The Properties of Transformation Digraphs D^{xyz}*

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Abstract: In this paper, we investigate some basic properties of these eight kinds of transformation digraphs.

Key words: Transformation digraphs; Connectivity; Isomorphism

1 Introduction

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [1]. We consider only strict digraph D(digraph contains no loops and no parallel arcs) with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, we denote the indegree, the outdegree of v, the minimum indegrees and outdegrees in D by $d_D^-(v)$, $d_D^+(v)$, $\delta^-(D)$ and $\delta^+(D)$, respectively. We denote the minimum degree of D by $\delta(D) = min\{\delta^-(D), \delta^+(D)\}$. $\overrightarrow{K_n}$, $\overrightarrow{C_n}$ and N_n denote the complete digraph, directed cycle and empty digraph of order n, respectively.

Let $D = (V(D), A(D))$ be a digraph, where $|V(D)| = n$, $|A(D)| = m$ and $V(D) = \{v_1, v_2, \cdots, v_n\}$. The *line digraph* of D, denoted by $L(D)$, is

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the digraph with vertex set $V(L(D)) = \{a_{ij} | (v_i, v_j) \text{ is an arc in } D\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D)$ if and only if $v_j = v_s$ in D.

In [4], Wu and Meng introduced transformation graphs and investigated some basic properties of them. In [2], the authors gave the definition of transformation digraphs and discussed the properties of D^{++-}. In [3], the authors obtained the spectra of transformation digraphs of a regular digraph.

Definition 1.1. *Let $D = (V(D), A(D))$ be a digraph, x, y, z be three variables taking values $-$ or $+$. The transformation digraph of D, denoted by D^{xyz}, is a digraph with vertex set $V(D^{xyz}) = V(D) \cup A(D)$. For any vertex $a, b \in V(D^{xyz})$, $(a, b) \in A(D^{xyz})$ if and only if one of the following four cases holds:*

(i) If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$ in D if $x = +$ and $(a, b) \notin A(D)$ in D if $x = -$.

(ii) If $a \in A(D)$ and $b \in A(D)$, then the head of arc a is the tail of arc b in D if $y = +$ and the head of arc a is not the tail of arc b in D if $y = -$.

(iii) If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D if $z = +$ and a is not the tail of arc b in D if $z = -$.

(iv) If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D if $z = +$ and b is not the head of arc a in D if $z = -$.

Thus, as defined above, there are eight kinds of transformation digraphs, in which D^{+++} is usually known as the total digraph of D. and D^{---} is its complement. Also, D^{--+},D^{-+-} and D^{-++} are the complement of D^{++-}, D^{+-+} and D^{+--}, respectively. In this paper, we investigate some basic properties of these eight kinds of transformation digraphs.

First, we list the out-degree、 vertex number and edge number of transformation digraphs D^{xyz}. Let $A = \sum_{v \in V(D)} d_D^+(v) d_D^-(v)$.

TD	*vertex*	$d^+(v), v \in V(D)$	$d^+(a), a = (u,v) \in A(D)$	*edge number*
D^{+++}	$m+n$	$2d_D^+(v)$	$d_D^+(v)+1$	$3m+A$
D^{++-}	$m+n$	m	$d_D^+(v)+n-1$	$2mn-m+A$
D^{---}	$m+n$	$n+m-1-2d_D^+(v)$	$m+n-2-d_D^+(v)$	$n^2+m^2-4m-n+2mn-A$
D^{--+}	$m+n$	$n-1$	$m-d_D^+(v)$	n^2+m^2-n-A
D^{+-+}	$m+n$	$2d_D^+(v)$	$m-d_D^+(v)$	m^2+2m-A
D^{-+-}	$m+n$	$n+m-1-2d_D^+(v)$	$d_D^+(v)+n-1$	$n^2+2mn-n-3m+A$
D^{+--}	$m+n$	m	$m+n-2-d_D^+(v)$	$m^2+2mn-2m-A$
D^{-++}	$m+n$	$n-1$	$d_D^+(v)+1$	$n^2-n+m+A$

Tabel 1

2 Connectedness of D^{xyz}

The connectedness of D^{+++} and D^{++-} have been discussed.

