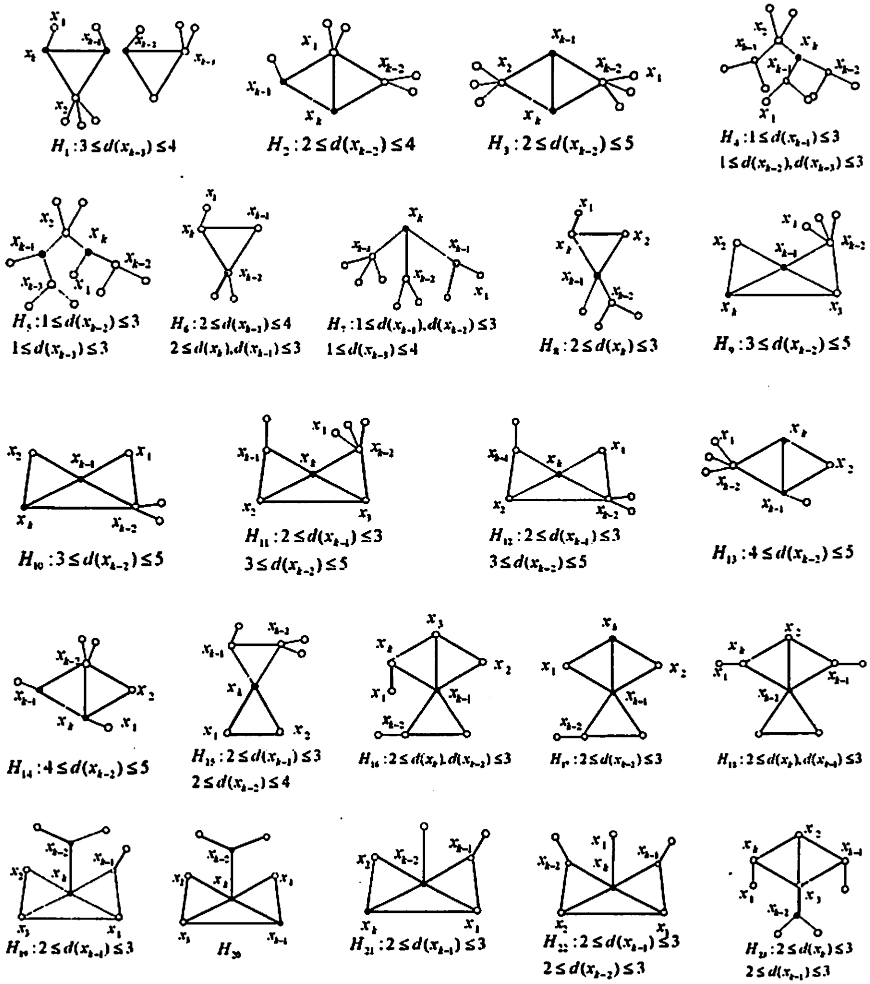
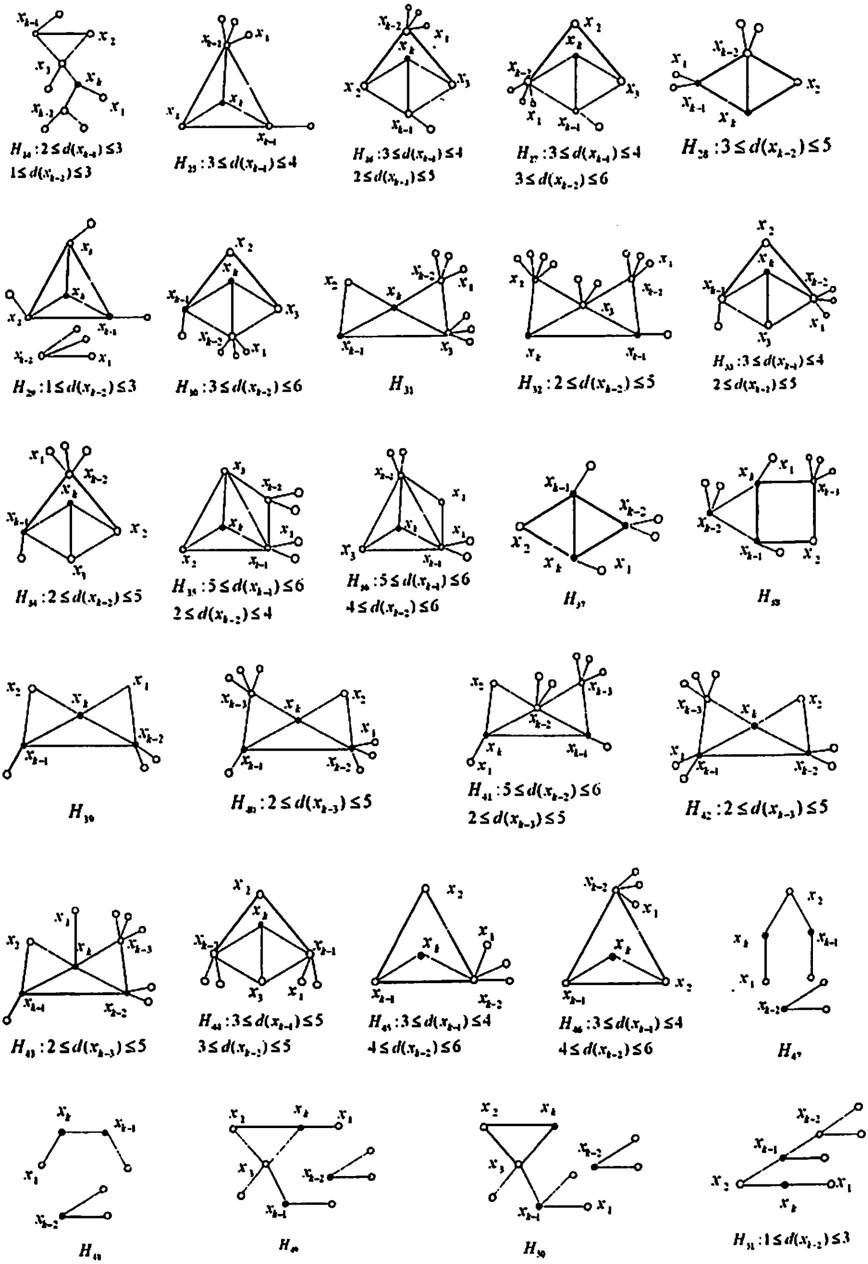


Figure 1



H24 : 2 ≤ d(x_{k-1}) ≤ 3
1 ≤ d(x_{k-2}) ≤ 3
H25 : 3 ≤ d(x_{k-1}) ≤ 4
H26 : 3 ≤ d(x_{k-1}) ≤ 4
2 ≤ d(x_{k-2}) ≤ 5
H27 : 3 ≤ d(x_{k-1}) ≤ 4
3 ≤ d(x_{k-2}) ≤ 6
H28 : 3 ≤ d(x_{k-2}) ≤ 5
H29 : 1 ≤ d(x_{k-2}) ≤ 3
H30 : 3 ≤ d(x_{k-2}) ≤ 6
H31
H32 : 2 ≤ d(x_{k-2}) ≤ 5
H33 : 3 ≤ d(x_{k-1}) ≤ 4
2 ≤ d(x_{k-2}) ≤ 5
H34 : 2 ≤ d(x_{k-2}) ≤ 5
H35 : 5 ≤ d(x_{k-1}) ≤ 6
2 ≤ d(x_{k-2}) ≤ 4
H36 : 5 ≤ d(x_{k-1}) ≤ 6
4 ≤ d(x_{k-2}) ≤ 6
H37
H38
H39
H40 : 2 ≤ d(x_{k-3}) ≤ 5
H41 : 5 ≤ d(x_{k-2}) ≤ 6
2 ≤ d(x_{k-3}) ≤ 5
H42 : 2 ≤ d(x_{k-3}) ≤ 5
H43 : 2 ≤ d(x_{k-3}) ≤ 5
H44 : 3 ≤ d(x_{k-1}) ≤ 5
3 ≤ d(x_{k-2}) ≤ 5
H45 : 3 ≤ d(x_{k-1}) ≤ 4
4 ≤ d(x_{k-2}) ≤ 6
H46 : 3 ≤ d(x_{k-1}) ≤ 4
4 ≤ d(x_{k-2}) ≤ 6
H47
H48
H49
H50
H51 : 1 ≤ d(x_{k-2}) ≤ 3

