Minimizing the least eigenvalue of bicyclic graphs with k pendant vertices

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Abstract

Let $\mathscr{B}(n,k)$ be the set of bicyclic graphs with n vertices and k pendant vertices. In this paper, we determine the unique graph with minimal least eigenvalue among all graphs in $\mathscr{B}(n,k)$. This extremal graph is the same as that on the Laplacian spectral radius as done by Ji-Ming Guo[The Laplacian spectral radius of bicyclic graphs with n vertices and k pendant vertices, Science China Mathematics, 53(8)(2010)2135-2142]. Moreover, the minimal least eigenvalue is a decreasing function on k.

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. Denote by $V(G)$ the vertex set and $E(G)$ the edge set of a graph G. The degree of a vertex v, written by $d_G(v)$ or $d(v)$, is the number of edges incident with v. A pendant vertex is a vertex of degree 1. The *girth* $g(G)$ of a graph G is the length of the shortest cycle in G, with the girth of an

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acyclic graph being infinite. Denote by C_n and P_n the cycle and the path on n vertices respectively.

Let A be the adjacency matrix of a graph G. Since A is symmetric and real, the eigenvalue of A, i.e., the zeros of the characteristic polynomial $\phi(G;\lambda) = det(\lambda I - A)$, can be arranged as follows: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. For connected graph G, the spectral radius $\rho(G) = \lambda_1(G)$ is simple and has a unique positive eigenvector up to multiples. We will refer to such an eigenvector as the Perron vector of G. Denote $\lambda_n(G)$ by $\lambda(G)$. Let X be a unit vector, by the Rayleigh-Ritz Theorem, $\lambda(G) = \min X^T A(G)X$. It is known [2] that $\lambda(G) = -\rho(G)$ for a bipartite graph G.

In [1], R.A. Brualdi and E.S. Solheid posed the problem of maximizing the spectral radius and determining the extremal graph for a given class of graphs, which became one of the classic problems of spectral graph theory. Recently, the investigation on minimizing the least eigenvalue of a given class of graphs is of great interest(for example, [3, 6, 7]). Let $\mathscr{B}(n,k)$ be the set of bicyclic graphs with n vertices and k pendant vertices. k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have almost equal lengths if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. Let $B_1(k)$ denote the bicyclic graph in $\mathscr{B}(n,k)$ obtained from the graph G_1(see Fig.1) by attaching k paths with almost equal lengths to vertex v. The main result of this paper is as follows:

Theorem 1.1 *Let B^* have minimal least eigenvalue in $\mathscr{B}(n,k)$, where $1 \leq k \leq n-7$. Then $B^* \cong B_1(k)$. Moreover, $\lambda(B_1(k))$ is a decreasing function on k.*

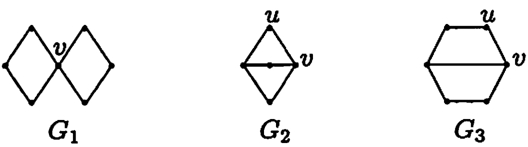

Fig.1. Bicyclic graphs G_1, G_2 and G_3.

2 Preliminaries

Let B be a bicyclic graph. The base of B, denoted by B_0, is the unique bicyclic subgraph of B containing no pendant vertices, and B can be obtained from B_0 by planting trees at some vertices of B_0.

Bicyclic graphs only have two types of bases. Denote by $B(p,l,q)$the graph obtained by joining a new path $P_l : v_0 v_1 \cdots v_{l-1}$ between two vertex-disjoint cycles C_p and C_q, where $v_0 \in V(C_p), v_{l-1} \in V(C_q), q \geq p \geq 3$ and $l \geq 1$. In particular, $l = 1$ means identifying v_0 with v_{l-1}. Denote by $P(p,q,l)$ the graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{l+1}$ with common endpoints, where $l \geq q \geq p \geq 1$ and at most one of them is 1. We introduce two subclasses of $\mathscr{B}(n,k)$: $\mathscr{B}_1(n,k) = \{B \in$

$\mathscr{B}(n,k)|B_0 = B(p,l,q)\}$ and $\mathscr{B}_2(n,k) = \{B \in \mathscr{B}(n,k)|B_0 = P(p,q,l)\}$. Then $\mathscr{B}(n,k) = \mathscr{B}_1(n,k) \cup \mathscr{B}_2(n,k)$.

Let G, H be two disjoint connected graphs with $u \in V(G)$ and $w \in V(H)$, we denote by $GuwH$ the graph obtained from G and H by identifying u with w.

Lemma 2.1 ([3]) *Let G, H be two disjoint nontrivial connected graphs with $u, v \in V(G)$ and $w \in V(H)$. Let X be a unit eigenvector corresponding to $\lambda(GuwH)$. If $|x_u| \leq |x_v|$, then $\lambda(GuwH) \geq \lambda(GvwH)$, and the equality holds if and only if X is also an eigenvector corresponding to $\lambda(GvwH)$, $x_u = x_v$ and $\sum_{i \in N_H(w)} x_i = 0$.*

Let G be a connected graph with $uv \in E(G)$. We denote by G_{uv} the graph obtained from G by subdividing the edge uv, that is, introducing a new vertex on the edge uv. A walk $v_1v_2 \cdots v_k$ $(k \geq 2)$ in a graph G is called an *internal path*, if these k vertices are distinct (except possibly $v_1 = v_k$), $d_G(v_1) > 2$, $d_G(v_k) > 2$ and $d_G(v_2) = \cdots = d_G(v_{k-1}) = 2$ (unless $k = 2$).

Let W_n $(n \geq 6)$ be the graph obtained from a path $v_1v_2 \cdots v_{n-4}$ by attaching two pendant vertices to v_1 and another two to v_{n-4}. Hoffman and Smith showed the following result.

Lemma 2.2 ([4]) *Let G be a connected graph with $uv \in E(G)$. If uv belongs to an internal path of G and $G \not\cong W_n$, then $\rho(G_{uv}) < \rho(G)$.*

Lemma 2.3 ([9]) *Let u, v be two distinct vertices of a connected graph G, $\{v_i | i = 1, 2, \ldots, s\} \subseteq N_G(v) \backslash (N_G(u) \cup \{u\})$, and $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G. Let $G^* = G - \sum_{i=1}^{s} v_iv + \sum_{i=1}^{s} v_iu$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 2.4 ([6])*Let $f_1(x) = x, f_i(x) = x - \frac{1}{f_{i-1}(x)}, i \geq 2$. For $x \leq -2$, then $f_i(x) < f_{i+1}(x) < -1$.*

Lemma 2.5 ([6])*Let v_0 be a vertex of a connected graph G with at least two vertices. Let G_l $(l \geq 1)$ be the graph obtained from G by attaching a new path $P : v_0v_1 \cdots v_l$ of length l at v_0, where $v_1, \ldots, v_l$ are distinct new vertices. Let X be a unit eigenvector of $\lambda(G_l)$. If $\lambda(G_l) \leq -2$, then we have (i)$x_{v_i} = f_{l-i}(\lambda)x_{v_{i+1}} (0 \leq i \leq l-1)$, where $f_i(x)$ is a function on x defined in Lemma 2.4 and $\lambda = \lambda(G_l)$.*
(ii)For any fixed i $(i = 0, 1, \ldots, l-1)$, we have $|x_{v_{i+1}}| \leq |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \leq 0$, with equalities if and only if $x_{v_0} = 0$.