Theorem 2.1. *For a digraph D, D^{+++} is strongly connected if and only if D is strongly connected.*

Theorem 2.2. *[2] For a digraph D, D^{++-} is strongly connected if and only if D has at least one arc.*

Theorem 2.3. *For a digraph D, D^{+-+} is strongly connected if and only if $\delta(D) \geq 1$.*

Proof. The 'only if' part is obvious. We now show the 'if' part. For any two vertices $u, v \in V(D)$, if there is a directed path in D from u to v, then there is a directed path in D^{+-+}. If there is no a directed path in D from u to v, then there are two arcs $a = (u, u_1)$ and $b = (v_1, v)$ in $A(D)$ $(u_1 \neq v_1)$ since $\delta(D) \geq 1$, hence $(a, b) \in A(D^{+-+})$. Thus, (u, a, b, v) is a directed path in D^{+-+} from u and v.

For any two vertices $a, b \in A(D)$, let $a = (u, u_1), b = (v_1, v)$. If $u_1 \neq v_1$, then $(a, b) \in A(\overline{L(D)})$. If $u_1 = v_1$, then (a, u_1, b) is a directed path in D^{+-+} from a to b.

For any two vertices $u \in V(D), b \in A(D)$, then there is an arc $a = (u, u_1)$ in D for $\delta(D) \geq 1$. By the above argument there is a directed path P_1 from a to b, hence $(u, a) \cup P_1$ is a directed path in D^{+-+} from u to b. Similarly, there is an arc $c = (u_2, u)$ in D for $\delta(D) \geq 1$. Hence $P_2 \cup (c, u)$ is a directed path in D^{+-+} from b to u (P_2 is a directed path from b to c). □

Theorem 2.4. *For a digraph D, D^{+--} is strongly connected if and only if D has at least one arc.*

Proof. If D is empty, then D^{+--} is not strongly connected. Now, we show that if D has at least one arc, then D^{+--} is strongly connected.

Take an arc $a = (u_1, v_1) \in A(D)$, then (u_1, v_1, a, u_1) is a 3-cycle in D^{+--}.

For any two vertices $u, v \in V(D)$, if $(u, v) \in A(D)$, then $(u, v) \in A(D^{+--})$. Now we consider $(u, v) \notin A(D)$, if $u = u_1, v \neq v_1$, then

(u, v_1, a, v) is a directed path from u to v. If $u \neq u_1, v = v_1$, then (u, a, u_1, v) is a directed path from u to v. If $u \neq u_1, v \neq v_1$, then (u, a, v) is a directed path from u to v.

For any two vertices $b, c \in A(D)$, if $(b, c) \notin A(L(D))$, then $(b, c) \in A(D^{+--})$. Now we consider $(b, c) \in A(L(D))$. Let $b = (w, w_1) \in A(D)$, $c = (w_1, w_2) \in A(D)$, then (b, w, w_1, w_2, c) is a directed path from b to c.

For any two vertices $u \in V(D), b \in A(D)$, let $b = (w, w_1) \in A(D)$. If $u = w$, then (u, w_1, b) is a directed path from u to b. If $u \neq w$, then $(u, b) \in A(D^{+--})$. Similarly, if $u = w_1$, then (b, w, u) is a directed path from b to u. If $u \neq w_1$, then $(b, u) \in A(D^{+--})$. Thus, D^{+--} is strongly connected. □

Theorem 2.5. *For a digraph D, D^{-++} is strongly connected for any digraph D.*

Proof. If D is empty, then it is obvious. For any two vertices $u, v \in V(D)$, if $(u, v) \notin A(D)$, then $(u, v) \in A(D^{-++})$. If $a = (u, v) \in A(D)$, then (u, a, v) is a directed path in D^{-++} from u to v. For any two vertices $a = (u, u_1), b = (v_1, v) \in A(D)$, if $u_1 = v_1$, then $(a, b) \in A(D^{-++})$; if $u_1 \neq v_1$, by the above argument, there is a directed path P from u_1 to v_1, then $(a, u_1) \cup P \cup (v_1, b)$ is a directed path from a to b. For any two vertices $u \in V(D)$, $b = (v_1, v) \in A(D)$, then $P_1 \cup (v_1, b)$ is a directed path from u to b (P_1 is a directed path from u to v_1). Similarly, $(b, v) \cup P_2$ is a directed path from b to u (P_2 is a directed path from v to u). □

By the similar argument, we have the Theorem 2.6.

