Existence for perfect $T(K_{1,k})$-triple systems*

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Abstract: Let G be a subgraph of K_n. The graph obtained from G by replacing each edge with a 3-cycle whose third vertex is distinct from other vertices in the configuration is called a $T(G)$-triple. An edge-disjoint decomposition of $3K_n$ into copies of $T(G)$ is called a $T(G)$-triple system of order n. If, in each copy of $T(G)$ in a $T(G)$-triple system, one edge is taken from each 3-cycle (chosen so that these edges form a copy of G) in such a way that the resulting copies of G form an edge-disjoint decomposition of K_n, then the $T(G)$-triple system is said to be perfect. The set of positive integers n for which a perfect $T(G)$-triple system exists is called its spectrum. Earlier papers by authors including Billington, Lindner, Küçükçifçi and Rosa determined the spectra for cases where G is any subgraph of K_4. In this paper, we will focus in star graph $K_{1,k}$ and discuss the existence for perfect $T(K_{1,k})$-triple system. Especially, for prime powers k, its spectra are completely determined.

Keywords: $T(G)$-triple; perfect $T(G)$-triple system; star graph.

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1 Introduction

Denote an edge in K_n on vertices x and y by xy or yx, and denote a 3-cycle on vertices x, y, z by (x, y, z) or (x, z, y) (or any cyclic shift of these). Let G be a subgraph of K_n. Let $T(G) = \{(a, b, c) : ab \in E(G)\}$ be a collection of 3-cycles satisfying:

(i) if $(a, b, c) \in T(G)$ and $ab \in E(G)$, then $c \notin V(G)$, and

(ii) if $(a_i, b_i, c_i) \in T(G)$, $i = 1, 2$, with $a_1b_1, a_2b_2 \in E(G)$, then $c_1 \neq c_2$.

The graph formed in this way, by taking a triangle or triple on each edge of G, will be called a $T(G)$-triple. In a $T(G)$-triple, the vertices and edges of G are called *interior*, but the vertices and edges of $T(G) - G$ are called *exterior*.

A *$T(G)$-triple system* of order n, denoted by $T(G, n)$ briefly, is a pair $(X, \mathcal{B})$ where X is the vertex set of K_n and $\mathcal{B}$ is an edge-disjoint collection of $T(G)$-triples which partitions the edges set of $3K_n$. If the interior edges of the $T(G)$-triples (which form the copies of G) partition the edge set of K_n (with vertex set X), then $(X, \mathcal{B})$ is said to be a *perfect* $T(G)$-triple system. The *spectrum* for perfect $T(G)$-triple system is the set of all positive integers n for which there exists a perfect $T(G)$-triple system of order n. The concepts of $T(G)$-triple, $T(G)$-triple system and perfect $T(G)$-triple system were firstly introduced by S. Küçükçifçi and C. C. Lindner in [4].

A *holey* $T(G)$-triple system with m h-holes, denoted by $T(G, h^m)$, is a pair $(\{S_1, \cdots, S_m\}, \mathcal{A})$, where each S_i is a h-set (or hole), these S_i are pairwise disjoint and $\mathcal{A}$ is a collection of $T(G)$-triples which partitions all edges joining the vertices in distinct holes. An *incomplete* $T(H)$-triple system on the set $X - Y$, denoted by $T(H, v : h)$, is a trio $(X, Y, \mathcal{C})$, where $Y \subset X$, $|X| = v$, $|Y| = h$ and $\mathcal{C}$ is a collection of $T(H)$-triples which partitions the edges of X, that are not in Y.

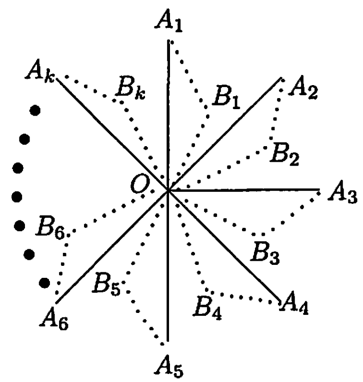
Lemma 1.1 *Let H be a simple graph with e edges. If there exists a $T(H, v)$, then $2e|v(v-1)$ and v is odd. Specially, the orders $v \equiv 1$ mod $2e$ and the orders $v \equiv e$ mod $2e$ (for odd e) satisfy the necessary conditions.*

Proof. First, it is easy to see that the degree of each exterior-vertex is two and the degree of each interior-vertex is even. Thus, the greatest common divisor d of the degrees of all vertices is 2. By the definition of $T(H)$-triple system and the necessary condition for existence of graph design, we have

$$3e|3\binom{v}{2} \text{ and } d|3(v-1) \Longrightarrow 2e|v(v-1) \text{ and } v \text{ is odd.}$$

Of course, the orders $v \equiv 1 \bmod 2e$ and the orders $v \equiv e \bmod 2e$ (for odd e) satisfy the above conditions. ■

To date, the spectrum for perfect $T(G)$-triple system has been determined for all subgraphs G of K_4 (see [1, 2, 4, 5]). In this paper, we will focus in star graph $K_{1,k}$ and discuss the existence of perfect $T(K_{1,k})$-triple system. The block in a $T(K_{1,k}, v)$ can be expressed two forms in right.



$$(O : A_1, A_2, \cdots, A_k; B_1, B_2, \cdots, B_k)$$

$$\begin{pmatrix} O & \begin{matrix} A_1 & A_2 & \cdots & A_k \\ B_1 & B_2 & \cdots & B_k \end{matrix} \end{pmatrix}$$

Sometimes, the vertex sequences $A_1, A_2, \cdots, A_k$ and $B_1, B_2, \cdots, B_k$ can be replaced by some integer intervals. Let a, b be integers, and $a \le b$. So-called *integer interval* $[a, b]$ represents the ordered set $\{a, a+1, \cdots, b-1, b\}$. If $a \equiv b \bmod t$, the *generalized integer interval* $[a, b]_t$ represents the ordered set $\{a, a+t, \cdots, b-t, b\}$. Furthermore, denote $[a, b]^- = \{b, b-1, \cdots, a+1, a\}$ and $[a, b]_t^- = \{b, b-t, \cdots, a+t, a\}$.

In this paper, the construction for perfect $T(H)$-triple system will use the difference method. The elements in $Z_n^* = Z_n \backslash \{0\} = \{1, 2, \cdots, n-1\}$ can be written as

$$\{1, 2, \cdots, \tfrac{n-1}{2}, -1, -2, \cdots, -\tfrac{n-1}{2}\} \text{ for odd } n; \text{ or}$$
$$\{1, 2, \cdots, \tfrac{n-2}{2}, \tfrac{n}{2}, -1, -2, \cdots, -\tfrac{n-2}{2}\} \text{ for even } n.$$

Using this notation, the ordered differences in Z_n are $\{0, 1, 2, \cdots, n-1\}$, but the unordered differences in Z_n are $\{1, 2, \cdots, \frac{n-1}{2}\}$ or $\{1, 2, \cdots, \frac{n-2}{2}, \frac{n}{2}\}$. The equivalent transformation from an ordered difference set A to an unordered difference set B is written as $A \to B$.

Lemma 1.2 *Let n be positive integers, then*

(1). In Z_{4n} or Z_{4n+2}, $[1-2n, 2n-3]_4 \to [1, 2n-1]_2$,
$[3-2n, 2n-1]_4 \to [1, 2n-1]_2$;

(2). In Z_{4n+2}, $[2,4n-2]_4 \to [2,2n]_2$ *and* $[4,4n]_4 \to [2,2n]_2$;

(3). In Z_{6n+4}, $[1-3n,3n-3]_4 \to [1,3n-1]_2$ *for* $n \equiv 0$ *mod 4*,

$[-3n-1,3n-1]_4 \to [1,3n+1]_2$ *for* $n \equiv 2$ *mod 4*;

(4). For $a \in Z_{2n+1}$, $1 \le a \le n$, $[a,2n-a]_2 \to [a,n]$ *for odd* a,

$[a,2n+2-a]_2 \to [a-1,n]$ *for even* a.