Lemma 2.6 *Let $g_1(x) = \frac{x}{2}$, $g_i(x) = x - \frac{1}{g_{i-1}(x)}$ $(i \geq 2)$. Let $h_1(x) = x - 1$, $h_i(x) = x - \frac{1}{h_{i-1}(x)}$ $(i \geq 2)$. If $x < -2$, then for any i, $g_{i+1}(x) < g_i(x) < -1$ and $h_i(x) < h_{i+1}(x) < -1$.*

Proof. We use induction on i to show the result. Initially, $g_1(x) = \frac{x}{2}$ and $g_2(x) = x - \frac{2}{x}$. Clearly, $g_2(x) < g_1(x) < -1$, since $x < -2$. Now suppose that the result holds for $i > 1$. Then by the induction hypothesis, $g_{i+1}(x) = x - \frac{1}{g_i(x)} < x+1 < -1$ and $g_{i+2}(x) - g_{i+1}(x) = \frac{1}{g_i(x)} - \frac{1}{g_{i+1}(x)} < 0$. For $h_i(x)$, we can get the result likewise. □

Lemma 2.7 ([6]) *Let A be an $n \times n$ real symmetric matrix and λ be the least eigenvalue of A. If $X \in \mathbb{R}^n$ is a unit vector such that $\lambda = X^T AX$, then $AX = \lambda X$.*

Lemma 2.8 *Let G be a connected graph, $C_l = v_0 v_1 \cdots v_{l-1} v_0$ be a cycle and Gv_0C_l be the graph obtained by identifying v_0 with some vertex of G(see Fig.2). Then there exists an eigenvector X of $\lambda(Gv_0C_l)$ such that $x_{v_i} = x_{v_{l-i}}$, where $1 \le i \le l-1$.*

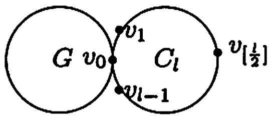

Fig.2. Gv_0C_l.

Proof. Let X be a unit eigenvector of $\lambda(Gv_0C_l)$. Note that there is an automorphism φ of Gv_0C_l which maps v_i to v_{l-i} for $1 \le i \le l-1$ and preserves other vertices. Define a new vector $\bar{X}$ such that $\bar{X}_v = x_{\varphi(v)}$ for each vertex $v \in V(Gv_0C_l)$.

If $\bar{X}+X \neq 0$, then $\bar{X}+X$ is clearly a desirable eigenvector of $\lambda(Gv_0C_l)$. Otherwise $\bar{X} + X = 0$, then $x_v = 0$ for each vertex $v \in V(G)$. Note that X is an eigenvector of $\lambda(Gv_0C_l)$, without loss of generality, assume that $x_{v_k} \neq 0$ for some $v_k \in V(C_l) \setminus \{v_0\}$. Let ψ be a circular permutation of C_l such that $\psi(v_i) = v_{i+k(mod\ l)} (0 \le i \le l-1)$ and define a new vector Y such that $y_{v_i} = x_{\psi(v_i)}$ for each $v_i \in V(C_l)$ and $y_v = x_v$ for each $v \notin V(C_l)$. We can observe that $\lambda(Gv_0C_l) = X^TAX = 2\sum_{uv \in E(Gv_0C_l)} x_u x_v = Y^TAY$, and hence by Lemma 2.7, Y is also a unit eigenvector of $\lambda(Gv_0C_l)$ with $y_{v_0} = x_{v_k} \neq 0$. Similar to $\bar{X}$, define a vector $\bar{Y}$, then $\bar{Y} + Y \neq 0$ is a desirable eigenvector of $\lambda(Gv_0C_l)$. □

Lemma 2.9 *Let X be a unit eigenvector of Gv_0C_l corresponding to $\lambda(Gv_0C_l)$ such that $x_{v_i} = x_{v_{l-i}}$, where $1 \le i \le l-1$. $g_i(x)$ and $h_i(x)$ are two functions on x defined in Lemma 2.6 and $\lambda = \lambda(Gv_0C_l)$. If $\lambda < -2$, then*
(i) For even l, $x_{v_i} = g_{\frac{l}{2}-i}(\lambda)x_{v_{i+1}}$, and thus $|x_{v_{i+1}}| \le |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \le 0$, with equalities if and only if $x_{v_0} = 0$, where $0 \le i \le \frac{l-2}{2}$;
(ii) For odd l, $x_{v_i} = h_{\frac{l-1}{2}-i}(\lambda)x_{v_{i+1}}$, and hence $|x_{v_{i+1}}| \le |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \le 0$, with equalities if and only if $x_{v_0} = 0$, where $0 \le i \le \frac{l-3}{2}$.

Proof. For even l. From $A(Gv_0C_l)X = \lambda X$, we have $\lambda x_{v_{\frac{l}{2}}} = x_{v_{\frac{l-2}{2}}} + x_{v_{\frac{l+2}{2}}} = 2x_{v_{\frac{l-2}{2}}}$ and $\lambda x_{v_i} = x_{v_{i-1}} + x_{v_{i+1}}$, where $i = 1, 2, \ldots, \frac{l-2}{2}$. By these two equations, it is easy to show that the first half of (i) holds.