Figure 1

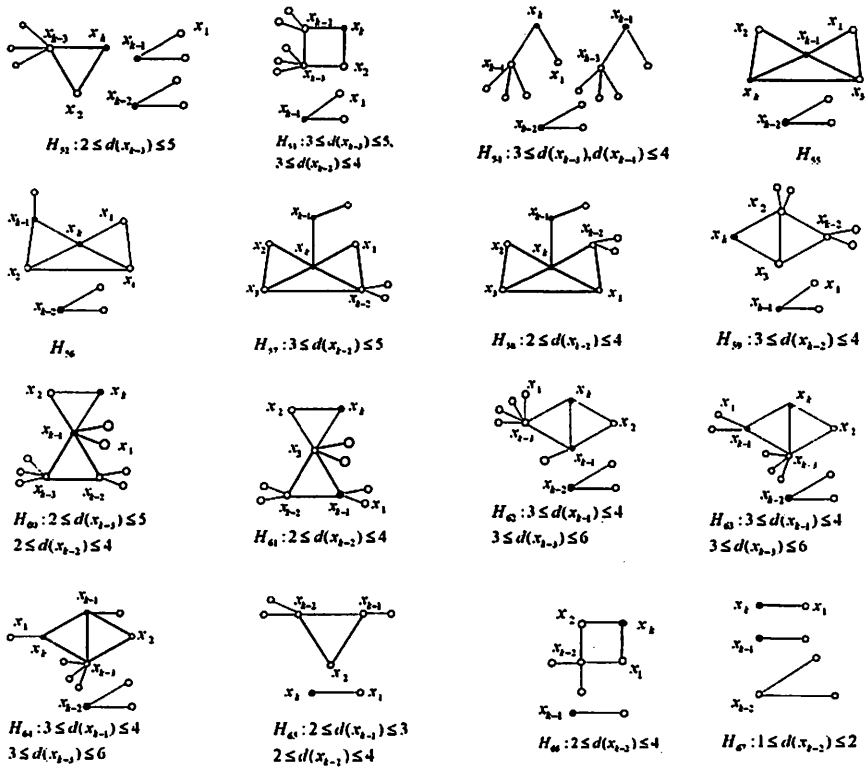


Figure 1

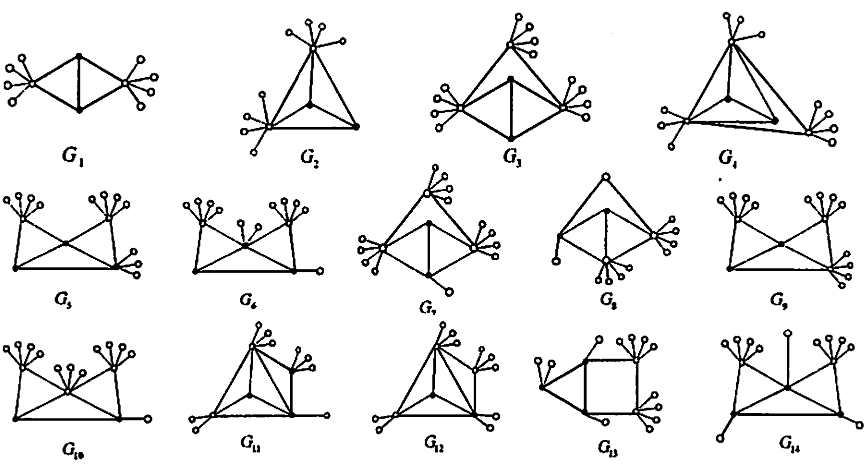


Figure 2

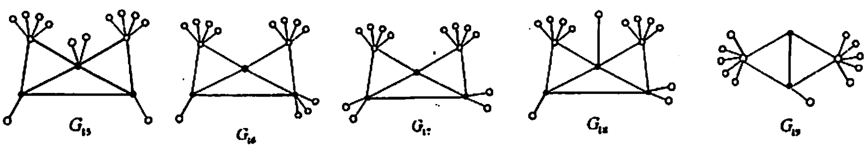


Figure 2

Each configuration in Figure 1 and Figure 2 represents subgraphs for which: (1)hollow vertices may be not distinct while solid vertices are distinct. (2)the degree of the solid vertex is fixed, and (3)except for special pointed, the degree of a hollow vertex may be any integer from $[d, \Delta(G)]$, where d is the number of edges incident to the hollow vertex in the configuration. (4) the order of the vertices on the boundary of a 4-face can be exchanged if the vertices are not common vertices of the 4-face and other face.

Proof. Let G be a minimal counterexample on the number of vertices. Then G without 6- and 7-cycles does not contain $H_1 \sim H_{67}$ in Figure 1.

For G contains no structures $H_1 \sim H_3$, we can obtain the following property.

Claim 1 Any two of $(3,3,5^+)$-faces can not simultaneously appear in G except that the structure G_1 (in Figure 2).

We call a face a *special face* if it belongs to $(3,3,5^+)$-faces. In the following, we call a 3-vertex a *special 3-vertex* if it is incident to a special face, otherwise, it is called a *simple 3-vertex*. For convenience, let $n_3(v)$, $m_3(v)$ and $m_4(v)$ denote the number of simple 3-vertices adjacent to v, the number of 3-faces incident to v and the number of 4-faces incident to v for each $v \in V(G)$ respectively. Let $n_i(f)$ denote the number of i-vertices incident to f.

Since G contains no structures H_4 and H_5, we can conclude the following properties.

Claim 2 For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 3-vertex which is adjacent to two 3^{--}-vertices, then it is not adjacent to other 3-vertex.

Claim 3 For any $v \in V(G)$, v is adjacent to at most one simple 3-vertex which is adjacent to other 3^{--}-vertex.

By Euler's formula $|V|-|E|+|F| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E|$, we have

$$\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -10(|V| - |E| + |F|) = -20.$$

Define an initial charge function w on $V(G) \cup F(G)$ by setting $w(v) = 3d(v) - 10$ if $v \in V(G)$ and $w(f) = 2d(f) - 10$ if $f \in F(G)$, so that $\sum_{x \in V(G) \cup F(G)} W(x) = -20$.

We divide the proof into the following four cases by Corollary 2.2.

Case 1 $\delta(G) = 3$.

For G contains no structure H_6. G has the following properties.

Claim 1.1 All 3-faces in G are $(3,3,5^+)$-, $(3,4^+,4^+)$- or $(4^+,4^+,4^+)$-faces.

Now redistribute the charge according to the following discharging rules.

*R*1 Transfer charge 1 from each 5^+-vertex to every adjacent simple 3-vertex v which is adjacent to exactly two 3^{--}-vertices.

*R*2 Transfer charge $\frac{1}{2}$ from each 4^+-vertex to every adjacent simple 3-vertex v which is adjacent to exactly one 3^{--}-vertex.

*R*3 Transfer charge $\frac{1}{3}$ from each 4^+-vertex to every adjacent simple 3-vertex v which is not adjacent to any 3^{--}-vertex.

*R*4 Transfer charge $\frac{3}{4}$ from each 8^+-face f to every adjacent 3-face and 4-face via each common edge.

*R*5 If f is a 4-face incident to v, then v gives f charge $\frac{2}{3}$ if $d(v) \geq 6$.

*R*6 If f is a 3-face incident to a vertex v, then v gives f charge $\frac{3}{2}$ if $d(v) = 4$ and f is a $(3^-,4,4)$-face, $\frac{5}{6}$ if $d(v) = 4$ and f is a $(3^-,4,5^+)$-face, $\frac{2}{3}$ if $d(v) = 4$ and f is a $(4,4,4^+)$- or $(4,5,5^+)$-face, $\frac{7}{4}$ if $d(v) = 5$ and f is a $(3^-,3^+,5)$-face, $\frac{7}{6}$ if $d(v) = 5$ and f is an other face, $\frac{7}{4}$ if $d(v) = 6$, 2 if $d(v) \geq 7$.

Let the new charge of each element x be $w'(x)$ for each $x \in V(G) \cup F(G)$. Particularly, we use w'_s denotes the total new charge of all the special 3-vertices and all the special faces in G.

Case 1.1 $\delta(G) = 3$ and there are at least two 3-vertices in G.

Now let us check the charge of each element $x \in V(G) \cup F(G)$.

Suppose $d(v) = 3$. Then $w(v) = -1$. Since G contains no structure H_7, v is adjacent to at least one 5^+-vertex or is adjacent to at least two 4^+-vertex. If v is a simple 3-vertex, we have $w'(v) = -1 + 1 = 0$ by *R*1, $w'(v) = -1 + \frac{1}{2} \times 2 = 0$ by *R*2 or $w'(v) = -1 + \frac{1}{3} \times 3 = 0$ by *R*3. Otherwise, we have $w'(v) = w(v) = -1$.