Proof. (1) When $n \equiv 0$ mod 2, then

$$[1-2n,2n-3]_4 = [1-2n,-3]_4 \cup [1,2n-3]_4 \to [3,2n-1]_4 \cup [1,2n-3]_4 = [1,2n-1]_2,$$

$$[3-2n,2n-1]_4 = [3-2n,-1]_4 \cup [3,2n-1]_4 \to [1,2n-3]_4 \cup [3,2n-1]_4 = [1,2n-1]_2;$$

When $n \equiv 1$ mod 2, then

$$[1-2n,2n-3]_4 = [1-2n,-1]_4 \cup [3,2n-3]_4 \to [1,2n-1]_4 \cup [3,2n-3]_4 = [1,2n-1]_2;$$

$$[3-2n,2n-1]_4 = [3-2n,-3]_4 \cup [1,2n-1]_4 \to [3,2n-3]_4 \cup [1,2n-1]_4 = [1,2n-1]_2.$$

(2) When $n \equiv 0$ mod 2, then

$$[2,4n-2]_4 = [2,2n-2]_4 \cup [2n+2,4n-2]_4 \to [2,2n-2]_4 \cup [4,2n]_4 = [2,2n]_2,$$

$$[4,4n]_4 = [4,2n]_4 \cup [2n+4,4n]_4 \to [4,2n]_4 \cup [2,2n-2]_4 = [2,2n]_2.$$

When $n \equiv 1$ mod 2, then

$$[2,4n-2]_4 = [2,2n]_4 \cup [2n+4,4n-2]_4 \to [2,2n]_4 \cup [4,2n-2]_4 = [2,2n]_2,$$

$$[4,4n]_4 = [4,2n-2]_4 \cup [2n+2,4n]_4 \to [4,2n-2]_4 \cup [2,2n]_4 = [2,2n]_2.$$

(3) When $n \equiv 2$ mod 4, then

$$[-3n-1,3n-1]_4 \to [-3n-1,-3]_4 \cup [1,3n-1]_4 \to [3,3n+1]_4 \cup [1,3n-1]_4 = [1,3n+1]_2.$$

When $n \equiv 0$ mod 4, then

$$[1-3n,3n-3]_4 \to [1-3n,-3]_4 \cup [1,3n-3]_4 \to [3,3n-1]_4 \cup [1,3n-3]_4 = [1,3n-1]_2.$$

(4) When $n \equiv 0$ mod 2, then

$$[a,2n+2-a]_2 = [a,n]_2 \cup [n+2,2n+2-a]_2 \to [a,n]_2 \cup [a-1,n-1]_2 = [a-1,n] \text{ for even } a;$$

$$[a,2n-a]_2 = [a,n-1]_2 \cup [n+1,2n-a]_2 \to [a,n-1]_2 \cup [a+1,n]_2 = [a,n] \text{ for odd } a.$$

When $n \equiv 1$ mod 2, then

$$[a,2n+2-a]_2 = [a,n-1]_2 \cup [n+1,2n+2-a]_2 \to$$

$$[a, n-1]_2 \cup [a-1, n]_2 = [a-1, n], \text{ for even } a;$$
$$[a, 2n-a]_2 = [a, n]_2 \cup [n+2, 2n-a]_2 \rightarrow$$
$$[a, n]_2 \cup [a+1, n-1]_2 = [a, n], \text{ for odd } a. \qquad \blacksquare$$

In $Z_n \times Z_m$, the difference between $Z_n \times \{i\}$ and $Z_n \times \{j\}$, $i, j \in Z_m$, is denoted by (i, j)-difference, which is named *pure* $(i = j)$ or *mixed* $(i \neq j)$.

2 A recurrence method

Theorem 2.1 *Let H be a simple graph with e edges. If there exist $T(H, 2e+1), T(H, 4e+1)$ and $T(H, e^3)$, then there exists a $T(H, 2me+1)$ for any positive integer m.*

Construction. Take the vertex set $(Z_{2m} \times Z_e) \cup \{\infty\}$. The block set of $T(H, 2me+1)$ consists of $(2me+1)m$ blocks. From [3], for $m \geq 3$, there exist

$$3\text{-}GDD(2^m) = (Z_{2m}, \{G_j : 1 \leq j \leq m\}, \mathcal{B}) \text{ for } 3 \not| (m-2), \text{ where}$$
$$G_j = \{2j-1, 2j\},\ 1 \leq j \leq m;$$
$$3\text{-}GDD(2^{m-2}4^1) = (Z_{2m}, \{G_j : 0 \leq j \leq m-2\}, \mathcal{B}) \text{ for } 3|(m-2), \text{ where}$$
$$G_0 = \{1, 2, 3, 4\} \text{ and } G_j = \{2j+3, 2j+4\},\ 1 \leq j \leq m-2.$$

For the group G_0, $|G_0| = 4$, let $((G_0 \times Z_e) \cup \{\infty\}, \mathcal{A}_0)$ be a $T(H, 4e+1)$. For each group G_j, $|G_j| = 2$, let $((G_j \times Z_e) \cup \{\infty\}, \mathcal{A}_j)$ be a $T(H, 2e+1)$. For each triple $B \in \mathcal{B}$, let $(\{\{a\} \times Z_e : a \in B\}, \mathcal{C}_B)$ be a $T(H, e^3)$. Then,

$$\Omega = (\bigcup_{B \in \mathcal{B}} \mathcal{C}_B) \cup (\bigcup_{j \in J} \mathcal{A}_j)$$

forms a $T(H, 2me+1)$, where $J = \{1, \cdots, m\}$ if $3 \not| (m-2)$ or $J = \{0, \cdots, m-2\}$ if $3|(m-2)$.

Proof. First, we have the following enumeration:

$$|\mathcal{A}_0| = 2(4e+1), \quad |\mathcal{A}_j| = 2e+1 \text{ for } j \neq 0, \quad |\mathcal{C}_B| = \frac{3\binom{3}{2}e^2}{3e} = 3e,$$

$$|\mathcal{B}| = \begin{cases} \frac{\binom{m}{2}2^2}{3} = \frac{2m(m-1)}{3} & \text{if } 3 \not| (m-2) \\ \frac{\binom{m-2}{2}2^2 + 8(m-2)}{3} = \frac{2(m-2)(m+1)}{3} & \text{if } 3|(m-2) \end{cases},$$

$$|\Omega| = \begin{cases} \frac{2m(m-1)}{3} \cdot 3e + m(2e+1) = (2me+1)m. \\ \frac{2(m-2)(m+1)}{3} \cdot 3e + 2(4e+1) + (m-2)(2e+1) = (2me+1)m. \end{cases}$$

The number $|\Omega|$ is just the block number in a $T(H, 2me+1)$. Furthermore,

$\forall x \in Z_{2m}, \exists G_j$ containing $x \Longrightarrow \forall i \in Z_e$, $\{(x,i),\infty\}$ appears in three blocks of $\mathcal{A}_j$, where exactly one edge is interior.