Since $\lambda < -2$, then by Lemma 2.6, $g_i(\lambda) < -1$. Hence, from the first half of (i), we have if $x_{v_{i+1}} \neq 0$, then $|x_{v_{i+1}}| < |x_{v_i}|, x_{v_i}x_{v_{i+1}} < 0$, where $i = 0, 1, \ldots, \frac{l-2}{2}$. Also from the first half of (i), we have

$$\begin{aligned} x_{v_0} &= g_{\frac{l}{2}}(\lambda)x_{v_1} = g_{\frac{l}{2}}(\lambda)g_{\frac{l-2}{2}}(\lambda)x_{v_2} \\ &= \cdots = g_{\frac{l}{2}}(\lambda)g_{\frac{l-2}{2}}(\lambda)\cdots g_2(\lambda)g_1(\lambda)x_{v_{\frac{l}{2}}}. \end{aligned}$$

Thus $x_{v_0} = 0$ if and only if $x_{v_1} = x_{v_2} = \cdots = x_{v_{\frac{l}{2}}} = 0$. Then (i) holds. Similarly, we can prove that (ii) holds. □

Lemma 2.10 ([5]) *Let v be a vertex in a connected graph G and suppose that two new paths $P : vv_1v_2\cdots v_k$ and $Q : vu_1u_2\cdots u_m$ of length k, m $(k \geq m \geq 1)$ are attached to G at v, respectively, to form a new graph $G_{k,m}$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_m$ are distinct new vertices. Then for any $\lambda \geq \rho(G_{k,m})$, we have $\phi(G_{k+1,m-1}; \lambda) > \phi(G_{k,m}; \lambda)$. In particular, $\rho(G_{k,m}) > \rho(G_{k+1,m-1})$.*

Lemma 2.11 ([8]) *Let v be a vertex of G and $\mathscr{C}(v)$ be the set of all cycles containing v. Then*

$$\phi(G;\lambda) = \lambda\phi(G-v;\lambda) - \sum_{u\in N(v)} \phi(G-v-u;\lambda) - 2\sum_{Z\in\mathscr{C}(v)} \phi(G-V(Z);\lambda).$$

Lemma 2.12 ([5]) *Let G and H be two connected graphs such that $\phi(G;\lambda) > \phi(H;\lambda)$ for $\lambda \geq \rho(H)$ or $\lambda = \rho(G)$, then $\rho(G) < \rho(H)$.*

Let $B_2(k)$ denote the bicyclic graph in $\mathscr{B}(n,k)$ obtained from the graph G_2(see Fig.1) by attaching k paths with almost equal lengths to vertex v. Note that both $B_1(k)$ and $B_2(k)$ exist if and only if $k \geq 1, n \geq k+7$.

Lemma 2.13 *If both $B_1(k)$ and $B_2(k)$ exist, then $\rho(B_1(k)) > \rho(B_2(k))$.*

Proof. Denote by l the maximal number of vertices of a path attached to the vertex v of $B_2(k)$. Note that $k \leq n-7$, then $l \geq 2$. Suppose that the number of such paths is t.

Case1. $t \geq 2$.

Let B_1 be the graph analogous to $B_1(k)$ in which all paths attached to vertex v have $l-1$ vertices. Let B_2 be the graph analogous to $B_2(k)$ in which all paths attached to vertex v have l vertices. Evidently, B_1 is an induced subgraph of $B_1(k)$ and $B_2(k)$ is an induced subgraph of B_2. Therefore, $\rho(B_1) \leq \rho(B_1(k))$ with equality if and only if $n = (l-1)k+7$. Also, $\rho(B_2(k)) \leq \rho(B_2)$ with equality if and only if $n = lk+5$. Thus for the proof of Lemma 2.13 it is sufficient to show that $\rho(B_2) < \rho(B_1)$. Let $r = \rho(B_2)$, by Lemma 2.12, it suffices to show that $\phi(B_1; r) < 0$.

By applying Lemma 2.11 to the vertex v of B_1 and B_2, respectively,

$$\phi(B_1;\lambda) = \lambda^2(\lambda^2-2)\phi(P_{l-1};\lambda)^{k-1}[(\lambda^3-6\lambda)\phi(P_{l-1};\lambda)-k(\lambda^2-2)\phi(P_{l-2};\lambda)];$$

$$\phi(B_2;\lambda) = \lambda^2\phi(P_l;\lambda)^{k-1}[(\lambda^3-6\lambda)\phi(P_l;\lambda)-k(\lambda^2-3)\phi(P_{l-1};\lambda)].$$

Note that $(r^3-6r)\phi(P_l;r) - k(r^2-3)\phi(P_{l-1};r) = 0$, where $l \geq 2$. Then for $l \geq 3$ we have $(r^3-6r)\phi(P_{l-1};r) - k(r^2-3)\phi(P_{l-2};r) = 0$. Hence $\phi(B_1;r) = -kr^2(r^2-2)\phi(P_{l-1};r)^{k-1}\phi(P_{l-2};r) < 0$ since $r > 2$. When $l = 2$, $\phi(B_1;r) = r^2(r^2-2)\phi(P_1;r)^{k-1}[(r^3-6r)\phi(P_1;r)-k(r^2-2)\phi(P_0;r)] < 0$ since $k = \frac{(r^3-6r)(r^2-1)}{(r^2-3)r}$.

Case2. $t = 1$.

In this case, it is straightforward to check that the maximal number of vertices of a path attached to the vertex v of $B_2(k)$ is l, while the minimal number of vertices of a path attached to the vertex v of $B_1(k)$ is $l-2$, where $l \geq 3$. Let B_1' be the graph analogous to $B_1(k)$ in which all paths attached to vertex v have $l-2$ vertices and B_2 be the graph analogous to $B_2(k)$ in which all paths attached to vertex v have l vertices. Evidently, B_1' is an induced subgraph of $B_1(k)$ and $B_2(k)$ is an induced subgraph of B_2. Therefore, $\rho(B_1') \leq \rho(B_1(k))$ with equality if and only if $n = (l-2)k+7$. Also, $\rho(B_2(k)) \leq \rho(B_2)$ with equality if and only if $n = lk+5$. Thus for the proof of Lemma 2.13 it is sufficient to show that $\rho(B_2) < \rho(B_1')$. Let $r = \rho(B_2)$, by Lemma 2.12, it suffices to show that $\phi(B_1';r) < 0$. Similar to Case1, we can easily prove the result. □

Let $B_3(k)$ denote the bicyclic graph in $\mathscr{B}(n,k)$ obtained from the graph G_3(see Fig.1) by attaching k paths with almost equal lengths to vertex v. Note that both $B_1(k)$ and $B_3(k)$ exist if and only if $k \geq 1, n \geq k+7$.

Lemma 2.14 *If both $B_1(k)$ and $B_3(k)$ exist, then $\rho(B_1(k)) > \rho(B_3(k))$.*

Proof. Denote by l the maximal number of vertices of a path attached to the vertex v of $B_3(k)$. Note that $k \leq n-7$, then $l \geq 2$.