Suppose $d(v) = 4$. Then $w(v) = 2$ and $m_3(v) \leq 3$.

Case 1.1.1 $m_3(v) = 3$, then $n_3(v) \leq 1$ for G contains no structure H_8. If $m_3(v) = 3$, $n_3(v) = 1$ and v is incident to two 3-faces each of which is incident to the 3-vertex, then we have $w'(v) \geq 2 - \frac{5}{6} \times 2 - \frac{1}{3} = 0$ for G contains no structures H_9, H_{10} and by *R*6, *R*3. If $m_3(v) = 3$, $n_3(v) = 1$ and v is incident to only one 3-face which is incident to a 3-vertex, then we have $w'(v) \geq 2 - \frac{5}{6} - \frac{1}{2} = \frac{2}{3} > 0$ for G contains no structures H_{11}, H_{12} and by *R*6, *R*2. If $m_3(v) = 3$, $n_3(v) = 0$, then we have $w'(v) \geq 2 - \frac{2}{3} \times 3 = 0$ by *R*6.

Case 1.1.2 $m_3(v) = 2$, then $n_3(v) \leq 1$ for G contains no structure H_8. If $m_3(v) = 2$, $n_3(v) = 1$ and v is incident to two 3-faces each of which is incident to the 3-vertex, then we have $w'(v) \geq 2 - \frac{5}{6} \times 2 - \frac{1}{3} = 0$ for G contains no structure H_{13} and by *R*6, *R*3. If $m_3(v) = 2$, $n_3(v) = 1$ and v is incident to only one 3-face which is incident to the 3-vertex, then we have $w'(v) \geq 2 - \frac{5}{6} - \frac{2}{3} - \frac{1}{2} = 0$ for G contains no structures H_{14}, H_{15} and by $R6$, $R2$. If $m_3(v) = 2$, $n_3(v) = 1$ and v is not incident to any 3-face which is incident to a 3-vertex, then we have $w'(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{2} = \frac{1}{6} > 0$ for G contains no structure H_7 and by $R6$, $R2$. If $m_3(v) = 2$, $n_3(v) = 0$, then we have $w'(v) \geq 2 - \frac{2}{3} \times 2 = \frac{2}{3} > 0$ by $R6$.

Case 1.1.3 $m_3(v) = 1$, then $n_3(v) \leq 2$ for G contains no structure H_8. If $m_3(v) = 1$, $n_3(v) = 2$, then we have $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2} > 0$ for G contains no structure H_8 and by $R6$, $R2$, $R3$ and Claim 3. If $m_3(v) = 1$, $n_3(v) \leq 1$, then $w'(v) \geq 2 - \frac{3}{2} - \frac{1}{2} = 0$ by $R6$ and $R2$.

Case 1.1.4 $m_3(v) = 0$, then $n_3(v) \leq 4$. So $w'(v) \geq 2 - \frac{1}{2} - \frac{1}{3} \times 3 = \frac{1}{2} > 0$ by $R2$, $R3$ and Claim 3.

Suppose $d(v) = 5$. Then $w'(v) = 5$, $m_3(v) \leq 3$.

Case 1.2.1 $m_3(v) = 3$, then $n_3(v) \leq 1$ for G contains no structures $H_{16} \sim H_{22}$. If $m_3(v) = 3$, $n_3(v) = 1$ and v is incident to two 3-faces each of which is incident to the 3-vertex, then $w'(v) \geq 5 - \frac{7}{4} \times 2 - \frac{7}{6} - \frac{1}{3} = 0$ by $R6$ and $R3$. If $m_3(v) = 3$, $n_3(v) = 1$ and v is incident to only one 3-face which is incident to the 3-vertex, then $w'(v) \geq 5 - \frac{7}{4} - \frac{7}{6} \times 2 - \frac{1}{2} = \frac{5}{12} > 0$ by $R6$ and $R2$. If $m_3(v) = 3$, $n_3(v) = 1$ and v is not incident to any 3-face which is incident to a 3-vertex, then $w'(v) \geq 5 - \frac{7}{6} \times 3 - 1 = \frac{1}{2} > 0$ by $R6$ and $R1$. If $m_3(v) = 3$, $n_3(v) = 0$, then we have $w'(v) \geq 5 - \frac{7}{6} \times 3 = \frac{3}{2} > 0$ by $R6$.

Case 1.2.2 $m_3(v) = 2$, then $n_3(v) \leq 4$. So $w'(v) \geq 5 - \frac{7}{4} \times 2 - \frac{1}{3} \times 3 - \frac{1}{2} = 0$ by $R6$, $R3$, $R2$ and Claim 4.

Case 1.2.3 $m_3(v) \leq 1$, then $n_3(v) \leq 5$. So $w'(v) > 5 - \frac{7}{4} - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{17}{12} > 0$ by $R6$, $R3$, $R2$ and Claim 4.

Suppose $d(v) = 6$. Then $w(v) = 8$, $m_3(v) \leq 4$.

If $m_3(v) = 4$, then $m_4(v) = 0$, $n_3(v) \leq 3$ for G contains no structure H_{23}. So $w'(v) \geq 8 - \frac{7}{4} \times 4 - \frac{1}{3} \times 3 = 0$ for G contains no structure H_{24} and by $R6$, $R3$. If $m_3(v) = 3$, then $m_4(v) \leq 1$, $n_3(v) \leq 4$. So $w'(v) \geq 8 - \frac{7}{4} \times 3 - \frac{2}{3} - \frac{1}{3} \times 4 = \frac{3}{4} > 0$ by $R6$, $R5$ and $R3$. If $m_3(v) = 2$, then $m_4(v) \leq 2$, $n_3(v) \leq 5$. We have $w'(v) \geq 8 - \frac{7}{4} \times 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{4}{3} > 0$ by $R6$, $R5$, $R3$ and $R2$. If $m_3(v) \leq 1$, then $m_4(v) \leq 6$, $n_3(v) \leq 6$. We have $w'(v) > 8 - \frac{7}{4} - \frac{2}{3} \times 6 - \frac{1}{3} \times 5 - \frac{1}{2} = \frac{1}{12} > 0$ by $R6$, $R5$, $R3$ and $R2$.

Suppose $d(v) = 7$. Then $w(v) = 11$, $m_3(v) \leq 5$.

If $m_3(v) = 5$, then $m_4(v) = 0$, $n_3(v) \leq 3$ for G contains no structure H_{23}. We have $w'(v) \geq 11 - 2 \times 5 - \frac{1}{3} \times 3 = 0$ for G contains no structure H_{24} and by $R6$, $R3$. If $m_3(v) = 4$, then $m_4(v) \leq 1$, $n_3(v) \leq 4$ for G contains no structure H_{23}. So $w'(v) > 11 - 2 \times 4 - \frac{2}{3} - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{5}{6} > 0$ by $R6$, $R5$, $R3$ and $R2$. If $m_3(v) = 3$, then $m_4(v) \leq 2$, $n_3(v) \leq 5$. So $w'(v) > 11 - 2 \times 3 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{11}{6} > 0$ by $R6$, $R5$, $R3$ and $R2$. If $m_3(v) = 2$, then $m_4(v) \leq 2$, $n_3(v) \leq 6$. So $w'(v) > 11 - 2 \times 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 6 = \frac{11}{3} > 0$ by $R6$, $R5$ and $R3$. If $m_3(v) \leq 1$, then $m_4(v) \leq 7$, $n_3(v) \leq 7$. So

$w'(v) > 11 - 2 - \frac{2}{3} \times 7 - \frac{1}{3} \times 7 = 2 > 0$ by $R6$, $R5$ and $R3$.

Suppose $d(v) = 8$. Then $w(v) = 14$, $m_3(v) \leq 6$.

If $m_3(v) = 6$, then $m_4(v) = 0$, $n_3(v) \leq 4$ for G contains no structure H_{23}. So $w'(v) \geq 14 - 2 \times 6 - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{1}{2} > 0$ by $R6$, $R3$, $R2$ and Claim 3. If $m_3(v) = 5$, then $m_4(v) = 0$, $n_3(v) \leq 4$ for G contains no structure H_{23}. So $w'(v) > 14 - 2 \times 5 - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{5}{2} > 0$ by $R6$, $R3$ and $R2$. If $m_3(v) = 4$, then $m_4(v) \leq 2$, $n_3(v) \leq 5$ for G contains no structure H_{23}. So $w'(v) > 14 - 2 \times 4 - \frac{2}{3} \times 2 - \frac{1}{3} \times 5 = 3 > 0$ by $R6$, $R5$, $R3$. If $m_3(v) = 3$, then $m_4(v) \leq 2$, $n_3(v) < 8$. We have $w'(v) > 14 - 2 \times 3 - \frac{2}{3} \times 2 - \frac{1}{3} \times 8 = 4 > 0$ by $R6$, $R5$, $R3$. If $m_3(v) \leq 2$, then $m_4(v) \leq 8$, $n_3(v) \leq 8$. We have $w'(v) > 14 - 2 \times 2 - \frac{2}{3} \times 8 - \frac{1}{3} \times 8 = 2 > 0$ by $R6$, $R5$, $R3$.