$\forall\ (x,i) \neq (x',i') \in Z_{2m} \times Z_e$, $x \in G_j$ and $x' \in G_{j'}$,

if $j \neq j'$, then $\exists B \in \mathcal{B}$ such that $x, x' \in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{C}_B$, where exactly one edge is interior.

if $j = j'$, then $x, x' \in G_j \Longrightarrow \{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{A}_j$, where exactly one edge is interior. ■

Theorem 2.2 *Let H be a simple graph with odd e edges. If there exist $T(H,3e), T(H,5e), T(H,e^3)$ and $T(H,3e:e)$, then there exists $T(H,2me+e)$ for any $m > 0$.*

Construction. Take the vertex set $(Z_{2m} \cup \{\infty\}) \times Z_e$. There are $(2m+1)\cdot\frac{(2m+1)e-1}{2}$ blocks in a $T(H,2me+e)$. From [3], for $m \geq 3$, there exist

$$3\text{-}GDD(2^m) = (Z_{2m}, \{G_j : 0 \leq j \leq m-1\}, \mathcal{B}) \text{ for } 3 \not|(m-2),$$
$$\text{where } G_j = \{2j+1, 2j+2\},\ 0 \leq j \leq m-1;$$
$$3\text{-}GDD(2^{m-2}4^1) = (Z_{2m}, \{G_j : 0 \leq j \leq m-2\}, \mathcal{B}) \text{ for } 3|(m-2),$$
$$\text{where } G_0 = \{1,2,3,4\} \text{ and } G_j = \{2j+3, 2j+4\},\ 1 \leq j \leq m-2.$$

For the group G_0, let $((G_0 \cup \{\infty\}) \times Z_e, \mathcal{A}_0)$ be a $T(H,3e)$ if $3 \not|(m-2)$ or a $T(H,5e)$ if $3|(m-2)$. For each group G_j, $j \neq 0$, there exists a $T(H,3e:e) = (((G_j \cup \{\infty\}) \times Z_e, \{\infty\} \times Z_e), \mathcal{A}_j)$. For each triple $B \in \mathcal{B}$, there exists a $T(H,e^3) = (\{\{a\} \times Z_e : a \in B\}, \mathcal{C}_B)$. Then,

$$\Omega = (\bigcup_{B \in \mathcal{B}} \mathcal{C}_B) \cup (\bigcup_{j=0}^{s} \mathcal{A}_j)$$

forms a $T(H,2me+e)$, where $s = m-1$ if $3 \not|(m-2)$ or $s = m-2$ if $3|(m-2)$.

Proof. First, we have the following enumeration:

$$|\mathcal{A}_0| = \begin{cases} \frac{3(3e-1)}{2} & \text{if } 3 \not|(m-2) \\ \frac{5(5e-1)}{2} & \text{if } 3|(m-2) \end{cases},\ |\mathcal{A}_j| = \frac{3(\binom{2e}{2}+2e^2)}{3e} = 4e-1 \text{ for } j \neq 0,$$

$$|\mathcal{B}| = \begin{cases} \frac{\binom{m}{2}2^2}{3} = \frac{2m(m-1)}{3} & \text{if } 3 \not|(m-2) \\ \frac{\binom{m-2}{2}2^2+8(m-2)}{3} = \frac{2(m-2)(m+1)}{3} & \text{if } 3|(m-2) \end{cases},$$

$$|\mathcal{C}_B| = \frac{3\binom{3}{2}e^2}{3e} = 3e,$$

$$|\Omega| = \begin{cases} \frac{2m(m-1)}{3} \cdot 3e + \frac{3(3e-1)}{2} + (4e-1)(m-1) \\ \frac{2(m-2)(m+1)}{3} \cdot 3e + \frac{5(5e-1)}{2} + (4e-1)(m-2) \end{cases} = \frac{(2m+1)(2me+e-1)}{2}.$$

The number $|\Omega|$ is just the block number in a $T(H, 2me+e)$. Further,

$\forall i \neq i' \in Z_e$, $\{(\infty,i)(\infty,i')\}$ appears in three blocks of $\mathcal{A}_0$, where exactly one edge is interior.

$\forall x \in Z_{2m}, \exists G_j$ containing $x \Longrightarrow \forall i, i' \in Z_e$, $\{(x,i),(\infty,i')\}$ appears in three blocks of $\mathcal{A}_j$, where exactly one edge is interior.

$\forall\ (x,i) \neq (x',i') \in Z_{2m} \times Z_e$, $x \in G_j$ and $x' \in G_{j'}$,

if $j \neq j'$, then $\exists B \in \mathcal{B}$ such that $x, x' \in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{C}_B$, where exactly one edge is interior.

if $j = j'$, then $x, x' \in G_j \Longrightarrow \{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{A}_j$, where exactly one edge is interior. ■

3 Main constructions

Theorem 3.1 *There exist $T(K_{1,k}, 2km+1)$ for integers $m, k \geq 1$.*

Construction. Take the vertex set Z_{2km+1}. The block set consists of the following m base blocks module $2km+1$, where $0 \leq i \leq m-1$.

$B_i = (0 : ki+1, ki+2, \cdots, ki+k; -(ki+1), -(ki+2), \cdots, -(ki+k))$.

Proof. The interior differences in the base block B_i are

$$ki+1, ki+2, \cdots, ki+k,$$

so the interior differences of all base blocks B_i, $0 \leq i \leq m-1$, exactly cover the integer interval $[1, km]$. But, the exterior differences in the base block B_i are

$$ki+1, ki+2, \cdots, ki+k \text{ and } 2(ki+1), 2(ki+2), \cdots, 2(ki+k),$$

so the exterior differences of all base blocks B_i, $0 \leq i \leq m-1$, exactly cover the intervals $[1, km]$ and $[2, 2km]_2 = [1, km]$, see Lemma 1.2. ■

Theorem 3.2 *There exists a $T(K_{1,k}, k^3)$ for any odd integer $k \geq 1$.*

Construction. Total $3k$ blocks on the set $Z_k \times Z_3$ are given by the following base block module $(k, 3)$:

$B = (0_0 : (k-1)_1, 0_1, 1_1, \cdots, (k-2)_1; (k-1)_2, (k-2)_2, (k-3)_2, \cdots, 2_2, 1_2, 0_2)$,

Proof. The interior $(0,1)$-mixed differences in B exactly cover the integer interval $[0, k-1]$. B mod $(-,3)$ will give the interior $(1,2)$- and $(2,0)$-mixed differences with the same values. The exterior mixed differences in B are

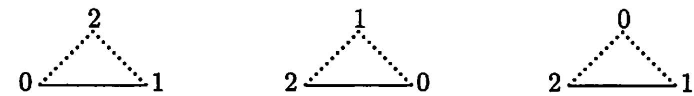
$(2,0)$-mixed differences:$-[0, k-1] = [0, k-1]$;

$(1,2)$-mixed differences:$\{i-(k-2-i)\}_{i\in Z_k} = \{2i+2\}_{i\in Z_k} = [0, k-1]$.

But, B mod $(-,3)$ will give the exterior $(0,1)$- and $(1,2)$-mixed differences ($(2,0)$- and $(0,1)$-mixed differences) respectively, with the same values. ■

Theorem 3.3 *There exists a $T(K_{1,k}, 3k : k)$ for any odd integer $k \geq 1$.*

Construction. For $k = 1$, a $T(K_{1,1}, 3 : 1)$ is just a $T(K_{1,1}, 3)$, which consists of three blocks as follows.



Below, for $k \geq 3$, define the following $4k-1$ blocks A_i, B_i, B_i' and A_i' on the set $Z_{2k} \cup \{\infty_0, \cdots, \infty_{k-1}\}$, where $\{\infty_0, \cdots, \infty_{k-1}\}$ is a k-hole, $0 \leq i \leq k-1$ for A_i, B_i, B_i', and $1 \leq i \leq k-1$ for A_i'.

$$A_i = \left(\begin{array}{cc} & [0, 2k-2]_2 \\ \infty_i & \\ & [k+2i, 3k-2+2i]_2 \end{array} \right),$$

$$A_i' = \left(\begin{array}{cc} & [1, 2k-1]_2 \\ \infty_i & \\ & [k+1+2i, 3k-1+2i]_2 \end{array} \right),$$

$$B_i = \left(\begin{array}{cccc} & k+2i & [1+2i, k-2+2i]_2 & [2+2i, k-1+2i]_2 \\ 2i & & & \\ & \infty_0 & \infty_1, \cdots\cdots, \infty_{\frac{k-1}{2}} & [k+1+2i, 2k-2+2i]_2^- \end{array} \right),$$

$$B_i' = \left(\begin{array}{cccc} & \infty_0 & [2+2i, k-1+2i]_2 & [3+2i, k+2i]_2 \\ 1+2i & & & \\ & k+1+2i & \infty_{\frac{k+1}{2}}, \cdots\cdots, \infty_{k-1} & [k+2+2i, 2k-1+2i]_2^- \end{array} \right).$$

Proof. The appearance of the pair containing ∞_i is as follows.