Theorem 2.6. *For a digraph D, D^{--+} is strongly connected for any digraph D .*

Theorem 2.7. *For a digraph D, D^{-+-} is strongly connected if and only if D is not a star (out-star and in-star).*

Proof. The 'only if' part is obvious. We now show the 'if' part. For any two vertices $u, v \in V(D)$, if $(u, v) \notin A(D)$, then $(u, v) \in A(D^{-+-})$, now we consider the case that $a = (u, v) \in A(D)$. If there is a vertex $u_1 (\neq v) \in V(D)$ such that $(u, u_1) \notin A(D)$, then $(u, u_1) \in A(D^{-+-})$, if $(u_1, v) \notin A(D)$, then $(u_1, v) \in A(D^{-+-})$, thus (u, u_1, v) is a directed path in D^{-+-}, if $b = (u_1, v) \in A(D)$, then there is an isolated vertex x or an arc c such that

v is not the head of c since D is not an in-star, thus, (u, x, v), (u, c, v) or (u, b, u_1, c, v) is a directed path in D^{-+-}. For any vertex $w(\neq u) \in V(D)$, if $(u, w) \in A(D)$, then there is an arc $c = (w, z) \in A(D)$ such that $w \neq u$, and $z \in V(D)$ since D is not an out-star. Let $(u, w) = d$, if $z = v$, then (u, c, w, d, v) is a directed path from u to v, if $z \neq v$, then (u, c, v) is a directed path from u to v.

For any two vertices $a, b \in A(D)$, let $a = (u, u_1), b = (v_1, v)$, by the above argument, there is a directed path P from u to v in D^{-+-}, then $(a, u) \cup P \cup (v, b)$ is a directed path from a to b.

For any two vertices $u \in V(D)$, $b \in A(D)$, let $b = (v_1, v)$, if $u \neq v_1$, then $(u, b) \in A(D^{-+-})$, if $u = v_1$, then (u, P_1, v, b) is a directed path from u to b (Let P_1 be a directed path from v_1 to v). Similarly, if $u \neq v$, then $(b, u) \in A(D^{-+-})$, if $u = v$, then (b, v_1, P_1, u) is a directed path from b to u. □

By the similar argument, we have Theorem 2.8.

Theorem 2.8. *For a digraph D, D^{---} is strongly connected if and only if D is not a star (out-star and in-star).*

3 Regularity of D^{xyz}

Theorem 3.1. *For a digraph D of order n, then D^{+++} and D^{---} are regular if and only if $D \cong C_n$.*

Proof. By Table 1, for any vertex $v \in V(D)$, we have $d^+_{D^{+++}}(v) = 2d^+_D(v)$, $d^-_{D^{+++}}(v) = 2d^-_D(v)$. For any arc $a = (u_1, u_2) \in A(D)$, we have $d^+_{D^{+++}}(a) = d^+_D(u_2) + 1$, $d^-_{D^{+++}}(a) = d^-_D(u_1) + 1$. If D^{+++} is regular, then $d^+_D(v) = d^-_D(v) = 1$ for every vertex $v \in V(D)$, hence $D \cong \overrightarrow{C_n}$. If $D \cong \overrightarrow{C_n}$, then D^{+++} and D^{---} are regular. □

By the similar argument, we have the following Theorems.

Theorem 3.2. *For a digraph D of order n, then D^{++-} and D^{--+} are regular if and only if D is an $m - n + 1$-regular digraph.*

Theorem 3.3. *For a digraph D of order n, then D^{+-+} and D^{-+-} is regular if and only if D is an $m/3$-regular digraph.*

Theorem 3.4. *For a digraph D of order n, then D^{+--} and D^{-++} is regular if and only if D is an $n-2$-regular digraph.*

4 Isomorphism of D^{xyz}

Theorem 4.1. *Let D be a digraph, then the transformation digraphs $D^{xyz} \cong D$ if and only if $D \cong N_n$ when $x = +$, $D \cong K_1$ when $x = -$.*

Proof. It is clear that if $x = +$ and $D \cong N_n$, or $x = -$ and $D \cong K_1$, then $D^{xyz} \cong D$. On the other hand, if $D^{xyz} \cong D$, then D has no arc, hence D is an empty digraph N_n. Thus, if $x = +$ and $D^{xyz} \cong D$, then $D \cong N_n$. If $x = -$ and $|V(D)| = n \geq 2$, then $D^{xyz} = \overline{N_n} = \overrightarrow{K_n}$, it is not isomorphic to D. Therefore, if $x = -$ and $D^{xyz} \cong D$, then $D \cong K_1$. □

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