Suppose $d(v) \geq 9$. Then $w(v) = 3d(v) - 10$. Since $n_3(v) \leq m_3(v) + d(v) - \frac{4}{3}m_3(v)$, we have

$$n_3(v) \leq d(v) - \frac{1}{3}m_3(v).$$

And since $m_4(v) \leq d(v) - \frac{4}{3}m_3(v)$, we have $w'(v) \geq 3d(v) - 10 - 2m_3(v) - \frac{2}{3} \times m_4(v) - \frac{1}{2} - \frac{1}{3}(n_3(v) - 1) \geq 3d(v) - 10 - 2m_3(v) - \frac{2}{3} \times (d(v) - \frac{4}{3}m_3(v)) - \frac{1}{2} - \frac{1}{3}(d(v) - \frac{1}{3}m_3(v) - 1) = 2d(v) - m_3(v) - \frac{61}{6}$ by $R6$, $R5$, $R2$, $R3$ and Claim 4. And since

$$m_3(v) \leq \frac{3}{4}d(v).$$

We have $w'(v) \geq \frac{9}{8}d(v) - \frac{61}{6} \geq \frac{13}{12} > 0$.

Suppose $d(f) = 3$. Then $w(f) = -4$, $n_3(f) \leq 2$ by Claim 1.1.

Case 1.3.1 $n_3(f) = 2$, then $n_4(f) = 0$, f is a special face $(3, 3, 5^+)$-face.

Case 1.3.1.1 f is not adjacent to any special 3-face, we have $w'(f) \geq -4 + \frac{7}{4} = -\frac{9}{4}$ by $R6$.

Case 1.3.1.2 f is adjacent to a special 3-faces, i.e. the cluster G_1 in Figure 2 for G contains no structures H_2 and H_3. If G_1 is neither adjacent to a 3-face nor adjacent to a 4-face, then we have $w'(G_1) \geq -4 \times 2 + \frac{7}{4} \times 2 + \frac{3}{4} \times 4 = -\frac{3}{2} >$ by $R6$ and $R4$. If G_1 is adjacent to a 3-face and not adjacent to a 4-face, we consider cluster G_2 in Figure 2 for G contains no structure H_{25}, we have $w'(G_2) \geq -4 \times 3 + 2 \times 4 + \frac{3}{4} \times 2 = -\frac{5}{2}$ by $R6$ and $R4$. If G_1 is adjacent to a 4-face and not adjacent to a 3-face, we consider cluster G_3 in Figure 2 for G contains no structure H_{26} and H_{27}, we have $w'(G_3) \geq -4 \times 2 - 2 + 2 \times 2 + \frac{2}{3} \times 3 + \frac{3}{4} \times 4 = -1$ by $R6$, $R5$ and $R4$. If G_1 is not only adjacent to a 3-face but also adjacent to a 4-face, we consider cluster G_4 in Figure 2 for G contains no structures H_{25} and H_{26}, we have $w'(G_4) \geq -4 \times 3 - 2 + 2 \times 4 + \frac{2}{3} \times 3 + \frac{3}{4} \times 2 = -\frac{5}{2}$ $R6$, $R5$ and $R4$.

Case 1.3.2 $n_3(f) = 1$ and f is a $(3, 4, 4)$-, $(3, 4, 5)$-face, then f is not adjacent to any 3-face for G contains no structures H_{13}, H_{14} and H_{28}. So $w'(f) \geq -4 + \frac{3}{2} \times 2 + \frac{3}{4} \times 2 = \frac{1}{2} > 0$ or $w'(f) \geq -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$ by $R5$ and $R4$.

Case 1.3.3 $n_3(f) = 1$ and f is a $(3,4,6)$-face. For G contains no structure H_{29}, f is adjacent to at most two 3-faces. If f is not adjacent to any 3-face, then f is adjacent to at most one 4-face for G contains no 7-cycle. We have $w'(f) \geq -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$ by $R6$, $R4$. If f is adjacent to a 3-face, then f is not adjacent to any 4-face for G contains no structures H_{27} and H_{30}. We have $w'(f) \geq -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$ by $R6$, $R4$. If f is adjacent to two 3-faces, i.e. G_5 and G_6 in Figure 2 for G contains no structures H_{29}, H_{31} and H_{32}. For G contains no 6-cycle, G_5 and G_6 are not adjacent to any 4-face or 5-face. We have $w'(G_5) \geq -4 \times 3 + \frac{7}{4} + \frac{5}{6} \times 2 + \frac{7}{4} \times 3 + \frac{3}{4} \times 5 = \frac{5}{12} > 0$, $w'(G_6) \geq -4 \times 3 + \frac{7}{4} + \frac{7}{4} \times 3 + \frac{7}{4} + \frac{5}{6} + \frac{3}{4} \times 5 = \frac{4}{3} > 0$ by $R6$ and $R4$.

Case 1.3.4 $n_3(f) = 1$, f is a $(3,4,7^+)$-face. For G contains no structure H_{29}, f is adjacent to at most two 3-faces.

Case 1.3.4.1 f is not adjacent to any 3-face, we have $w'(f) \geq -4 + \frac{5}{6} + 2 + \frac{3}{4} \times 2 = \frac{1}{3} > 0$ by $R6$ and $R4$.

Case 1.3.4.2 f is a $(3,4,7^+)$-face and is adjacent to one 3-face. If f is not adjacent to any 4-face, then we have $w'(f) \geq -4 + \frac{5}{6} + 2 + \frac{3}{4} \times 2 = \frac{1}{3} > 0$ by $R6$ and $R4$. Otherwise, we consider G_7 and G_8 (in Figure 2) for G contains no structures H_{26}, H_{27}, H_{33} and H_{34}. We have $w'(G_7) \geq -4 \times 2 - 2 + \frac{5}{6} \times 2 + 2 \times 2 + \frac{2}{3} \times 3 + \frac{3}{4} \times 4 = \frac{2}{3} > 0$, $w'(G_8) \geq -4 \times 2 - 2 + \frac{5}{6} + 2 \times 3 + \frac{2}{3} + \frac{3}{4} \times 4 = \frac{1}{2} > 0$ by $R6$ and $R4$.

Case 1.3.4.3 f is a $(3,4,7^+)$-face and adjacent to two 3-faces, then we need to consider the following three structures G_9, G_{10} in Figure 2 for G contains no structures H_{29}, H_{31} and H_{32}. We have $w'(G_9) \geq -4 \times 3 + \frac{7}{4} \times 2 + 2 \times 2 + \frac{5}{6} \times 2 + \frac{3}{4} \times 5 = \frac{11}{12} > 0$, $w'(G_{10}) \geq -4 \times 3 + \frac{7}{4} \times 2 + 2 \times 3 + \frac{5}{6} + \frac{3}{4} \times 5 = \frac{25}{12} > 0$ by $R6$ and $R4$.

Case 1.3.5 $n_3(f) = 1$, f is a $(3,5,5^+)$- or $(3,6,6^+)$-face. If f is adjacent to at most two 3-faces, we have $w'(f) \geq -4 + \frac{7}{4} \times 2 + \frac{3}{4} = \frac{1}{4} > 0$ by $R6$ and $R4$. If f is adjacent to three 3-faces i.e. G_{11} and G_{12} in Figure 2 for G contains no structures H_{35} and H_{36}, then $w'(G_{11}) \geq -4 \times 4 + \frac{7}{4} \times 4 + 2 \times 3 + \frac{7}{6} \times 2 + \frac{3}{4} \times 4 = \frac{7}{3} > 0$, $w'(G_{12}) \geq -4 \times 4 + \frac{7}{4} \times 5 + 2 \times 3 + \frac{7}{6} + \frac{3}{4} \times 4 = \frac{35}{12} > 0$, by $R6$ and $R4$.

Case 1.3.6 $n_3(f) = 1$, f is a $(3,7^+,7^+)$-face, we have $w'(f) \geq -4 + 2 \times 2 = 0$ by $R6$.