<u>interior</u> $(i=0):$ $[0, 2k-2]_2 \cup \{1+2i\}_{i=0}^{k-1} = [0, 2k-1]$, in A_0 and all B_i';

$(i \neq 0):$ $[0, 2k-2]_2 \cup [1, 2k-1]_2 = [0, 2k-1]$, in A_i and A_i'.

<u>exterior</u> $(i=0):$ $[k, 3k-2]_2 \cup \{2j\}_{j=0}^{k-1} \cup \{k+2j\}_{j=0}^{k-1} \cup \{k+1+2j\}_{j=0}^{k-1} =$
$2 \times ([k, 3k-2]_2 \cup [0, 2k-2]_2) \equiv 2 \times [0, 2k-1]$,
in A_0, all B_j and B_j';

$(1 \leq i \leq k-1):$ $[k+2i, 3k-2+2i]_2 \cup [k+1+2i, 3k-1+2i]_2$
$= [k+2i, 3k-1+2i] \equiv [0, 2k-1]$, in A_i and A_i';

$(1 \le i \le \frac{k-1}{2})$: $\{2j\}_{j=0}^{k-1} \cup \{2j+2i-1\}_{j=0}^{k-1} = [0, 2k-2]_2 \cup$
$[2i-1, 2i+2k-3]_2 \equiv [0, 2k-1]$, in all B_j;
$(\frac{k+1}{2} \le i \le k-1)$: $\{1+2j\}_{j=0}^{k-1} \cup \{2i+2j-k+1\}_{j=0}^{k-1} =$
$[1, 2k-1]_2 \cup [2i-k+1, 2i+k-1]_2 \equiv [0, 2k-1]$, in all B'_j.

The appearance of the difference $[d] = \{\{x, x+d\} : x \in Z_{2k}\}$, $1 \le d \le k$, is as follows.

<u>interior</u> $(d = k)$: $\{2i, 2i+k\}_{i=0}^{k-1}$ in all B_i;
$(1 \le d \le k-2,\ d \text{ odd})$: $\{\{2i, 2i+d\} : d \in [1, k-2]_2\}_{i=0}^{k-1}$ in all B_i,
$\{\{1+2i, 1+2i+d\} : d \in [1, k-2]_2\}_{i=0}^{k-1}$ in all B'_i;
$(2 \le d \le k-1,\ d \text{ even})$: $\{\{2i, 2i+d\} : d \in [2, k-1]_2\}_{i=0}^{k-1}$ in all B_i,
$\{\{1+2i, 1+2i+d\} : d \in [2, k-1]_2\}_{i=0}^{k-1}$ in all B'_i.

<u>exterior</u> $(d = k)$: $\{2j, 2j+k\}_{j=0}^{k-1} \cup \{1+2i, 1+2i+k\}_{i=0}^{k-1}$, in A_0 and all B'_i;
$(1 \le d \le k-2,\ d \text{ odd})$: $\{\{2j, 2j+k+2i\}\}_{j=0}^{k-1}$ in all A_i,
$\{\{1+2j, 1+2j+k+2i\}\}_{j=0}^{k-1}$ in all A'_i,
where $i \in [1, k-1]$ and $d \in \{k+2i\}_{i=1}^{k-1}$
$= [k+2, 3k-2]_2 \to 2 \times [1, k-2]_2$;
$(2 \le d \le k-1,\ d \text{ even})$: $\{\{2i, 2i+d\}\}_{i=0}^{k-1}$ in all B_i,
$\{\{1+2i, 1+2i+d\}\}_{i=0}^{k-1}$ in all B'_i, where
$d \in [k+1, 2k-2]_2 \to [2, k-1]_2$;
$\{\{k+2i+1-2j, k+2i-1+2j\}\}_{j=1}^{\frac{k-1}{2}}$ in all B_i,
$\{\{k+2i+2-2j, k+2i+2j\}\}_{j=1}^{\frac{k-1}{2}}$ in all B'_i, where $i \in [0, k-1]$
and $d \in \{4j-2\}_{j=1}^{\frac{k-1}{2}} = [2, 2k-4]_4 \to [2, k-1]_2$. ■

Theorem 3.4 *There exists a $T(K_{1,k}, 3k)$ for any $k \equiv 1 \bmod 4$.*

Construction. First, a $T(K_{1,1}, 3)$ has been given in Theorem 3.3. Below, let $k = 4t+1$, $t > 0$. Take the vertex set $(Z_{6t+1} \times Z_2) \cup \{\infty\}$. The block set consists of the following 3 base blocks module $6t+1$, where the interval $[a, b] = \{a_0, \cdots, b_0\}$ and the interval $[\overline{a}, \overline{b}] = \{a_1, \cdots, b_1\}$.

$$B_1 = \begin{pmatrix} 0 & \begin{matrix} \infty & [5t+1, 6t] & [3t+1, 4t] & [4t+1, 5t] & [\overline{2t+1}, \overline{3t}] \\ \overline{0} & [\overline{t+1}, \overline{2t}]^- & [\overline{5t+1}, \overline{6t}]^- & [1, t]^- & [\overline{1}, \overline{t}]^- \end{matrix} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \overline{0} & \begin{matrix} \infty & [1, t] & [2t+1, 3t] & [\overline{t+1}, \overline{2t}] & [\overline{3t+1}, \overline{4t}] \\ 0 & [t+1, 2t]^- & [5t+1, 6t]^- & [3t+1, 4t]^- & [\overline{1}, \overline{t}]^- \end{matrix} \end{pmatrix},$$

$$B_3 = \left(\bar{0} \begin{array}{ccccc} 0 & [5t+1,6t] & [4t+1,5t] & [t+1,2t] & [\bar{1},\bar{t}] \\ \infty & [1,t]^- & [2t+1,3t]^- & \overline{[5t+1,6t]}^- & \overline{[4t+1,5t]}^- \end{array} \right).$$

Proof. Obvious, the appearance of the pair containing ∞ satisfy conditions. Below, list tables of the interior differences and exterior differences in $Z_{6t+1} \times Z_2$, respectively. In these tables, the notation (i,i)-PD represents (i,i)-pure difference for $i = 0, 1$, and the notation $(0,1)$-MD represents $(0,1)$-mixed difference.

The interior differences of the base blocks B_i

	$(0,0)$-PD	$(1,1)$-PD	$(0,1)$-MD
B_1	$[3t+1,6t]$ $\to [1,3t]$		$[2t+1,3t]$
B_2		$[t+1,2t]$ $[3t+1,4t] \to [2t+1,3t]$	$[5t+1,6t]$ $[3t+1,4t]$
B_3		$[1,t]$	$[4t+1,5t]$, $[0,2t]$

The exterior differences of the base blocks B_i

	(0,0)-PD	(1,1)-PD	(0,1)-MD
B_1	$[1,t]$ $[t+2,3t]_2$	$[t+1,3t-1]_2$	$[t+1,2t], [5t+1,6t]$ $[1,t], [t+2,3t]_2,$ $[t+1,3t-1]_2, 0$
B_2	$[1,2t-1]_2$ $[2t+1,3t]$	$[1,t]$ $[2t+1,3t]$	$[4t+1,5t], [1,t], 0$ $[2t+1,3t]$, $[3t+2,5t]_2$
B_3	$[2,2t]_2$ $[t+1,3t-1]_2$	$[1,t]$ $[t+1,2t]$ $[t+2,3t]_2$	$[5t+1,6t]$, $[3t+1,4t]$ $[3t+1,5t-1]_2$

■

Theorem 3.5 *There exists a $T(K_{1,k}, 3k)$ for any $k \equiv 3$ mod 4.*

Construction. Let $k = 4t+3$. Below, in the procedure from the base block B to block $B+i$, we use the following notations:

$$\begin{pmatrix} a \\ b(d) \end{pmatrix} \text{ means the blocks } B+i \text{ take} \begin{cases} \binom{a}{b} & \text{for } 0 \le i \le 3t+1 \\ \binom{a}{d} & \text{for } 3t+2 \le i \le 6t+3 \end{cases},$$

$$\begin{pmatrix} a(b) \\ c \end{pmatrix} \text{ means the blocks } B+i \text{ take} \begin{cases} \binom{a}{c} & \text{for } 0 \le i \le 3t+1 \\ \binom{b}{c} & \text{for } 3t+2 \le i \le 6t+3 \end{cases},$$

$\binom{a[b]}{c[d]}$ means the blocks $B+i$ take $\begin{cases} \binom{a}{c} & \text{for} \quad 0 \le i \le \frac{3t}{2} \\ \binom{a}{d} & \text{for} \quad \frac{3t+2}{2} \le i \le 3t+1 \\ \binom{b}{d} & \text{for} \quad 3t+2 \le i \le 6t+3 \end{cases}$.