Let B_1 be the graph analogous to $B_1(k)$ in which all paths attached to vertex v have $l-1$ vertices. Let B_3 be the graph analogous to $B_3(k)$ in which all paths attached to vertex v have l vertices. Evidently, B_1 is an induced subgraph of $B_1(k)$ and $B_3(k)$ is an induced subgraph of B_3. Therefore, $\rho(B_1) \leq \rho(B_1(k))$ with equality if and only if $n = (l-1)k+7$. Also, $\rho(B_3(k)) \leq \rho(B_3)$ with equality if and only if $n = lk+6$. Thus for the proof of Lemma 2.14 it is sufficient to show that $\rho(B_3) < \rho(B_1)$. By

Lemma 2.12, it suffices to show that $\phi(B_1;\rho(B_3)) < 0$. Similar to Case1 of Lemma 2.13, we can easily get the result. □

Let $\bar{B}_2(k)$ denote the bicyclic graph in $\mathscr{B}(n,k)$ obtained from the graph G_2(see Fig.1) by attaching k paths with almost equal lengths to vertex u. Let $\bar{B}_3(k)$ denote the bicyclic graph in $\mathscr{B}(n,k)$ obtained from the graph G_3(see Fig.1) by attaching k paths with almost equal lengths to vertex u.

Lemma 2.15 $\rho(B_2(k)) > \rho(\bar{B}_2(k))$ *and* $\rho(B_3(k)) > \rho(\bar{B}_3(k))$.

Proof. Assume that $k(k\geq 1)$ paths in $B_2(k)$ and $\bar{B}_2(k)$ are $P_{l_1},\ldots,P_{l_k}$. Apply Lemma 2.11 to the vertex v of $B_2(k)$ and the vertex u of $\bar{B}_2(k)$,

$$\begin{aligned}\phi(B_2(k);\lambda) &= (\lambda^5-6\lambda^3)\phi(P_{l_1};\lambda)\cdots\phi(P_{l_k};\lambda)-(\lambda^4-3\lambda^2)\\ &\quad \sum_{i=1}^{k}\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda),\end{aligned}$$

$$\begin{aligned}\phi(\bar{B}_2(k);\lambda) &= (\lambda^5-6\lambda^3)\phi(P_{l_1};\lambda)\cdots\phi(P_{l_k};\lambda)-(\lambda^4-4\lambda^2)\\ &\quad \sum_{i=1}^{k}\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda).\end{aligned}$$

For $\lambda\geq\rho(B_2(k))\geq 2.5576$, we have

$$\phi(\bar{B}_2(k);\lambda)-\phi(B_2(k);\lambda)=\lambda^2\sum_{i=1}^{k}\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda)>0.$$

By Lemma 2.12, $\rho(B_2(k)) > \rho(\bar{B}_2(k))$. Similarly, we can get $\rho(B_3(k)) > \rho(\bar{B}_3(k))$.

3 Characterization of the extremal graph

Let B^* have minimal least eigenvalue in $\mathscr{B}(n,k)$, and $B_1(k)\in\mathscr{B}(n,k)$. Note that $B_1(k)$ is a bipartite graph, hence $\lambda(B^*)\leq\lambda(B_1(k))=-\rho(B_1(k))< -\rho(G_1)\approx -2.4495$. Note that $B_1(k)$ exists if and only if $k\geq 1, n\geq k+7$. For any $B\in\mathscr{B}(n,k)$, we know that $B\in\mathscr{B}_1(n,k)$ or $B\in\mathscr{B}_2(n,k)$.

Lemma 3.1 *For any graph* $B\in\mathscr{B}_1(n,k)$*, there exists a graph* $\bar{B}\in\mathscr{B}_{11}(n,k)$*(see Fig.3)such that* $\lambda(B)\geq\lambda(\bar{B})$.

Proof. For any graph $B\in\mathscr{B}_1(n,k)$, B can be obtained from $B(p,l,q)$ by planting trees at a subset V_0 of $V(B(p,l,q))$, where $P_l: v_0v_1\cdots v_{l-1}$ is the unique path between C_p and C_q, any $u \in V_0$ is called the root of tree T_u, or the root-vertex of B. Let X be a unit eigenvector of B corresponding to $\lambda(B)$. Without loss of generality, let $|x_v| = \max\{|x_u||u \in V_0\}$. Let B_1 be the graph obtained from $B(p,l,q)$ by planting all the trees T_u at vertex u to form a new big tree T with root v. From a repeated use of Lemma 2.1, we have $\lambda(B) \geq \lambda(B_1)$. Consider the graph B_1. If $v \notin V(P_l)$, without loss of generality, we can assume that $v \in V(C_p) \setminus \{v_0\}$. Let X' be a unit eigenvector of B_1 corresponding to $\lambda(B_1)$. Denote $N(v) \cap V(T) = \{z_1, \ldots, z_{d_v-2}\}$ and $N(v_0) \cap V(P_l \cup C_q) = \{w_1, \ldots, w_{d_{v_0}-2}\}$. Define

$$B_2 = \begin{cases} B_1 - \{vz_1, \ldots, vz_{d_v-2}\} + \{v_0z_1, \ldots, v_0z_{d_v-2}\} & if \quad |x'_{v_0}| \geq |x'_v| \\ B_1 - \{v_0w_1, \ldots, v_0w_{d_{v_0}-2}\} + \{vw_1, \ldots, vw_{d_{v_0}-2}\} & if \quad |x'_{v_0}| < |x'_v| \end{cases}$$

Then in either case, the unique root-vertex $v = v_0 \in V(P_l)$ in B_2, and by Lemma 2.1, $\lambda(B_1) \geq \lambda(B_2)$. If $v \in V(P_l)$, then $B_2 = B_1$, and $\lambda(B_1) = \lambda(B_2)$. Hence for any graph B_1, we can always find a graph B_2 with $\lambda(B_1) \geq \lambda(B_2)$, and the unique root-vertex $v = v_i \in V(P_l)$ in B_2, where $0 \leq i \leq l-1$.

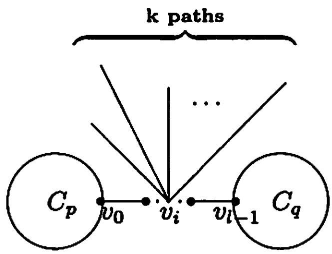

Fig.3. $\mathscr{B}_{11}(n,k)$.

Consider the graph B_2, let t be the cardinality of the vertices whose degrees are no less than 3 in $V(T) \setminus \{v_i\}$ and X'' be a unit eigenvector corresponding to $\lambda(B_2)$, and now we distinguish the following three cases:

Case1. $t = 0$. In this case, $\bar{B} = B_2 \in \mathscr{B}_{11}(n,k)$, and $\lambda(B_2) = \lambda(\bar{B})$.