Case 1.3.7 $n_3(f) = 0$, f is a $(4,4,4)$-face, then f is not adjacent to any 3-face for G does not contain structure H_{37}. If f is not adjacent to any 4-face, then we have $w'(f) = -4 + \frac{2}{3} \times 3 + \frac{3}{4} \times 3 = \frac{1}{4} > 0$ by $R6$ and $R4$. Otherwise, we consider G_{13} for G contains no structure H_{38}, so $w'(G_{13}) \geq -4 - 2 + \frac{2}{3} \times 3 + \frac{2}{3} \times 2 + \frac{3}{4} \times 5 = \frac{13}{12} > 0$ by $R6$, $R5$ and $R4$.

Case 1.3.8 $n_3(f) = 0$, f is a $(4,4,5)$- or $(4,4,6)$-face. If f is adjacent to at most one 3-face, we have $w'(f) \geq -4 + \frac{2}{3} \times 2 + \frac{7}{6} + \frac{3}{4} \times 2 = 0$ by $R6$ and $R4$. If f is adjacent to two 3-faces i.e. G_{14}, G_{15} and G_{16} in Figure 2 for G contains no structures H_{39}, H_{40} and H_{41}, we have $w'(G_{14}) \geq -4 \times 3 + \frac{7}{4} \times 2 +$

$\frac{7}{6}\times 3+\frac{2}{3}\times 4+\frac{3}{4}\times 5=\frac{17}{12}>0$, $w'(G_{15})\geq -4\times 3+\frac{7}{4}\times 2+\frac{7}{4}\times 3+\frac{2}{3}\times 2+\frac{3}{4}\times 5=\frac{11}{6}>0$, $w'(G_{16})\geq -4\times 3+\frac{7}{4}\times 2+\frac{7}{4}\times 2+\frac{2}{3}\times 4+\frac{3}{4}\times 5=\frac{17}{12}>0$ by $R6$ and $R4$.

Case 1.3.9 $n_3(f)=0$, f is a $(4,4,7^+)$-face, then we have $w'(f)\geq -4+\frac{2}{3}\times 2+2+\frac{3}{4}=\frac{1}{12}>0$ by $R6$ and $R4$.

Case 1.3.10 $n_3(f)=0$, f is a $(4,5,5)$-face. If f is adjacent to at most one 3-face, we have $w'(f)\geq -4+\frac{7}{6}\times 2+\frac{2}{3}+\frac{3}{4}\times 2=\frac{1}{2}>0$ by $R6$ and $R4$. If f is adjacent to two 3-faces i.e. G_{17} or G_{18} in Figure 2 for G contains no structures H_{42} and H_{43}. We have $w'(G_{17})\geq -4\times 3+\frac{7}{4}\times 2+\frac{2}{3}\times 3+\frac{7}{6}\times 4+\frac{3}{4}\times 5=\frac{23}{12}>0$, or $w'(G_{18})\geq -4\times 3+\frac{7}{4}\times 2+\frac{7}{6}\times 5+\frac{2}{3}\times 2+\frac{3}{4}\times 5=\frac{29}{12}>0$ by $R6$ and $R4$.

Case 1.3.11 $n_3(f)=0$, f is a $(4,5,6^+)$- or $(4,6^+,6^+)$-face, then we have $w'(f)\geq -4+\frac{7}{6}+\frac{7}{4}+\frac{2}{3}+\frac{3}{4}=\frac{1}{3}>0$ by $R6$ and $R4$.

Case 1.3.12 $n_3(f)=0$, f is a $(5^+,5^+,5^+)$-face, then we have $w'(f)\geq -4+\frac{7}{6}\times 3+\frac{3}{4}=\frac{1}{4}>0$ by $R6$ and $R4$.

Suppose $d(f)=4$. Then $w(f)=-2$. If f is not adjacent to any 3-face, then all faces which are adjacent to f are 8^+-faces for G contains no 6- and 7-cycles. So $w'(f)\geq -2+\frac{3}{4}\times 4=1>0$ by $R4$. If f is adjacent to only one 3-face, then other faces which are adjacent to f are 8^+-faces for G contains no 6- and 7-cycles. So $w'(f)\geq -2+\frac{3}{4}\times 3=\frac{1}{4}>0$ by $R4$. If f is adjacent to two 3-face, then f is adjacent to at least one 6^+-vertex and other adjacent faces are 8^+-faces for G contains no structure H_{44} and contains no 6- and 7-cycles. So $w'(f)\geq -2+\frac{2}{3}+\frac{3}{4}\times 2=\frac{1}{6}>0$ by $R5$ and $R4$..

Suppose $d(f)=5$. Then $w'(f)=w(f)=0$.

Suppose $d(f)\geq 8$. Then $w'(f)\geq w(f)-\frac{3}{4}\times d(f)=2d(f)-10-\frac{3}{4}d(f)=\frac{5}{4}d(f)-10\geq\frac{5}{4}\times 8-10=0$ by $R4$.

From the above discussion, If there is at least two 3-vertices in G, we can obtain that if x is neither a special 3-vertex nor a special face, then $w'(x)\geq 0$ for each $x\in V(G)\cup F(G)$. Furthermore, we have $w'_s\geq -1\times 2-\min\{-\frac{5}{2},-\frac{9}{4},-\frac{3}{2},-1\}=-\frac{9}{2}$ by Claim 2. So we can obtain $\sum_{x\in V(G)\cup F(G)}w'(x)\geq -\frac{9}{2}>-20$, a contradiction.

Case 1.2 $\delta(G)=3$ and there is only one 3-vertex in G.

There is no special 3-face and special 3-vertex in G. The discussion is similar to Case 1.1, it is clear that we only need to check the charge of 3-face in Case 1.3.3 and Case 1.3.4 of Case 1.1, i.e. $(3,4,6)$- and $(3,4,7^+)$-faces such that they are adjacent to two 3-faces each of which is incident to the 3-vertex. For G contains no structure H_{25}, we only consider the structure G_{19}, $w'(G_{19})\geq -4\times 2+2\times 2=-4$ by $R6$. So we can obtain $\sum_{x\in V(G)\cup F(G)}w'(x)\geq -4>-20$, a contradiction.

Case 2 $\delta(G)=2$ and there are at most two 2-vertices in G.

The discharging rules are the same as Case 1 except for the two cases. (1). A 4-face have a common 2-vertex with a 3-face, (2). Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of G is larger than or equal to zero. For convenience, let w'_{t1} (w'_{t2}) denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

Case 2.1 There exists one 2-vertex in G.

Case 2.1.1 The 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, for G contains no structure H_{45} and H_{46}, we can obtain that the 3-face and 4-face is $(2,3,7^+)$- and $(2,3,6^+,7^+)$ respectively; the 3-face and 4-face is $(2,4^+,7^+)$- and $(2,4^+,6^+,7^+)$ respectively; the 3-face and 4-face is $(2,5^+,5^+)$- and $(2,3^+,5^+,5^+)$-face respectively. So $w'_{t1} \geq -4-4-2+2+\frac{2}{3} = -\frac{22}{3}$, $w'_{t1} \geq -3-2-2+2+\frac{5}{6}+\frac{2}{3} = -\frac{13}{2}$, $w'_{t1} \geq -3-2-2+\frac{7}{4}\times 2 = -\frac{13}{2}$ by $R6$ and $R4$.

Case 2.1.2 The 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is $(2,3^+,5^+)$- or $(2,4^+,4^+)$-face for G contains no structure H_6. We have $w'(t1) \geq -4-4+\frac{7}{4} = -\frac{25}{4}$ or $w'(t1) \geq -4-4+\frac{3}{2}\times 2 = -5$ by $R6$.

Case 2.1.3 The 2-vertex is incident a 4-face and not incident to a 3-face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the two 4-faces are adjacent to at least two 8^+-face for G contains no 6- and 7-cycles. We have $w'_(t1) \geq -2-2-4+\frac{3}{4}\times 2 = -\frac{13}{2}$ by $R4$. We can obtain that $w'_{t1} \geq \min\{-\frac{22}{3}, -\frac{13}{2}, -\frac{25}{4}, -5,\}$ i.e. $w'_{t1} \geq -\frac{22}{3}$, so $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -\frac{9}{2}+w'_{t1} \geq -\frac{22}{3}-\frac{9}{2} = -\frac{81}{6} > -20$, a contradiction.

Case 2.2 There exist two 2-vertices in G.

Case 2.2.1 The two 2-vertices are incident to a same 3-face, then f is a $(2,2,5^+)$-face for G contains no structure H_7. So $w'_{t2} \geq -2\times 4-4+\frac{7}{6} = -\frac{65}{6}$ by $R6$.