For $t=0$, take $Z_3 \times Z_3$, the following blocks mod $(3,-)$:

$$(0_0 : 1_0, 0_1, 1_1; 2_0, 2_2, 1_2),\ (0_1 : 1_1, 1_2, 2_2; 2_1, 1_0, 0_0),$$
$$(0_2 : 1_2, 0_1, 0_0; 2_2, 2_0, 2_1),\ (0_0 : 2_1, 1_2, 2_2; 0_2, 1_1, 0_1).$$

For $t=1$, take $(Z_{10} \times Z_2) \cup \{\infty\}$, the following blocks mod $(10,-)$:

$$(0 : 7, 9, 6, 8, \overline{9}, \infty, 5(\overline{0}); 1, 2, \overline{3}, \overline{4}, \overline{2}, \overline{0}(5), \overline{5}),$$
$$(\overline{0} : 7, 9, 6, 8, 3, \infty, 0(\overline{5}); \overline{1}, \overline{2}, 4, 2, \overline{4}, \overline{5}(0), 5),$$
$$(\overline{0} : \overline{6}, \overline{8}, 2, 4, 5, \overline{9}, \overline{7}; 8, 9, 3, 1, \infty, \overline{2}, \overline{1}).$$

For $t=2$, take $(Z_{16} \times Z_2) \cup \{\infty\}$, the following blocks mod $(16,-)$:

$$(0 : 8[\overline{8}], \infty, 1, 2, 4, 6, \overline{7}, \overline{13}, 3, 5, 7; 12[\overline{12}], 15, 11, \overline{5}, 13, 14, \overline{3}, \overline{6}, \overline{14}, \overline{15}, \overline{16}),$$
$$(\overline{0} : \overline{8}[8], \infty, 11, 0, 2, 4, 6, \overline{5}, \overline{7}, \overline{4}, \overline{6}; \overline{12}[12], 10, \overline{13}, 7, 5, 3, 1, 14, 15, \overline{14}, \overline{15}),$$
$$(\overline{0} : \overline{1}, \overline{3}, 14, \overline{2}, 1, 5, \overline{9}, 10, 12, 13, 15; 4, \infty, 2, \overline{13}, \overline{8}, \overline{10}, \overline{4}, 8, 6, \overline{14}, \overline{15}).$$

Below, for the case $t \ge 3$, take the vertex set $(Z_{6t+4} \times Z_2) \cup \{\infty\}$. The block set consists of three base blocks module $(6t+4,-)$.

Case odd $t \ge 3$:

$$B_1 = \left(0 \begin{array}{cccccc} \infty & 3t+2(\overline{0}) & [3t+4, 6t+3]_2 & [3t+3, 6t+2]_2 & [\overline{5t+4}, \overline{6t+3}]_2 & [\overline{3t+4}, \overline{4t+1}]_2 \\ \overline{0}(3t+2) & \overline{3t+2} & [1, \frac{3t+1}{2}] & [\overline{\tfrac{3t+3}{2}}, \overline{3t+1}] & [\overline{t+1}, \overline{\tfrac{3t+1}{2}}] & [\overline{1}, \overline{\tfrac{t-1}{2}}] \end{array} \right),$$

$$B_2 = \left(\overline{0} \begin{array}{cccccccc} \infty & 0(\overline{3t+2}) & [3t+4, 6t+3]_2 & [3t+3, 6t+2]_2 & [t+2, 2t+1]_2 & X_1 & Y_1 \\ \overline{3t+2}(0) & 3t+2 & [\overline{1}, \overline{\tfrac{3t+1}{2}}] & [2, 3t+1]_2^- & [\overline{2t+2}, \overline{\tfrac{5t+3}{2}}] & X_2 & Y_2 \end{array} \right),$$

$$B_3 = \left(\overline{0} \begin{array}{cccccc} 3t+2 & [\overline{3t+3}, \overline{6t+2}]_2 & [2, 3t+1]_2 & Z_1 & W_1 \\ \infty & [\frac{9t+7}{2}, 6t+3] & [1, 3t]_2^- & Z_2 & W_2 \end{array} \right), \text{ where}$$

for $t \equiv 1 \bmod 4$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} [\overline{\tfrac{3t+3}{2}}, \overline{2t-1}]_2 \\ [\overline{\tfrac{21t+15}{4}}, \overline{\tfrac{11t+5}{2}}] \end{pmatrix}, \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} [\overline{\tfrac{5t+5}{2}}, \overline{3t}]_2 \\ [\overline{\tfrac{23t+17}{4}}, \overline{6t+3}] \end{pmatrix},$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{9t+9}{2}}, \overline{6t+3}]_2 \\ [\overline{\frac{3t+5}{4}}, \overline{\frac{3t+1}{2}}] \end{pmatrix}, \quad \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{7t+7}{2}}, \overline{4t+3}]_2 \\ [\overline{\frac{t+3}{4}}, \overline{\frac{t+1}{2}}] \end{pmatrix};$$

for $t \equiv 3 \bmod 4$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{3t+5}{2}}, \overline{2t+1}]_2 \\ [\overline{\frac{21t+17}{4}}, \overline{\frac{11t+7}{2}}] \end{pmatrix}, \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{5t+7}{2}}, \overline{3t}]_2 \\ [\overline{\frac{23t+19}{4}}, \overline{6t+3}] \end{pmatrix},$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{9t+7}{2}}, \overline{6t+3}]_2 \\ [\overline{\frac{3t+3}{4}}, \overline{\frac{3t+1}{2}}] \end{pmatrix}, \quad \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} [\overline{\frac{7t+5}{2}}, \overline{4t+1}]_2 \\ [\overline{\frac{t+1}{4}}, \overline{\frac{t-1}{2}}] \end{pmatrix}.$$

<u>Case $t \equiv 2 \bmod 4$ and $t \geq 6$:</u>

$$B_1 = \left(\begin{array}{cccccc} \bar{0} & \overline{3t+2}[3t+2] & \infty & 4t+3 & [0, 3t]_2 \\ & \overline{\frac{9t+6}{2}}[\frac{9t+6}{2}] & \frac{9t+2}{2} & \overline{5t+3} & [1, 3t+1]_2^- \end{array} \right.$$

$$\left. \begin{array}{cc} [\overline{5}, \overline{3t+1}]_2 & [\overline{t+2}, \overline{3t}]_2 \\ [\frac{9t+10}{2}, 6t+3] & [\overline{5t+4}, \overline{6t+3}] \end{array} \right),$$

$$B_2 = \left(\begin{array}{ccccccc} 0 & 3t+2[\overline{3t+2}] & \infty & 1 & 2 & \overline{4t+5} & [4, 3t]_2 \\ & [\frac{9t+8}{2}, 6t+2] & \frac{9t+6}{2}[\overline{\frac{9t+6}{2}}] & 6t+3 & \frac{9t+4}{2} & \overline{\frac{3t+4}{2}} & \overline{2t+2} \end{array} \right.$$

$$\left. \begin{array}{ccc} [\overline{2t+3}, \overline{3t+3}]_2 & [\overline{3t+7}, \overline{4t+1}]_2 & [3, 3t+1]_2 \\ [\overline{t+1}, \overline{\frac{3t+2}{2}}] & [\overline{\frac{3t+6}{2}}, \overline{2t}] & [\overline{\frac{9t+10}{2}}, \overline{6t+4}] \end{array} \right),$$

$$B_3 = \left(\begin{array}{cccccc} \bar{0} & \bar{1} & 2t+1 & [\frac{9t+10}{2}, 6t+2]_2 & [3t+4, \frac{9t+6}{2}]_2 & [\overline{2}, \overline{t}]_2 \\ & \frac{3t+2}{2} & \overline{4t+2} & [2, \frac{3t-2}{2}]_2^- & [\frac{3t+6}{2}, 3t+2]_2^- & [\overline{\frac{9t+8}{2}}, \overline{5t+3}] \end{array} \right.$$

$$\left. \begin{array}{cccc} [1, 2t-3]_2 & \bar{3} & 3t-1 & [4t+5, 6t+3]_2 \\ [\overline{3t+2}, \overline{4t}] & \infty & \overline{\frac{9t+2}{2}} & [\overline{5t+4}, \overline{6t+3}] \end{array} \right).$$