Case2. $t = 1$. We can assume that there exists one vertex $v \in V(T) \setminus \{v_i\}$ with $d(v) \geq 3$ in B_2, then there is a unique path with the length at least 1 joining v_i and v. Denote $N(v_i) = \{w_1, w_2, w_3, \ldots, w_p\}$ and $N(v) = \{z_1, z_2, z_3, \ldots, z_q\}$. Assume that w_1, z_1 belong to the path joining v_i and v, w_2 lies on the path from v_0 to v_i. Define

$$\bar{B} = \begin{cases} B_2 - \{vz_3, \ldots, vz_q\} + \{v_iz_3, \ldots, v_iz_q\} & if \quad |x''_{v_i}| \geq |x''_v|; \\ B_2 - \{v_iw_3, \ldots, v_iw_p\} + \{vw_3, \ldots, vw_p\} & if \quad |x''_{v_i}| < |x''_v|. \end{cases}$$

Then in either case, $\bar{B} \in \mathscr{B}_{11}(n,k)$ and by Lemma 2.1, $\lambda(B_2) \geq \lambda(\bar{B})$.

Case3. $t > 1$. Suppose that $u, v \in V(T) \setminus \{v_i\}$ are two vertices of B_2 whose degrees are 3 or greater, and $|x''_u| \geq |x''_v|$. Since T is a tree, there is a path between u and v and only one neighbor of v, say w, is on the path. Assume $\{v_1, v_2, \ldots, v_{d_v-2}\} \subset N(v) \setminus \{w\}$. Delete the edges (v, v_j) and add the edges (u, v_j) $(1 \leq j \leq d_v - 2)$, then we get a new bicyclic graph B_1'. Obviously B_1' still has k pendant vertices. By Lemma 2.1, we have $\lambda(B_2) \geq \lambda(B_1')$ and the cardinality of the vertices of degree 3 or greater decreases to $t-1$. If $t-1 > 1$, to B_1' repeat the above step until the cardinality is only one. So we get bicyclic graphs $B_2', B_3', \ldots, B_{t-1}'$, and $\lambda(B_1') \geq \lambda(B_2') \geq \cdots \geq \lambda(B_{t-1}')$. Moreover, each B_i' has k pendant vertices. Refer to Case2, there exists $\bar{B} \in \mathscr{B}_{11}(n,k)$ such that $\lambda(B_{t-1}') \geq \lambda(\bar{B})$. By the above cases, we complete the proof. □

For any graph $B \in \mathscr{B}_2(n,k)$, B can be obtained from $P(p,l,q)$ by planting trees at a subset V_0 of $V(P(p,l,q))$, where any $u \in V_0$ is called the root of tree T_u, or the root-vertex of B. Similar to the proof of Lemma 3.1, we have the following result.

Lemma 3.2 *For any graph $B \in \mathscr{B}_2(n,k)$, there exists a graph $\bar{B} \in \mathscr{B}_{21}(n,k) \cup \mathscr{B}_{22}(n,k)$ (see Fig.4) such that $\lambda(B) \geq \lambda(\bar{B})$.*

Lemma 3.3 *Let B^* have minimal least eigenvalue in $\mathscr{B}(n,k)$, where $1 \leq k \leq n-7$. Then B^* contains no C_3.*

Proof. Note that $\mathscr{B}(n,k) = \mathscr{B}_1(n,k) \cup \mathscr{B}_2(n,k)$. Suppose that B^* contains C_3, then for the proof of Lemma 3.3 it suffices to find a graph $B \in \mathscr{B}_1(n,k) \cup \mathscr{B}_2(n,k)$ such that $\lambda(B^*) > \lambda(B)$.

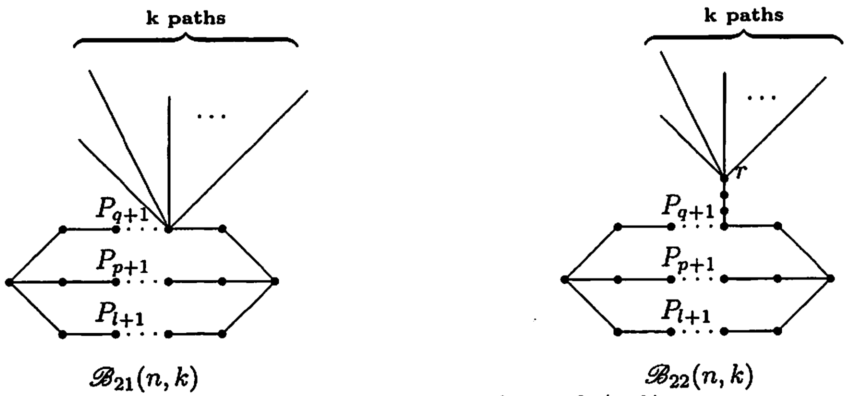

Fig.4. Two classes of graphs in $\mathscr{B}_2(n,k)$.

If $B^* \in \mathscr{B}_1(n,k)$. By Lemma 3.1, we may assume without loss of generality that $B^* \in \mathscr{B}_{11}(n,k)$. By the definition of the graph $B(p,l,q)$, we have $q \geq p \geq 3$ and $l \geq 1$. Suppose that B^* contains C_3, then $p = 3$. Let $N(v_0) \cap V(C_3) = \{u_1, u_2\}$, $C_q = v_{l-1}w_1w_2 \cdots w_{q-1}v_{l-1}$ be the other cycle of B^*. By Lemma 2.8, there exists a unit eigenvector X of $\lambda(B^*)$

such that $x_{u_1} = x_{u_2}$ and $x_{w_i} = x_{w_{q-i}}$, where $1 \leq i \leq q-1$. Note that $\lambda(B^*) < -2.4495$, by Lemmas 2.5, 2.9 and $AX = \lambda(B^*)X$, then $x_{v_0}, x_{v_{l-1}} \neq 0$(otherwise $X = 0$). Now we distinguish the following two cases:

Case1. $l = 1$, then we have $v_{l-1} = v_i = v_0$. According to Lemmas 2.5 and 2.9, then each component of X is not equal to zero.

Case1.1. $q \geq 5$ is odd.

Case1.1.1. $x_{u_1} x_{w_{\frac{q-1}{2}}} < 0$. Let $B = B^* - u_1u_2 - w_{\frac{q-1}{2}} w_{\frac{q+1}{2}} + u_1 w_{\frac{q-1}{2}} + u_2 w_{\frac{q+1}{2}}$. Clearly $B \in \mathscr{B}_1(n,k)$, and

$$\begin{aligned} \lambda(B) - \lambda(B^*) &\leq X^T(A(B) - A(B^*))X \\ &= -2(x_{u_1} - x_{w_{\frac{q-1}{2}}})^2 < 0, \end{aligned}$$

hence $\lambda(B^*) > \lambda(B)$.