Case 2.2.2 Two 2-vertices are incident to a same 4-face. If each 2-vertex is incident to another 4-face, we have $w'_{t2} \geq -4\times 2-2\times 3 = -14$. If one of the two 2-vertices is incident to another 3-face, the other 2-vertex is incident to another 4-face, then the 3-face which is incident to the 2-vertex is a $(2,5^+,5^+)$-face for G contains no structure H_{46}. We have $w'_{t2} \geq -4\times 3-2\times 2+\frac{7}{4}\times 2 = -\frac{25}{2}$ by $R6$.

Case 2.2.3 Two 2-vertices are not incident to a same face, from the discussion in Case 2.1, we have $w'_{t2} \geq -\frac{22}{3}\times 2 = -\frac{44}{3}$. Clearly, we have $w'_{t2} \geq -\frac{44}{3}$ in the rest cases. From the above discussion, we have $w'_{t2} \geq -\frac{44}{3}$. So $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -\frac{9}{2}-\frac{44}{3} = -\frac{115}{6} > -20$, a contradiction.

Case 3 There are at least three 2-vertices in G.

For G contains no structures $H_{47} \sim H_{54}$. G has the following properties.

Claim 3.1 Any vertex v is adjacent to at most one 2-vertex.

Claim 3.2 Two 2-vertices are not adjacent to each other.

Claim 3.3 For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 2-vertex, then it is not incident to a 3-face which is incident to a 3-vertex.

Claim 3.4 If v is adjacent to a 3-vertex, then it is not incident to a 3-face which is incident to a 2-vertex.

Claim 3.5 If a vertex v is adjacent to a 2-vertex, then it is not adjacent to a 3-vertex which is adjacent to other 3^{--}-vertex.

Claim 3.6 3-faces in G which is incident to a 2-vertex are $(2, 6^+, 6^+)$-faces.

Claim 3.7 4-faces which is incident to 2-vertices in G are $(2, 3, 6^+, 6^+)$-, $(2, 4, 6^+, 6^+)$- or $(2, 5^+, 5^+, 5^+)$-faces.

Claim 3.8 There is at most one 2-vertex which is adjacent to a k-vertex $(3 \leq k \leq 4)$in G.

We call a 2-vertex a *special 2-vertex* if it is adjacent to a k-vertex $(3 \leq k \leq 4)$, otherwise a simple 2-vertex. Let $n_2(v)$ denote the number of simple 2-vertices adjacent to v. We call a 4-face a simple 4-face if it is incident to a 2-vertex. We use $m_4'(v)$ denote the number of simple 4-faces which is incident to v for each $v \in V(G)$.

Now redistribute the charge according to the following discharging rules.

R1', *R2'*, *R3'* and *R4'* are the same as *R1*, *R2*, *R3* and *R4* in Case 1.

R5' is the same as *R5* except that $d(v) = 5$ and f is a simple 4-face. Transfer charge $\frac{2}{3}$ from each 5-vertex to every incident simple 4-face.

R6' If f is a 3-face incident to a vertex v, then v gives f charge $\frac{3}{2}$ if $d(v) = 4$ and f is a $(3, 4, 4)$-face, $\frac{5}{6}$ if $d(v) = 4$ and f is a $(3, 4, 5^+)$-face, $\frac{2}{3}$ if $d(v) = 4$ and f is an other face, $\frac{7}{4}$ if $d(v) = 5$ and f is a $(3, 3^+, 5)$-face, $\frac{7}{6}$ if $d(v) = 5$ and f is a $(4, 4, 5)$- or $(4, 5, 5^+)$- or $(5, 5, 5^+)$-face, $\frac{2}{3}$ if $d(v) = 5$ and f is a $(5, 6^+, 6^+)$-face, 2 if $d(v) = 6$ and f is a $(2, 6, 6^+)$-face, $\frac{7}{4}$ if $d(v) = 6$ and f is a $(3, 3^+, 6)$-face, $\frac{3}{2}$ if $d(v) = 6$ and f is a $(4, 4, 6)$- or $(4, 5, 6)$-face, $\frac{4}{3}$ if $d(v) = 6$ and f is an other face, 2 if $d(v) = 7$ and f is a $(2, 7, 6^+)$- or $(3, 3^+, 7)$-, or $(4, 4, 7)$-face, $\frac{7}{4}$ if $d(v) = 7$ and f is an other face, 2 if $d(v) \geq 8$.

R7' Transfer charge 2 from each 5^+-vertex to every adjacent 2-vertex.

For any face $f \in F(G)$, if $d(f) = 5$, $d(f) \geq 8$, the discussion is similar to the corresponding situation in Case 1. For any vertex $v \in V(G)$, if $d(v) = 3$, the discussion is also similar to the corresponding situation in Case 1. In the following, we discuss the rest cases.

Suppose $d(v) = 2$, then $w'(v) = -4$. Except the special 2-vertex, we have $w'(v) = -4 + 2 \times 2 = 0$ by *R7'*.

Suppose $d(v) = 4$. Then $w(v) = 2$ and $m_3(v) \leq 3$.

First, we focus on the situation $n_2(v) = 0$. If $m_3(v) = 3$, then $n_3(v) = 0$ for G contains no structures H_{55} and H_{56}. We have $w'(v) \geq 2 - \frac{2}{3} \times 3 =$

0 by $R6'$. If $m_3(v) \leq 2$, then the following discussion is similar to the corresponding situation in Case 1.

In the following, we focus on the situation $n_2(v) = 1$. By Claim 3.6, we have $m_3(v) \leq 2$. If $m_3(v) = 2$, then $n_3(v) = 0$ by Claim 3.3. We have $w'(v) \geq 2 - \frac{2}{3} \times 2 = \frac{2}{3} > 0$ by $R6'$. If $m_3(v) = 1$, then $n_3(v) \leq 1$ by Claim 3.3. We have $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{3} = \frac{1}{2} > 0$ by $R6'$, $R3'$, Claim 3.3 and Claim 3.5. If $m_3(v) = 0$, then $n_3(v) \leq 3$. So $w'(v) \geq 2 - \frac{1}{3} \times 3 = 1 > 0$ by $R3'$ and Claim 3.5.

Suppose $d(v) = 5$. Then $w(v) = 5$, $m_3(v) \leq 3$. If $n_2(v) = 0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v) = 1$.

Case 3.1.1 $m_3(v) = 3$, then $m'_4(v) = 0$, $n_3(v) = 0$ by Claim 3.3. We have $w'(v) \geq 5 - \frac{7}{6} \times 2 - \frac{2}{3} - 2 = 0$ for G contains no structures H_{57}, H_{58} and by $R6'$, $R7'$.

Case 3.1.2 $m_3(v) = 2$, then $m'_4(v) \leq 1$. If $m'_4(v) = 1$, then $n_3(v) = 0$ by Claim 3.3, Claim 3.6 and Claim 3.7. So $w'(v) \geq 5 - \frac{7}{6} \times 2 - \frac{2}{3} - 2 = 0$ by $R6'$, $R5'$ and $R7'$. If $m'_4(v) = 0$, then $n_3(v) \leq 1$ by Claim 3.3. So $w'(v) \geq 5 - \frac{7}{6} \times 2 - \frac{1}{3} - 2 = \frac{1}{3} > 0$ by Claim 3.5 and by $R6'$, $R3'$, $R7'$.

Case 3.1.3 $m_3(v) = 1$, then $m'_4(v) \leq 2$. If $m'_4(v) = 2$, then $n_3(v) = 0$ by Claim 3.3 and Claim 3.7. So $w'(v) \geq 5 - \frac{7}{6} - \frac{2}{3} \times 2 - 2 = \frac{1}{2} > 0$ by $R6'$, $R5'$ and $R7'$. If $m'_4(v) \leq 1$, then $n_3(v) \leq 2$ by Claim 3.3. So $w'(v) > 5 - \frac{7}{6} - \frac{2}{3} - \frac{1}{3} \times 2 - 2 = \frac{1}{2} > 0$ by Claim 3.5 and by $R6'$, $R5'$, $R3'$ and $R7'$.

Case 3.1.4 $m_3(v) = 0$, then $m'_4(v) \leq 2$, $n_3(v) \leq 4$. We have $w'(v) > 5 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - 2 = \frac{1}{3} > 0$ by Claim 3.5 and by $R5'$, $R3'$, $R7'$.

Suppose $d(v) = 6$. Then $w(v) = 8$, $m_3(v) \leq 4$. If $n_2(v) = 0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v) = 1$.