<u>Case $t \equiv 0 \bmod 4$ and $t \geq 4$:</u>

$$B_1 = \left(\begin{array}{ccccccc} 0 & 3t+2[\overline{3t+2}] & 1 & 2 & \infty & [4, 3t]_2 & [3, 3t-3]_2 \\ & \frac{9t+6}{2}[\overline{\frac{9t+6}{2}}] & \frac{9t+4}{2} & \overline{\frac{9t+10}{2}} & 6t+3 & [\frac{9t+8}{2}, 6t+2] & [\overline{\frac{3t+6}{2}}, \overline{3t}] \end{array} \right.$$

$$\left. \begin{array}{cccccc} [\overline{4}, \overline{t}]_2 & 3t+5 & 3t+3 & \overline{3t+1} & \overline{\frac{3t+4}{2}} & [\overline{4t+6}, \overline{5t+2}]_2 \\ [\overline{3t+4}, \overline{\frac{7t+4}{2}}] & \bar{2} & \bar{1} & 3t+1 & \overline{\frac{15t+12}{4}} & [\overline{5t+5}, \overline{\frac{11t+6}{2}}] \end{array} \right),$$

$$B_2 = \left(\begin{array}{ccccc} \bar{0} & \overline{3t+2}[3t+2] & 3t+1 & \overline{6t+1} & [1, \frac{3t-2}{2}]_2 \\ & \overline{\frac{9t+6}{2}}[\frac{9t+6}{2}] & \overline{\frac{3t+2}{2}} & \infty & [\frac{9t+10}{2}, 6t+3]_2^- \end{array} \right.$$

$$
\begin{array}{cccc}
[\frac{3t+2}{2}, 3t-1]_2 & [\overline{4t+4}, \overline{5t}]_2 & \overline{6t+3} & 6t+2 \\
[3t+3, \frac{9t+2}{2}]_2^- & [\overline{\frac{t+2}{2}}, \overline{t-1}] & \frac{9t+4}{2} & \overline{3t+1}
\end{array}
$$

$$
\left.\begin{array}{cccc}
[2t, 3t]_2 & [3t+4, \frac{9t}{2}]_2 & [\frac{9t+8}{2}, 5t+2]_2 & [4, t]_2 \\
[\bar{t}, \overline{\frac{3t}{2}}] & [\overline{\frac{3t+4}{2}}, \overline{\frac{9t}{4}}] & [\overline{\frac{9t+8}{4}}, \overline{\frac{5t+2}{2}}] & [\bar{2}, \overline{\frac{t}{2}}]
\end{array}\right),
$$

$$
B_3 = \left(\begin{array}{cccccc}
 & \infty & 0 & 2 & [3t+5, 6t+3]_2 & [\bar{2}, \overline{t+2}]_2 \\
\bar{0} & & & & & \\
 & \frac{3t+4}{2} & \overline{3t+2} & \bar{1} & [3t+4, 6t+2]_2^- & [\overline{\frac{9t+8}{2}}, \overline{5t+4}]
\end{array} \right.
$$

$$
\left.\begin{array}{cc}
[\overline{2t+2}, \overline{3t}]_2 & [5, \overline{3t+1}]_2 \\
[\overline{\frac{11t+8}{2}}, \overline{6t+3}] & [\frac{3t+6}{2}, 3t+1]
\end{array}\right).
$$

Proof. For the case $t = 0, 1, 2, 3$, the check is immediate. For $t \geq 4$, the appearance of the pair containing ∞ satisfy conditions. Below, consider the interior and exterior differences in $Z_{6t+4} \times Z_2$, respectively. The symbols $\widehat{3t+2}$ and $\widehat{0}$ repress the semi-orbit for PD $3t+2$ and a half of the orbit for MD 0, respectively. For even t, there is one symbol $\binom{a[b]}{c[d]}$ in B_1 and B_2:

$$
\left(\begin{array}{cc} & 3t+2[\overline{3t+2}] \\ 0 & \\ & \frac{9t+6}{2}[\overline{\frac{9t+6}{2}}] \end{array} \right) \text{ or } \left(\begin{array}{cc} & \overline{3t+2}[3t+2] \\ \bar{0} & \\ & \overline{\frac{9t+6}{2}}[\frac{9t+6}{2}] \end{array} \right).
$$

It is not difficult to verify that the exterior edges of this part in all blocks generated by bases B_1 and B_2 are just all pairs in (0,0)-PD $\frac{3t+2}{2}$, (1,1)-PD $\frac{3t+2}{2}$, (0,1)-MD $\frac{3t+2}{2}$ and (0,1)-MD $\frac{9t+6}{2}$. For even t, the rest of the differences except the above four are listed in the following tables.

The interior differences for odd $t \geq 3$

	(0, 0)-PD	(1, 1)-PD	(0, 1)-MD
B_1	$[1, 3t]_2$ $\widehat{3t+2}$ $[2, 3t+1]_2$		$[5t+4, 6t+3]_2$ $[3t+4, 4t+1]_2$ $\widehat{0}$
B_2		$\widehat{3t+2}$ $\begin{cases} [\frac{3t+3}{2}, 2t-1]_2, [\frac{5t+5}{2}, 3t]_2 & t \equiv 1(4) \\ [\frac{3t+5}{2}, 2t+1]_2, [\frac{5t+7}{2}, 3t]_2 & t \equiv 3(4) \end{cases}$	$[1, 3t]_2, \widehat{0}$ $[2, 3t+1]_2$ $[4t+3, 5t+2]_2$
B_3		$[2, 3t+1]_2$ $\begin{cases} [1, \frac{3t-1}{2}]_2, [2t+1, \frac{5t+1}{2}]_2 & t \equiv 1(4) \\ [1, \frac{3t+1}{2}]_2, [2t+3, \frac{5t+3}{2}]_2 & t \equiv 3(4) \end{cases}$	$[3t+3, 6t+2]_2$ $3t+2$

The exterior differences for odd $t \geq 3$

	(0,0)-PD	(1,1)-PD	(0,1)-MD
B_1	$[1, \frac{3t+1}{2}]$ $\widehat{3t+2}$ $[\frac{3t+3}{2}, 3t+1]$	$[\frac{3t+3}{2}, 2t+1], \widehat{3t+2}$ $[\frac{5t+5}{2}, 3t+1]$	$[\frac{3t+3}{2}, 3t+1]$ $[t+1, \frac{3t+1}{2}]$ $[1, \frac{t-1}{2}]$ $0, 3t+2$ $[3t+3, \frac{9t+5}{2}]$
B_2	$[2, 6t]_4$ $\rightarrow [2, 3t+1]_2$ $\widehat{3t+2}$	$\widehat{3t+2}, [1, \frac{3t+1}{2}], [2t+2, \frac{5t+3}{2}]$ $\begin{cases} [\frac{t+3}{2}, \frac{3t+1}{4}], [\frac{9t+7}{4}, \frac{5t+1}{2}] & t \equiv 1(4) \\ [\frac{t+1}{2}, \frac{3t-1}{4}], [\frac{9t+9}{4}, \frac{5t+3}{2}] & t \equiv 3(4) \end{cases}$ $\begin{cases} [1, \frac{t-1}{4}], [\frac{11t+9}{4}, 3t+1] & t \equiv 1(4) \\ [1, \frac{t-3}{4}], [\frac{11t+11}{4}, 3t+1] & t \equiv 3(4) \end{cases}$	$[3t+3, 6t+2]_2$ $[\frac{3t+3}{2}, 3t+1]$ $[\frac{t+1}{2}, t]$ $0, 3t+2$
B_3	$[-3t, 3t-2]_4$ $\rightarrow [1, 3t]_2$	$\begin{cases} [\frac{3t+5}{4}, \frac{3t+1}{2}], [\frac{t+3}{4}, \frac{t+1}{2}] & t \equiv 1(4) \\ [\frac{3t+3}{4}, \frac{3t+1}{2}], [\frac{t+1}{4}, \frac{t-1}{2}] & t \equiv 3(4) \end{cases}$ $\begin{cases} [\frac{3t+3}{2}, \frac{9t+3}{4}], [\frac{5t+3}{2}, \frac{11t+5}{4}] & t \equiv 1(4) \\ [\frac{3t+3}{2}, \frac{9t+5}{4}], [\frac{5t+5}{2}, \frac{11t+7}{4}] & t \equiv 3(4) \end{cases}$	$[1, \frac{3t+1}{2}]$ $[3t+4, 6t+3]_2$ $[\frac{9t+7}{2}, 6t+3]$