Case1.1.2. $x_{u_1} x_{w_{\frac{q-1}{2}}} > 0$, then $q \geq 7$. Next, for convenience, let $\lambda(B^*) = \lambda$. By Lemma 2.9, we have $x_{v_0} = h_{\frac{q-1}{2}}(\lambda) h_{\frac{q-3}{2}}(\lambda) \cdots h_1(\lambda) x_{w_{\frac{q-1}{2}}}$ and $x_{v_0} = (\lambda - 1) x_{u_1}$. By Lemma 2.6, $h_i(\lambda) < -1$, hence

$$\begin{aligned} |x_{v_0}| &= |h_{\frac{q-1}{2}}(\lambda) h_{\frac{q-3}{2}}(\lambda) \cdots h_1(\lambda)||x_{w_{\frac{q-1}{2}}}| \\ &\geq |h_3(\lambda) h_2(\lambda) h_1(\lambda)||x_{w_{\frac{q-1}{2}}}| \\ &= |\lambda^3 - \lambda^2 - 2\lambda + 1||x_{w_{\frac{q-1}{2}}}|. \end{aligned}$$

Since $\lambda < -2.4495$, then $|\lambda^3 - \lambda^2 - 2\lambda + 1| > |\lambda - 1|$, and we have $|x_{w_{\frac{q-1}{2}}}| \neq |x_{u_1}|$. Let $B = B^* - u_1u_2 - w_{\frac{q-1}{2}} w_{\frac{q+1}{2}} + u_1 w_{\frac{q-1}{2}} + u_2 w_{\frac{q+1}{2}}$. Clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$, since $x_{u_1} = x_{u_1}$ and $x_{w_{\frac{q-1}{2}}} = x_{w_{\frac{q+1}{2}}}$.

Case1.2. $q \geq 6$ is even.

Case1.2.1. $\frac{q}{2}$ is odd, then $x_{u_1} x_{w_{\frac{q}{2}}} > 0$. By Lemma 2.9, we have $x_{v_0} = g_{\frac{q}{2}}(\lambda) g_{\frac{q-2}{2}}(\lambda) \cdots g_1(\lambda) x_{w_{\frac{q}{2}}}$ and $x_{v_0} = (\lambda - 1) x_{u_1}$. By Lemma 2.6, $g_i(\lambda) < -1$, hence

$$\begin{aligned} |x_{v_0}| &= |g_{\frac{q}{2}}(\lambda) g_{\frac{q-2}{2}}(\lambda) \cdots g_1(\lambda)||x_{w_{\frac{q}{2}}}| \\ &\geq |g_3(\lambda) g_2(\lambda) g_1(\lambda)||x_{w_{\frac{q}{2}}}| \\ &= |\frac{1}{2}(\lambda^3 - 3\lambda)||x_{w_{\frac{q}{2}}}|. \end{aligned}$$

Since $\lambda < -2.4495$, then $|\frac{1}{2}(\lambda^3 - 3\lambda)| > |\lambda - 1|$, and we have $|x_{w_{\frac{q}{2}}}| < |x_{u_1}|$. Let $B = B^* - u_1u_2 - w_{\frac{q-2}{2}} w_{\frac{q}{2}} + u_1 w_{\frac{q-2}{2}} + u_2 w_{\frac{q}{2}}$. Clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$.

Case1.2.2. $\frac{q}{2}$ is even, then $q \geq 8$. Note that $\frac{q-2}{2} \geq 3$ is odd, hence $x_{u_1}x_{w_{\frac{q-2}{2}}} > 0$. Similar to Case1.2.1, we can show that $|x_{w_{\frac{q-2}{2}}}| < |x_{u_1}|$. Let $B = B^* - u_1u_2 - w_{\frac{q-2}{2}}w_{\frac{q}{2}} + u_1w_{\frac{q-2}{2}} + u_2w_{\frac{q}{2}}$. Clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$.

Case1.3. $q = 3$ or 4. If there exists a pendant vertex s in B^* such that $d_{B^*}(s, v_0)$ is even, then by Lemma 2.5, $x_{v_0}x_s > 0$. Note that $x_{u_1}x_s < 0$. Let $B = B^* - u_1u_2 + u_1s$, clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$. Otherwise for any pendant vertex t in B^*, $d_{B^*}(t, v_0)$ is odd. Since $k \leq n - 7$, then there must exist a pendant path with length $s \geq 3$. We have $x_{u_1}x_s > 0$. And next we will show that $|x_s| < |x_{u_2}|$. By Lemma 2.5, we have $x_{v_0} = f_s(\lambda)f_{s-1}(\lambda)\cdots f_1(\lambda)x_s$ and $x_{v_0} = (\lambda - 1)x_{u_2}$. By Lemma 2.4, $f_i(\lambda) < -1$,

$$\begin{aligned} |x_{v_0}| &= |f_s(\lambda)f_{s-1}(\lambda)\cdots f_1(\lambda)||x_s| \\ &\geq |f_3(\lambda)f_2(\lambda)f_1(\lambda)||x_s| \\ &= |\lambda^3 - 2\lambda||x_s|. \end{aligned}$$

Since $\lambda < -2.4495$, then $|\lambda^3 - 2\lambda| > |\lambda - 1|$, and we have $|x_s| < |x_{u_2}|$. Let $B = B^* - u_1u_2 + u_1s$, clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$.

Case2. $l \geq 2$. We have $v_0 \neq v_{l-1}$.

Define a new vector Y from X as follows: $y_i = (-1)^{d_{B^*}(i,v_0)}|x_i|$ for each vertex $i \in V(B^*)$, then $y_{v_0} > 0$. By Lemma 2.7, Y is also a unit eigenvector of $\lambda(B^*)$.

Case2.1. $i = 0$, i.e. $v_i = v_0$.