If $m_3(v) = 4$, then $m_4(v) = 0$, $n_3(v) = 0$ by Claim 3.4. So $w'(v) \geq 8 - 2 - \frac{4}{3} \times 3 - 2 = 0$ for G contains no structures H_{59}, H_{60} and by $R6'$ and $R7'$. If $m_3(v) = 3$, then $m_4(v) \leq 1$, $n_3(v) \leq 1$. We have $w'(v) > 8 - 2 - \frac{3}{2} \times 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$ by $R6'$, $R5'$, $R7'$ and $R3'$. If $m_3(v) = 2$, then $m_4(v) \leq 2$, $n_3(v) \leq 2$. We have $w'(v) > 8 - 2 - \frac{3}{2} - \frac{2}{3} \times 2 - 2 - \frac{1}{3} \times 2 = \frac{1}{2} > 0$ by $R6'$, $R5'$, $R7'$ and $R3'$. If $m_3(v) = 1$, then $m_4(v) \leq 3$, $n_3(v) \leq 3$ by Claim 3.3. So $w'(v) > 8 - 2 - \frac{2}{3} \times 3 - 2 - \frac{1}{3} \times 3 = 1 > 0$ by $R6'$, $R5'$, $R7'$ and $R3'$. If $m_3(v) = 0$, then $m_4(v) < 6$, $n_3(v) \leq 5$. So $w'(v) > 8 - \frac{2}{3} \times 6 - \frac{1}{3} \times 6 - 2 = 0$ by $R5'$, $R3'$ and $R7'$.

Suppose $d(v) = 7$. Then $w(v) = 11$, $m_3(v) \leq 5$. If $n_2(v) = 0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v) = 1$.

If $m_3(v) = 5$, then $m_4(v) = 0$, $n_3(v) = 0$. So $w'(v) = 11 - 2 - \frac{7}{4} \times 4 - 2 = 0$ for G contains no structure H_{59}, H_{61} and by $R6'$, $R7'$. If $m_3(v) = 4$, then $m_4(v) \leq 1$, $n_3(v) = 0$. So $w'(v) > 11 - 2 \times 4 - \frac{2}{3} - 2 = \frac{1}{3} > 0$ by

$R6'$, $R5'$ and $R7'$. If $m_3(v) = 3$, then $m_4(v) \leq 2$, $n_3(v) \leq 2$. So $w'(v) > 11 - 2\times 3 - \frac{2}{3}\times 2 - 2 - \frac{1}{3}\times 2 = 1 > 0$ by $R6'$, $R5'$, $R7'$, $R3'$. If $m_3(v) = 2$, then $m_4(v) \leq 3$, $n_3(v) \leq 6$, we have $w'(v) > 11 - 2\times 2 - \frac{2}{3}\times 3 - 2 - \frac{1}{3}\times 6 = 1 > 0$ by $R6'$, $R5'$, $R7'$ and $R3'$. If $m_3(v) \leq 1$, then $m_4(v) < 7$, $n_3(v) \leq 6$, we have $w'(v) > 11 - 2 - \frac{2}{3}\times 7 - 2 - \frac{1}{3}\times 6 = \frac{1}{3} > 0$ by $R6'$, $R5'$, $R7'$ and $R3'$.

Suppose $d(v) \geq 8$. Then $w(v) = 3d(v) - 10$. If $n_2(v) = 0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v) = 1$. By Claim 3.3 and Claim 3.4, we have $n_3(v) \leq d(v) - \frac{4}{3}m_3(v) - 1$, $m_4(v) \leq d(v) - \frac{4}{3}m_3(v)$. So $w'(v) \geq 3d(v) - 10 - 2m_3(v) - \frac{2}{3}m_4(v) - 2 - \frac{1}{3}n_3(v) \geq 3d(v) - 10 - 2m_3(v) - \frac{2}{3}\times(d(v) - \frac{4}{3}m_3(v)) - 2 - \frac{1}{3}(d(v) - \frac{4}{3}m_3(v) - 1) = 2d(v) - \frac{2}{3}m_3(v) - \frac{35}{3}$ by $R6'$, $R5'$, $R7'$ and $R3'$. Since

$$m_3(v) \leq \frac{3}{4}d(v).$$

We obtain $w'(v) \geq \frac{3}{2}d(v) - \frac{35}{3} \geq \frac{1}{3} > 0$.

Suppose $d(f) = 3$. Then $w(f) = -4$ and $n_2(f) \leq 1$.

Case 3.2.1 $n_2(f) = 1$, then f is a $(2, 6^+, 6^+)$-face by Claim 3.6. So $w'(f) \geq -4 + 2\times 2 = 0$ by $R6'$.

Case 3.2.2 $n_2(f) = 0$, $n_3(f) = 2$, then $n_4(f) = 0$, f is a special face $(3, 3, 5^+)$-face. The discussion is similar to the corresponding situation in Case 1.

Case 3.2.3 $n_3(f) = 1$ and f is a $(3,4,4)$-, $(3,4,5)$-face. The discussion is similar to the corresponding situation in Case 1.

Case 3.2.4 $n_3(f) = 1$ and f is a $(3,4,6)$-face. For G contains no structures $H_{62} \sim H_{64}$, f is not adjacent to any 3-face, then $w'(f) \geq -4 + \frac{5}{6} + \frac{5}{3} + \frac{3}{4}\times 2 = 0$ by $R6'$ and $R4'$.

Case 3.2.5 $n_3(f) = 1$, f is a $(3, 4, 7^+)$-face. For G contains no structure H_{42}, f is adjacent to at most two 3-faces. If f is adjacent to at most one 3-faces, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces, then we need to consider the following two structures G_9, G_{10} in Figure 2 for G contains no structures H_{29}, H_{31} and H_{32}. We have $w'(G_9) \geq -4\times 3 + \frac{7}{4} + \frac{4}{3} + \frac{7}{4} + 2 + \frac{5}{6}\times 2 + \frac{3}{4}\times 5 = \frac{1}{4} > 0$, $w'(G_{10}) \geq -4\times 3 + \frac{7}{4} + \frac{4}{3} + 2\times 2 + \frac{7}{4} + \frac{5}{6} + \frac{3}{4}\times 5 = \frac{17}{12} > 0$ by $R6'$ and $R4'$.

Case 3.2.6 $n_3(f) = 1$, f is a $(3, 5, 5^+)$- or $(3, 6, 6^+)$-face. If f is adjacent to at most two 3-faces, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to three 3-faces i.e. G_{11} and G_{12} in Figure 2 for G contains no structures H_{35} and H_{36}, then $w'(G_{11}) \geq -4\times 4 + \frac{7}{4}\times 4 + 2\times 2 + \frac{7}{4} + \frac{2}{3}\times 2 + \frac{3}{4}\times 4 = \frac{13}{12} > 0$, $w'(G_{12}) \geq -4\times 4 + \frac{7}{4}\times 4 + 2\times 2 + \frac{7}{4} + \frac{4}{3} + \frac{2}{3} + \frac{3}{4}\times 4 = \frac{7}{4} > 0$, by $R6'$ and $R4'$.

Case 3.2.7 $n_3(f) = 1$, f is a $(3, 7^+, 7^+)$-face, the following discussion is similar to the corresponding situation in Case 1.

Case 3.2.8 $n_3(f) = 0$, f is a $(4, 4, 4)$-face, the following discussion is similar to the corresponding situation in Case 1.

Case 3.2.9 $n_3(f) = 0$, f is a $(4, 4, 5)$- or $(4, 4, 6)$-face. If f is adjacent to at most one 3-face, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces i.e. G_{14}, G_{15} and G_{16} in Figure 2 for G contains no structures $H_{39} \sim H_{41}$, we have $w'(G_{14}) \geq -4\times3+\frac{3}{2}\times2+\frac{7}{6}\times3+\frac{2}{3}\times4+\frac{3}{4}\times5 = \frac{11}{12} > 0$, $w'(G_{15}) \geq -4\times3+\frac{4}{3}\times2+\frac{4}{3}\times2+\frac{3}{2}+\frac{2}{3}\times4+\frac{3}{4}\times5 = \frac{5}{4} > 0$, $w'(G_{16}) \geq -4\times3+\frac{3}{2}+\frac{4}{3}\times2+\frac{3}{2}+\frac{2}{3}\times5+\frac{3}{4}\times5 = \frac{3}{4} > 0$ by $R6'$ and $R4'$.

Case 3.2.10 $n_3(f) = 0$, f is a $(4, 4, 7^+)$-face, the following discussion is similar to the corresponding situation in Case 1.