The interior differences for $t \equiv 2 \bmod 4$, $t \geq 6$

	(0,0)-PD	(1,1)-PD	(0,1)-MD
B_1		$\widehat{3t+2}, [t+2, 3t]_2$ $[5, 3t+1]_2$	$\widehat{3t+2}, 0$ $[3t+4, 6t+2]_2, 2t+1$
B_2	$\widehat{3t+2}, [4, 3t]_2$ $1, 2, [3, 3t+1]_2$		$[2t+3, 3t+3]_2, \widehat{3t+2}$ $[3t+7, 4t+1]_2, 4t+5$
B_3		$[2, t]_2, 1, 3$	$[2, \frac{3t-2}{2}]_2, [\frac{3t+2}{2}, 3t]_2$ $[4t+7, 6t+3]_2, [1, 2t-1]_2$ $3t+5, 4t+3$

The exterior differences for $t \equiv 2 \bmod 4$, $t \geq 6$

	(0,0)-PD	(1,1)-PD	(0,1)-MD
B_1	$[-3t-1, 3t-1]_4$ $\rightarrow [1, 3t+1]_2$	$[1, t], t+1$ $[2t+2, 3t+1]$	$[3t+3, 6t+3]_2, t, \frac{3t+6}{2}$ $[1, \frac{3t-2}{2}], [\frac{3t+8}{2}, 3t+2]$
B_2	$1, \frac{3t+4}{2}, \frac{3t+6}{2}, 3t+2$ $[\frac{3t+8}{2}, 3t+1], [2, \frac{3t}{2}]$	$[t+2, \frac{3t+4}{2}]$ $[\frac{3t+8}{2}, 2t+1], 2t+3$	$\frac{3t+4}{2}, [\frac{9t+10}{2}, 6t+4]$ $[t+1, \frac{3t+2}{2}], [\frac{3t+6}{2}, 2t]$ $[3t+3, \frac{9t+4}{2}], 2t+2, \frac{3t}{2}$
B_3	$[4, 3t-2]_4, [2, 3t]_4$	$[t+1, \frac{3t}{2}], [1, t]$ $[2t+4, 3t+2], \frac{3t+6}{2}$ $2t+2, [\frac{3t+4}{2}, 2t+1]$	$\frac{9t+6}{2}, [\frac{9t+10}{2}, 6t+2]_2$ $[3t+2, \frac{9t+2}{2}]_2$ $[2t+3, 3t+1], \frac{9t+8}{2}$ $[0, t-1], \frac{3t+4}{2}, 2t+1$

The interior differences for $t \equiv 0 \bmod 4$, $t \geq 4$

	$(0,0)$-PD	$(1,1)$-PD	$(0,1)$-MD
B_1	$\widehat{3t+2}, 1$ $2, 3t-1, 3t+1$ $[4, 3t]_2, [3, 3t-3]_2$		$\widehat{3t+2}, 3t+1, \frac{3t+4}{2}$ $[4, t]_2, [4t+6, 5t+2]_2$
B_2		$\widehat{3t+2}, 3$ $[t+4, 2t]_2, 1$	$[\frac{9t+10}{2}, 6t+3]_2$ $[3t+5, \frac{9t+6}{2}]_2$ $3t+3, [5t+4, 6t]_2, 2$ $[3t+4, 4t+4]_2, \widehat{3t+2}$ $[\frac{3t+8}{2}, 3t]_2, [t+2, \frac{3t}{2}]_2$
B_3		$[2, t+2]_2, [2t+2, 3t]_2$ $[5, 3t+1]_2$	$[1, 3t-1]_2, 0, 6t+2$

The exterior differences for $t \equiv 0 \bmod 4$, $t \geq 4$

	$(0,0)$-PD	$(1,1)$-PD	$(0,1)$-MD
B_1	$\frac{3t+4}{2}, 1, 3t+1$ $[2, \frac{3t}{2}], \frac{3t+6}{2}$ $[\frac{3t+8}{2}, 3t+2]$	$\frac{9t+4}{4}, [\frac{5t+4}{2}, 3t]$ $[\frac{t+2}{2}, t-1]$	$[\frac{3t+6}{2}, 3t], [3t+4, \frac{7t+4}{2}]$ $\frac{9t+10}{2}, 2, 1, \frac{15t+12}{4}$ $[5t+5, \frac{11t+6}{2}], [3, \frac{3t}{2}]$ $\frac{9t+6}{2}, 3t+1, 3t+2, 0$
B_2	$[2, 3t-2]_4$ $[4, 3t]_4$	$3t+1, \frac{3t+2}{2}, [2, \frac{t}{2}]$ $[\frac{t+2}{2}, t-1]$ $[t, \frac{3t}{2}], [\frac{3t+4}{2}, \frac{9t}{4}]$ $[\frac{9t+8}{4}, \frac{5t+2}{2}]$ $[2t+3, \frac{5t+2}{2}]$	$\frac{3t+4}{2}, [1, \frac{3t-2}{2}]_2, [\frac{3t+6}{2}, 3t+1]_2$ $3t+3, \frac{9t+8}{2}, [\frac{11t+8}{2}, 6t+2]$ $\frac{3t+2}{2}, [\frac{9t+8}{2}, 5t+4]$ $[\frac{15t+16}{4}, \frac{9t+4}{2}], [\frac{7t+6}{2}, \frac{15t+8}{4}]$
B_3	$[1-3t, 3t-3]_4$ $\to [1, 3t-1]_2$	$3t+2, 1, [t, \frac{3t}{2}]$ $[1, \frac{t}{2}], [\frac{3t+4}{2}, 2t+2]$ $[\frac{5t+4}{2}, 3t+1]$	$\frac{9t+4}{2}, [2, 3t]_2$ $[3t+3, \frac{9t+2}{2}], 3t+2$ $6t+3, [\frac{9t+12}{2}, 6t+3], 0$

■

Theorem 3.6 *There exists a $T(K_{1,k}, 5k)$ for any $k \equiv 1 \bmod 4$.*

Construction. For $k = 1$, a $T(K_{1,1}, 5)$ consists of two base blocks: $(0 : 1; 2), (0 : 2; 3)$ mod 5. Below, let $k = 4t + 1$. Take the vertex set $(Z_{10t+2} \times Z_2) \cup \{\infty\}$. The block set consists of the following five base blocks module $10t + 2$. In the base block B, we use the following notation:

$$\binom{a(b)}{c(d)} \text{ means the blocks } B+i \text{ take} \begin{cases} \binom{a}{c} & \text{for } \; 0 \leq i \leq 5t \\ \binom{b}{d} & \text{for } \; 5t+1 \leq i \leq 10t+1 \end{cases}.$$