Clearly $|y_{v_0}| > |y_{v_{l-1}}|$.(Otherwise $|y_{v_0}| \leq |y_{v_{l-1}}|$. Let B be the graph obtained from B^* by removing all the pendant paths at v_0 to v_{l-1}, clearly $B \in \mathscr{B}_1(n,k)$, and by Lemma 2.1, we have $\lambda(B^*) \geq \lambda(B)$. If $\lambda(B^*) = \lambda(B)$, then $y_{v_0} = y_{v_{l-1}} = 0$, hence $Y = 0$, a contradiction. So we have $\lambda(B^*) > \lambda(B)$.) When $q \geq 3$ is odd, by Lemma 2.9, $|y_{v_0}| = |h_1(\lambda)||y_{u_1}|$ and $|y_{v_{l-1}}| = |h_{\frac{q-1}{2}}(\lambda)h_{\frac{q-3}{2}}(\lambda)\cdots h_1(\lambda)||y_{w_{\frac{q-1}{2}}}| \geq |h_1(\lambda)||y_{w_{\frac{q-1}{2}}}|$, hence $|y_{u_1}| > |y_{w_{\frac{q-1}{2}}}|$. Let $B = B^* - u_1u_2 - w_{\frac{q-1}{2}}w_{\frac{q+1}{2}} + u_1w_{\frac{q-1}{2}} + u_2w_{\frac{q+1}{2}}$. Clearly $B \in \mathscr{B}_2(n,k)$ and $\lambda(B^*) > \lambda(B)$. When $q \geq 4$ is even. If $d_{B^*}(v_0, v_{l-1})$ is even, by the definition of Y, $y_{w_1}y_{v_0} < 0$ and $y_{w_1}y_{v_{l-1}} < 0$. Let $B = B^* - w_1v_{l-1} + w_1v_0$. Clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$. Otherwise $d_{B^*}(v_0, v_{l-1})$ is odd, then $y_{v_{l-1}} < 0$. By Lemma 2.9, $|y_{u_1}| > |y_{w_{\frac{q-2}{2}}}| > |y_{w_{\frac{q}{2}}}|$. Let $B = B^* - u_1u_2 - w_{\frac{q-1}{2}}w_{\frac{q+1}{2}} + u_1w_{\frac{q-1}{2}} + u_2w_{\frac{q+1}{2}}$. Clearly $B \in \mathscr{B}_2(n,k)$ and $\lambda(B^*) > \lambda(B)$.

Case2.2. $1 \leq i \leq l - 1$.

Clearly $|y_{v_0}| < |y_{v_i}|$.(Otherwise $|y_{v_0}| \geq |y_{v_i}|$. Let B be the graph obtained from B^* by removing all the pendant paths at v_i to v_0, clearly $B \in \mathscr{B}_1(n,k)$, and by Lemma 2.1, we have $\lambda(B^*) \geq \lambda(B)$. If $\lambda(B^*) = \lambda(B)$, then

$y_{v_0} = y_{v_i} = 0$, hence $Y = 0$, a contradiction. So we have $\lambda(B^*) > \lambda(B)$.) If $d_{B^*}(v_0, v_i)$ is even, by the definition of Y, $y_{u_1}y_{v_0} < 0$ and $y_{u_1}y_{v_i} < 0$. Let $B = B^* - u_1v_0 + u_1v_i$. Clearly $B \in \mathscr{B}_1(n,k)$ and $\lambda(B^*) > \lambda(B)$. Otherwise $d_{B^*}(v_0, v_i)$ is odd. Define a new vector Z from Y as follows: $z_{u_1} = -y_{u_1}$ and $z_i = y_i, i \in V(B^*) \setminus \{u_1\}$. Let $B = B^* - u_1v_0 + u_1v_i$, clearly $B \in \mathscr{B}_1(n,k)$. And

$$\begin{aligned}\lambda(B^*) - \lambda(B) &\geq Y^T A(B^*)Y - Z^T A(B)Z \\ &= 2(y_{u_1}^2 + y_{v_0}y_{u_1} + y_{u_1}^2 + y_{v_i}y_{u_1}) \\ &> 2|y_{u_1}|(|y_{v_i}| - |y_{v_0}|) > 0.\end{aligned}$$

If $B^* \in \mathscr{B}_2(n,k)$. By Lemma 3.2, we may assume without loss of generality that $B^* \in \mathscr{B}_{21}(n,k) \cup \mathscr{B}_{22}(n,k)$. By the definition of the graph $P(p,l,q)$, we have $l \geq q \geq p \geq 1$ and at most one of them is 1. Suppose that B^* contains C_3, then $p = 1, q = 2, l \geq 2$.

Let X be a unit eigenvector of B^* corresponding to $\lambda(B^*)$. Let u and v be two common endpoints of P_2, P_3, P_{l+1}, w be the other vertex of C_3. For even $l \geq 4$ or odd $l \geq 3$. If $x_u x_v \geq 0$, without loss of generality letting $x_u \geq 0, x_v \geq 0$, then except u, v the path P_{l+1} must contain a vertex with negative value given by X. Deleting uv and adding edge between u(or v)and the negative vertex, we get a graph $B \in \mathscr{B}_2(n,k)$ with $\lambda(B^*) \geq \lambda(B)$. If $\lambda(B^*) = \lambda(B)$, then x_u(or x_v)$= 0$. Considering the equation $A(B^*)X = \lambda(B^*)X$ at vertex u(or v), we can deduce that $X = 0$, a contradiction. Hence $\lambda(B^*) > \lambda(B)$. If $x_u x_v < 0$, without loss of generality letting $x_u > 0, x_v < 0, x_w \geq 0$, then except u, v the path P_{l+1} must contain a negative vertex. Deleting wu and adding edge between w and the negative vertex, we get a graph $B \in \mathscr{B}_2(n,k)$ with $\lambda(B^*) \geq \lambda(B)$. If $\lambda(B^*) = \lambda(B)$, then $x_w = 0$. Considering the equation $A(B^*)X = \lambda(B^*)X$ at vertex w, u, v, we can deduce that $X = 0$, a contradiction. Hence $\lambda(B^*) > \lambda(B)$.

For even $l = 2$. When v is the root-vertex of B^*. If $x_u x_v \geq 0$, without loss of generality letting $x_u \geq 0, x_v \geq 0$. Since $1 \leq k \leq n-7$, then there exists a pendant path containing a non-pendant negative vertex t such that $d(v,t) = 1$. Deleting uv and joining ut, we get a graph $B \in \mathscr{B}_2(n,k)$ with $\lambda(B^*) > \lambda(B)$. If $x_u x_v < 0$, without loss of generality letting $x_u < 0, x_v > 0$. If $x_w \geq 0$, deleting wv and joining wt, we get a graph $B \in \mathscr{B}_2(n,k)$ with $\lambda(B^*) > \lambda(B)$. Otherwise $x_w < 0$, there exists a pendant path containing a non-pendant positive vertex q such that $d(v,q) = 2$. Deleting wu and joining wq, we get a graph $B \in \mathscr{B}_1(n,k)$ with $\lambda(B^*) > \lambda(B)$. When w is the root-vertex of B^*, similar to the above proof, we can find $B \in \mathscr{B}_1(n,k) \cup \mathscr{B}_2(n,k)$ such that $\lambda(B^*) > \lambda(B)$. □

Lemma 3.4 *Let B^* have minimal least eigenvalue in $\mathscr{B}(n,k)$, where $1 \leq k \leq n-7$. If $B^* \in \mathscr{B}_1(n,k)$, then $p = q = 4$. If $B^* \in \mathscr{B}_2(n,k)$, then $B^* \cong B_2(k)$ or $B_3(k)$.*

Proof. By the Lemma 3.3, B^* contains no C_3.