Case 3.2.11 $n_3(f) = 0$, f is a $(4, 5, 5)$-face. If f is adjacent to at most one 3-face, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces i.e. G_{17} or G_{18} in Figure 2 for G contains no structures H_{42} and H_{43}. We have $w'(G_{17}) \geq -4\times3+\frac{3}{2}\times2+\frac{2}{3}\times3+\frac{7}{6}\times4+\frac{3}{4}\times5 = \frac{17}{12} > 0$, or $w'(G_{18}) \geq -4\times3+\frac{3}{2}+\frac{7}{6}\times5+\frac{4}{3}+\frac{2}{3}\times2+\frac{3}{4}\times5 = \frac{7}{4} > 0$ by $R6'$ and $R4'$.

Case 3.2.12 $n_3(f) = 0$, f is a $(4, 5, 6^+)$- or $(4, 6^+, 6^+)$-face, then we have $w'(f) \geq -4+\frac{7}{6}+\frac{3}{2}+\frac{2}{3}+\frac{3}{4} = \frac{1}{12} > 0$ or $w'(f) \geq -4+\frac{2}{3}+\frac{4}{3}\times2+\frac{3}{4} = \frac{1}{12} > 0$ by $R6'$ and $R4'$.

Case 3.2.13 $n_3(f) = 0$, f is a $(5, 5, 5^+)$-face, then we have $w'(f) \geq -4 + \frac{7}{6} \times 3 + \frac{3}{4} = \frac{1}{4} > 0$ by $R6'$ and $R4'$.

Case 3.2.14 $n_3(f) = 0$, f is a $(5^+, 6^+, 6^+)$-face, then we have $w'(v) \geq -4 + \frac{2}{3} + \frac{4}{3} \times 2 + \frac{3}{4} = \frac{1}{12} > 0$ by $R6'$ and $R4'$.

Suppose $d(f) = 4$. Then $w(f) = -2$, $n_2(f) \geq 1$ by Claim 3.7. If $n_2(f) = 1$, then f is a $(2, 3, 6^+, 6^+)$-, $(2, 4, 6^+, 6^+)$- or $(2, 5^+, 5^+, 5^+)$-face by Claim 3.7. Any 4-face which is incident to a 2-vertex is adjacent to at least one 8^+-face for G contains no 6- and 7-cycles. So $w'(f) \geq -2 + \frac{3}{4} + \frac{2}{3} \times 2 = \frac{1}{12} > 0$ by $R4'$ and $R5'$. If $n_2(f) = 0$, then the discussion is similar to the situation when $d(f) = 4$ in Case 1.

From the above discussion, we can obtain that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$ and x is none of a special 3-vertex, a special 2-vertex and a special face. Furthermore, we have $w'_s \geq -\frac{9}{2} - 4 = -\frac{17}{2}$ by Claim 1 and Claim 3.8. From the above discussion, we have $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -\frac{17}{2} > -20$, a contradiction.

Case 4 $\delta(G) = 1$.

Case 4.1 There is one 1-vertex and at most two 2-vertices in G.

When $d(v) = 1$, then $w(v) = -7$. If there is one 1-vertex in G, then 3-faces in G are $(3^-, 5^+, 5^+)$-faces or $(4^+, 4^+, 4^+)$-faces for G contains no structure H_{65} and the 4-faces which are incident to a 2-vertex in G are

$(2, 5^+, 5^+, 5^+)$-faces for G contains no structure H_{66}. Now there are not special 3-vertices and special faces in G. The discharging rules are the same as Case 1 except for the two cases. (1). A 4-face have a common 2-vertex with a 3-face. (2). Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of G is larger than or equal to zero. For convenience, let w'_{t1} (w'_{t2}) denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

Case 4.1.1 There is one 2-vertex in G. If the 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, the 3-face and 4-face is $(2, 5^+, 5^+)$- and $(2, 5^+, 5^+, 5^+)$-face respectively. So $w'_{t1} \geq -4-2-4+\frac{7}{4}\times 2+\frac{3}{4} = -\frac{23}{4} > -6$ by $R6$ and $R4$. If the 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is $(2, 5^+, 5^+)$-face. We have $w'_{t1} \geq -4-4+\frac{7}{4}\times 2 = -\frac{9}{2} > -6$ by $R6$. If the 2-vertex is incident a 4-face and not incident to a 3-face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the 4-face is a $(2, 5^+, 5^+, 5^+)$-face. We have $w'_{t1} \geq -4-2-2+\frac{3}{4}\times 3 = -\frac{23}{4} > -6$ by $R4$. We can obtain that $w'_{t1} \geq -6$, so $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7+w'_{t1} \geq -7-6 = -13 > -20$, a contradiction.

Case 4.1.2 There are two 2-vertices in G.

For two 2-vertices are not incident to a same 3- or 4-face, from the discussion in **Case** 4.1.1, we have $w'_{t2} \geq -6\times 2 = -12$. So $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7-12 = -19 > -20$, a contradiction.

Case 4.2 There is one 1-vertex and at least three 2-vertices in G.

For G contains no structure H_{65}, the 3-faces in G are $(3^-, 5^+, 5^+)$-faces or $(4^+, 4^+, 4^+)$-faces. Now there are not special 3-vertices and special faces in G. The discussion is same as the situation in Case 3, we have $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7-4 = -11 > -20$, a contradiction.

Case 4.3 There are at least two 1-vertices in G.

If there are two 1-vertices in G, then there is neither 2-vertex nor other 1-vertex in G for G contains no structure H_{67}. Furthermore, any 3-face in G is $(3, 5^+, 5^+)$- or $(4^+, 4^+, 4^+)$-face for G contains no structure H_{65}. Note that there are neither special 3-vertices nor special faces in G now. And the following discussion is the same as the situation in Case 1. These imply that $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7\times 2 = -14 > -20$, a contradiction. ∎

In the following, let us give the proof of the main theorems.

Theorem 2.8 *If G is a planar graph without* 6- *and* 7-*cycles and* $k \geq \max\{6, \Delta(G)\}$, *then G is equitably k-colorable.*

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most 4 vertices, then $\Delta(G) \leq 3$. So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least five vertices. By Lemma 2.7, G has one of the structures $H_1 \sim H_{67}$, taking one and the vertices are labeled as they are in Figure 1. If there are vertices labeled repeatedly, then we take the larger (x_i is larger than x_{i-1}). In the following, we show how to find S in Lemma 2.4. If G has one of H_2, H_6, H_{48} and H_{67}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If G has one of H_3, H_8, H_{10}, $H_{12} \sim H_{15}$, H_{17}, H_{18}, H_{21}, H_{25}, H_{28}, H_{37}, H_{39}, $H_{45} \sim H_{47}$, H_{51}, H_{65} and H_{66}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. If G has one of H_9, H_{11}, H_{16}, H_{19}, H_{20}, $H_{22} \sim H_{24}$, H_{26}, H_{27}, $H_{29} \sim H_{36}$, H_{44}, H_{49}, H_{50}, $H_{55} \sim H_{59}$, H_{61}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_3, x_2, x_1\}$. If G has H_7, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. If G has one of H_1, H_4, H_5, H_{38}, $H_{40} \sim H_{43}$, H_{52}, H_{53}, H_{60}, $H_{62} \sim H_{64}$, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_2, x_1\}$. If G has H_{54}, then let $S' = \{x_k, x_{k-1}, \cdots, x_{k-4}, x_1\}$. By Lemma 2.1, G is 3-degenerate, then we can find the remaining unspecified positions in S from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of $|V(G)|$ and $k \geq \Delta(G) \geq \Delta(G-S)$, $G-S$ is equitably k-colorable. So G is also equitably k-colorable by Lemma 2.4. ∎

Corollary 2.9 *Let G be a planar graph without 6- and 7-cycles. If $\Delta(G) \geq 6$, then $\chi_e(G) \leq \Delta(G)$.*

Corollary 2.10 *Let G be a planar graph without 6- and 7-cycles. If $\Delta(G) \geq 6$, then $\chi_e^*(G) \leq \Delta(G)$.*

Theorem 2.11 *If G is a planar graph without 6- and 7-cycles and $k \geq \max\{6, \Delta(G)\}$, then G is equitably k-choosable.*

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most 4 vertices, then $\Delta(G) \leq 3$. So G is equitably k-choosable by Lemma 2.5. Otherwise, the proof is similar to the proof of Theorem 2.8 by Lemma 2.6 and Lemma 2.7. ∎

Corollary 2.12 *Let G be a planar graph without 6- and 7-cycles. If $\Delta(G) \geq 6$, then G is equitable $\Delta(G)$-choosable.*

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