Case even $t > 0$:

$$B_1 = \left(\bar{0} \quad \begin{matrix} \infty & [7t+3, 10t+1]_2 & [\overline{2}, \overline{5t}]_2 \\ \frac{5}{2}t & [\overline{\tfrac{17}{2}t+2}, \overline{10t+1}] & [\frac{5}{2}t+1, 5t] \end{matrix} \right),$$

$$B_2 = \begin{pmatrix} \overline{0} & \overline{5t+1}(0) & [\overline{1}, \overline{5t-1}]_2 & [7t+2, 10t]_2 \\ & \infty(5t+1) & [\overline{\tfrac{15}{2}t+2}, \overline{10t+1}] & [\tfrac{7}{2}t+1, 5t] \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & \overline{0}(5t+1) & [3t+1, 5t-1]_2 & [\overline{3t+2}, \overline{6t}]_2 & [\overline{3t+1}, \overline{6t-1}]_2 \\ & 5t+1(\infty) & [9t+2, 10t+1] & [\tfrac{13}{2}t+2, 8t+1] & [\overline{\tfrac{3}{2}t, 3t-1}] \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & [\overline{6t+2}, \overline{10t}]_2 & [\overline{6t+1}, \overline{10t+1}]_2 \\ & [8t+2, 10t+1] & [\overline{3t}, \overline{5t}] \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & \infty & [1, 3t-1]_2 & [2, 5t]_2 \\ & \overline{\tfrac{5}{2}t+1} & [\tfrac{15}{2}t+2, 9t+1] & [\overline{\tfrac{5}{2}t+2}, \overline{5t+1}] \end{pmatrix}.$$

Case odd t:

$$B_1 = \begin{pmatrix} \overline{0} & \infty & [\overline{3}, \overline{5t}]_2 & [7t+2, 10t+1]_2 \\ & \overline{\tfrac{15t+3}{2}} & [\tfrac{5t+5}{2}, 5t+1] & [\overline{\tfrac{17t+3}{2}}, \overline{10t+1}] \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \overline{0} & \overline{5t+1}(0) & \overline{1} & [\overline{2}, \overline{5t-1}]_2 & [7t+3, 10t]_2 \\ & \infty(5t+1) & \tfrac{5t+3}{2} & [\overline{\tfrac{15t+5}{2}}, \overline{10t+1}] & [\tfrac{7t+3}{2}, 5t] \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & \overline{0}(5t+1) & [1, 5t]_2 & [2, 3t-1]_2 \\ & 5t+1(\infty) & [\overline{\tfrac{5t+1}{2}}, \overline{5t}] & [\tfrac{15t+5}{2}, 9t+1] \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & \infty & [3t+1, 5t-1]_2 & [\overline{3t+2}, \overline{6t-1}]_2 & [\overline{3t+1}, \overline{6t}]_2 \\ & \tfrac{5t+1}{2} & [9t+2, 10t+1] & [\overline{\tfrac{3t+1}{2}}, \overline{3t-1}] & [\tfrac{13t+3}{2}, 8t+1] \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & [\overline{6t+1}, \overline{10t+1}]_2 & [\overline{6t+2}, \overline{10t}]_2 \\ & [\overline{3t}, \overline{5t}] & [8t+2, 10t+1] \end{pmatrix}.$$

Theorem 3.7 *There exists a $T(K_{1,k}, 5k)$ for any $k \equiv 3 \bmod 4$.*

Construction. For $k = 3$, a $T(K_{1,3}, 15)$ on $(Z_7 \times Z_2) \cup \{\infty\}$ can be constructed as follows.

$(0_1 : \infty, 2_1, 5_0; 5_1, 6_1, 2_0), (0_0 : \infty, 1_0, 3_0; 5_0, 3_1, 4_1), (0_1 : 1_1, 3_1, 6_0; 3_0, 4_0, 6_1),$
$(0_0 : 0_1, 2_0, 3_1; \infty, 6_0, 1_1), (0_0 : 5_1, 4_1, 6_1; 2_1, 5_0, 6_0)$, mod $(7, -)$.

Below, let $k = 4t + 3,\ t > 0$. Take the vertex set $(Z_{10t+7} \times Z_2) \cup \{\infty\}$. The block set consists of the following five base blocks module $10t + 7$.

Case even $t > 0$:

$$B_1 = \begin{pmatrix} \overline{0} & \infty & [\overline{2}, \overline{5t+2}]_2 & [7t+5, 10t+5]_2 \\ & \overline{\frac{15}{2}t+5} & [\overline{\frac{15}{2}t+6}, \overline{10t+6}] & [\frac{7}{2}t+2, 5t+2] \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & \infty & [1, 5t+3]_2 & [2, 3t]_2 \\ & \frac{15}{2}t+5 & [\overline{\frac{5}{2}t+3}, \overline{5t+4}] & [\frac{15}{2}t+6, 9t+5] \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \overline{0} & [\overline{1}, \overline{5t+3}]_2 & [7t+6, 10t+6]_2 \\ & [\frac{5}{2}t+3, 5t+4] & [\overline{\frac{17}{2}t+6}, \overline{10t+6}] \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & \overline{0} & [3t+2, 5t+2]_2 & [\overline{3t+3}, \overline{6t+3}]_2 & [\overline{3t+4}, \overline{6t+2}]_2 \\ & \infty & [9t+6, 10t+6] & [\overline{\frac{3}{2}t+1}, \overline{3t+1}] & [\frac{13}{2}t+5, 8t+4] \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & [\overline{6t+5}, \overline{10t+5}]_2 & [\overline{6t+4}, \overline{10t+6}]_2 \\ & [\overline{3t+2}, \overline{5t+2}] & [8t+5, 10t+6] \end{pmatrix}.$$

Case odd t:

$$B_1 = \begin{pmatrix} \overline{0} & \infty & [\overline{1}, \overline{5t+2}]_2 & [7t+6, 10t+5]_2 \\ & \frac{5t+5}{2} & [\overline{\frac{15t+11}{2}}, \overline{10t+6}] & [\frac{7t+5}{2}, 5t+2] \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & \infty & [2, 5t+3]_2 & [1, 3t]_2 \\ & \overline{\frac{5t+5}{2}} & [\overline{\frac{5t+7}{2}}, \overline{5t+4}] & [\frac{15t+11}{2}, 9t+5] \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \overline{0} & [\overline{2}, \overline{5t+3}]_2 & [7t+5, 10t+6]_2 \\ & [\frac{5t+7}{2}, 5t+4] & [\overline{\frac{17t+11}{2}}, \overline{10t+6}] \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & \overline{0} & [3t+2, 5t+2]_2 & [\overline{3t+4}, \overline{6t+3}]_2 & [\overline{3t+3}, \overline{6t+2}]_2 \\ & \infty & [9t+6, 10t+6] & [\overline{\frac{3t+3}{2}}, \overline{3t+1}] & [\frac{13t+9}{2}, 8t+4] \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & [\overline{6t+5}, \overline{10t+5}]_2 & [\overline{6t+4}, \overline{10t+6}]_2 \\ & [\overline{3t+2}, \overline{5t+2}] & [8t+5, 10t+6] \end{pmatrix}.$$

The proofs of the Theorems 3.6 and 3.7 are simple, which are omitted.

4 Conclusion

Theorem 4.1 *A $T(K_{1,k}, v)$ exists for $v \equiv 1$ mod $2k$.*

Proof. By the directed construction Theorem 3.1. ■

Theorem 4.2 *For odd k, a $T(K_{1,k}, v)$ exists for $v \equiv k$ mod $2k$.*

Proof. By Theorems 3.2 – 3.7 and Theorem 2.2. ■

Theorem 4.3 *The spectrum for perfect $T(K_{1,2^t}, v)$ is $v \equiv 1$ mod 2^{t+1}. For odd prime power q, the spectrum for perfect $T(K_{1,q}, v)$ is $v \equiv 1, q$ mod $2q$.*

Proof. For prime power k, the necessary conditions to exist a $T(K_{1,k}, v)$ is that v is odd and $v \equiv 1, k$ mod $2k$ (see Lemma 1.1). Thus, when k is even, the first conclusion can be obtained by Theorem 4.1. And, when k is odd, the second conclusion can be obtained by Theorems 4.1 and 4.2. ■

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