When $B^* \in \mathscr{B}_1(n,k)$, by Lemma 3.1, we may assume without loss of generality that $B^* \in \mathscr{B}_{11}(n,k)$. By the definition of the graph $B(p,l,q)$, we have $q \geq p \geq 3$. Note that B^* contains no C_3, then $q \geq p \geq 4$. If $q \geq 5$. Let B be the graph obtained from B^* by contracting C_q to C_4 and adding a pendant path P_{q-4} to a pendant vertex of B^*, then $B \in \mathscr{B}_1(n,k)$ and B is bipartite. By Lemma 2.2, $\rho(B^*) < \rho(B)$. Thus $\lambda(B^*) \geq -\rho(B^*) > -\rho(B) = \lambda(B)$. Hence $p = q = 4$.

When $B^* \in \mathscr{B}_2(n,k)$, by Lemma 3.2, we may assume without loss of generality that $B^* \in \mathscr{B}_{21}(n,k) \cup \mathscr{B}_{22}(n,k)$. By the definition of the graph $P(p,l,q)$, we have $l \geq q \geq p \geq 1$ and at most one of them is 1. Note that B^* contains no C_3, then we distinguish the following two cases:

Case1. $2 \leq p \leq q \leq l$.

If $3 \leq p \leq q \leq l$. Let B be the graph obtained from B^* by contracting $P(p,l,q)$ to $P(2,2,2)$ and adding a pendant path $P_{p+q+l-6}$ to a pendant vertex of B^*, then $B \in \mathscr{B}_2(n,k)$ and B is bipartite. By Lemma 2.2, $\rho(B^*) < \rho(B)$. Thus $\lambda(B^*) \geq -\rho(B^*) > -\rho(B) = \lambda(B)$. If $p = 2, 3 \leq q \leq l$ or $p = 2, q = 2, 3 \leq l$, similar to the above proof, there exists $B \in \mathscr{B}_2(n,k)$ such that $\lambda(B^*) > \lambda(B)$. Hence $p = q = l = 2$. Clearly $P(2,2,2) = G_2$(see Fig.1). Note that at this moment B^* is bipartite, we consider $\rho(B^*)$. If $v \in G_2$ is the root-vertex of B^*, by Lemmas 2.2, 2.3 and 2.10, we have $B^* \cong B_2(k)$. If $u \in G_2$ is the root-vertex of B^*, then $B^* \cong \bar{B}_2(k)$. Combining Lemma 2.15, $\rho(B_2(k)) > \rho(\bar{B}_2(k))$. Hence $B^* \cong B_2(k)$.

Case2. $p = 1, 3 \leq q \leq l$.

Similar to the proof of Case1, then $p = 1, q = l = 3$. Clearly $P(3,1,3) = G_3$(see Fig.1). Note that at this moment B^* is bipartite, we consider $\rho(B^*)$. If $v \in G_3$ is the root-vertex of B^*, by Lemmas 2.2, 2.3 and 2.10, we have $B^* \cong B_3(k)$. If $u \in G_3$ is the root-vertex of B^*, then $B^* \cong \bar{B}_3(k)$. Combining Lemma 2.15, $\rho(B_3(k)) > \rho(\bar{B}_3(k))$. Hence $B^* \cong B_3(k)$. □

Let B^* have minimal least eigenvalue in $\mathscr{B}(n,k)$. When $B^* \in \mathscr{B}_1(n,k)$, clearly, B^* have minimal least eigenvalue in $\mathscr{B}_1(n,k)$. By Lemma 3.1, we may assume without loss of generality that $B^* \in \mathscr{B}_{11}(n,k)$. Let $\mathscr{B}_{11}^*(n,k)$ be the set of bicyclic graphs in $\mathscr{B}_{11}(n,k)$ with $p = q = 4$. According to Lemma 3.4, $B^* \in \mathscr{B}_{11}^*(n,k)$. Note that $\lambda(G) = -\rho(G)$ for any bipartite graph. Thus the problem minimizing the least eigenvalue in $\mathscr{B}_1(n,k)$ is equivalent to that of maximizing the spectral radius in $\mathscr{B}_{11}^*(n,k)$.

Theorem 3.5 *Let B^* have maximal spectral radius in $\mathscr{B}_{11}^*(n,k)$, where $1 \leq k \leq n-7$. Then $B^* \cong B_1(k)$.*

Proof. If $l = 1$, then we have $v_{l-1} = v_i = v_0$. From a repeated use of Lemma 2.10, we have $B^* \cong B_1(k)$. Otherwise $l \geq 2$, then $0 \leq i \leq l-1$. For any $B^* \in \mathscr{B}_{11}^*(n,k)$, by Lemmas 2.2 and 2.3, there exists $B \in \mathscr{B}_{11}^*(n,k)$ such that $\rho(B) > \rho(B^*)$, a contradiction. Hence $B^* \cong B_1(k)$. □

Proof of Theorem 1.1 According to Theorem 3.5 and Lemma 3.4, $B^* \cong B_1(k), B_2(k)$ or $B_3(k)$. By Lemmas 2.13 and 2.14, we have $\rho(B_1(k)) > \rho(B_2(k))$ and $\rho(B_1(k)) > \rho(B_3(k))$. Note that these three bicyclic graphs are all bipartite, hence $\lambda(B_1(k)) < \lambda(B_2(k))$ and $\lambda(B_1(k)) < \lambda(B_3(k))$. So $B^* \cong B_1(k)$.

Let $1 \leq k < n-7$. It follows that there exists a pendant path $P_l = v_1 v_2 \cdots v_l$ attached to the root vertex v_1 of $B_1(k)$ such that $l \geq 3$. Let $B = B_1(k) - \{v_{l-1}v_l\} + \{v_1 v_l\}$. Then $B \in \mathscr{B}_1(n, k+1)$. By Lemma 2.10, we have $\rho(B_1(k)) < \rho(B)$. Note that $B_1(k)$ and B are bipartite, then $\lambda(B_1(k)) > \lambda(B)$. By Theorem 3.5, we have $\lambda(B) \geq \lambda(B_1(k+1))$. Hence $\lambda(B_1(k)) > \lambda(B_1(k+1))$. Hence $\lambda(B_1(k))$ is a decreasing function on k. This completes the proof of Theorem 1.1. □

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