A note on enumeration of noncrossing partitions *

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Abstract. In this paper, we study the enumeration of noncrossing partitions with fixed points. The expressions of $f_m(x_1, x_2, x_3, 0, 0, \cdots, 0)$ and $f_m(x_1, x_2, 0, \cdots, 0, x_{p+3}, 0, \cdots, 0)$ are found and a new proof of the expression of $f_m(x_1, x_2, 0, 0, \cdots, 0)$ is obtained using diophantine equations.

Keywords: noncrossing partitions, fixed points, diophantine equations

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1 Introduction

A partition $\pi = B_1/B_2/\cdots/B_m$ of a totally ordered set X is called noncrossing partition (n.c.p.) iff there do not exist four elements $a < b < c < d$ of X such that $a, c \in B_i$, $b, d \in B_j$ and $i \neq j$. We denote by $NC(X)$ the set of all n.c.p. of X.

Many authors have worked on n.c.p. (see for example Kreweras [2] Sapounakis and Tsikouras [3]).

In [3], an interesting class of n.c.p. has been introduced. $\pi \in NC(X, A)$ is called noncrossing partition with fixed points the elements of a given set

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$A \subseteq X$ iff every block of π contains exactly one element of A. The set of all these n.c.p. is denoted by $NC(X, A)$. Our purpose is to evaluate the cardinality $|NC(X, A)|$.

Since the distribution of the elements of A in X determines the cardinality $|NC(X, A)|$, we can restrict the problem to the equivalent case. Let $[m] = \{1, 2, \cdots, m\}, X = [m] \cup Y$, where the elements of Y are distributed in the intervals $(i, i+1)$, $i \in [m-1]$ and $(m, +\infty)$, so that $X \cap (i, i+1) = X_i \ \forall i \in [m-1]$ and $X \cap (m, +\infty) = X_m$. A function f_m of m variables is defined with $f_m(x_1, x_2, \cdots, x_m) = |NC(X, [m])|$, where $x_i = |X_i|$ for every $i \in [m-1]$ and $x_m = |X_m|$.

In [3],[4] Sapounakis and Tsikouras have proved the following results.

Theorem 1.1 *For every $m \in \mathbb{N}$, with $m \geqslant 2$ we have that*
$$f_m(x_1, x_2, 0, 0, \cdots, 0) = \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}.$$

Theorem 1.2 *For every $m \in \mathbb{N}$, with $m \geqslant 4$ and for every $\rho \in \mathbb{N}^*$, with $2\rho \leq m-2$ we have that*
$$f_m(x_1, 0, \cdots, 0, x_{\rho+2}, 0, \cdots, 0) = \binom{x_1+x_{\rho+2}+m}{m} - \binom{x_1+m-1}{m} - \binom{x_{\rho+2}+m-1}{m} + \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-\delta} \binom{x_1+k-1}{k}\binom{x_{\rho+2}+m-k-1}{m-k}.$$

Using the formula $\binom{a+b+m+1}{m} = \sum_{k=0}^{m} \binom{a+k}{a}\binom{b+m-k}{b}$ (e.g. see [1]), we can obtain another expression of Theorem 1.2 as follows:
$$f_m(x_1, x_2, 0, 0, \cdots, 0) = \binom{x_1+x_2+m-1}{m-1} + \sum_{k=1}^{m-1} \binom{x_1+k-1}{k}\binom{x_2+m-k-1}{m-k}.$$

In section 2 of this paper, we first show a lemma about the number of nonnegative integer solutions of a diophantine equation. Using this lemma, we can easily prove the result obtained by Sapounakis and Tsikouras.

In section 3, the expressions of $f_m(x_1, x_2, x_3, 0, \cdots, 0)$ and $f_m(x_1, x_2, 0, \cdots, 0, x_{\rho+3}, 0, \cdots, 0)$ are presented.

2 A new proof of the expression of $f_m(x_1, x_2, 0, 0, \cdots, 0)$

We denote by $N[t_1 + t_2 + \cdots + t_r = n]$ the number of nonnegative integer solutions of the equation $t_1 + t_2 + \cdots + t_r = n$;

$N[t_1 + t_2 + \cdots + t_r = n; t_1 \leq n_1]$ the number of nonnegative integer solutions of the equation $t_1 + t_2 + \cdots + t_r = n$ with $t_1 \leq n_1$;

$N[t_1 + t_2 + \cdots + t_r = n; t_1 \le n_1, t_2 \le n_2]$ the number of nonnegative integer solutions of the equation $t_1 + t_2 + \cdots + t_r = n$ with $t_1 \le n_1$ and $t_2 \le n_2$.

Lemma 2.1 *For every $m \in \mathbb{N}$, with $m \geqslant 2$*

$$N[t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2; t_1 \le x_1, t_2 \le x_2] = \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}.$$

Proof. It is well-known that, $N[t_1+t_2+\cdots+t_{m+1} = x_1+x_2] = \binom{x_1+x_2+m}{m}$. On the other hand, $N[t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2;\ t_1 > x_1]$

$$= N[t_1 + t_2 + \cdots + t_{m+1} = x_2 - 1] = \binom{x_2+m-1}{m},$$

and $N[t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2; t_2 > x_2]$

$$= N[(t_1 + t_2 + \cdots + t_{m+1} = x_1 - 1] = \binom{x_1+m-1}{m}.$$

Then $N[t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2;\ t_1 \le x_1, t_2 \le x_2]$

$$= N[t_1+t_2+\cdots+t_{m+1} = x_1+x_2] - N[t_1+t_2+\cdots+t_{m+1} = x_1+x_2; t_1 > x_1] - N[t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2; t_2 > x_2]$$
$$= \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}. \qquad \blacksquare$$

Using Lemma 2.1, we prove Theorem 1.1.

Proof. Here, we deal with the set $NC(X, [m])$ with $X = [m] \cup \bigcup_{i=1}^{2} X_i$, $X_i = X \cap (i, i+1)$, $x_i = |X \cap (i, i+1)|$ $(i = 1, 2)$.

For every n.c.p. $\pi = B_1/B_2/\cdots/B_m \in NC(X, [m])$ with $i \in B_i$ for every $i \in [m]$,

$$\text{let } n_i = \begin{cases} |B_i| - 1\ , & i \in \{1, 3, 4, \cdots m\} \\ |B_2 \cap X_1|, & i = 2 \\ |B_2 \cap X_2|, & i = m + 1. \end{cases}$$

Then the sequence (n_i), $i \in [m+1]$ is a nonnegative integer solution of the equation $t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2$ with $t_2 \le x_1$ and $t_{m+1} \le x_2$.

Conversely, if (n_i), $i \in [m+1]$ is a nonnegative integer solution of the equation $t_1 + t_2 + \cdots + t_{m+1} = x_1 + x_2$ with $t_2 \le x_1$ and $t_{m+1} \le x_2$. We define recursively the blocks of a n.c.p. $\pi = B_1/B_2/\cdots/B_m \in NC(X, [m])$ with $i \in B_i$ for every $i \in [m]$ as follows.

(1) B_2 contains 2, the last n_2 elements of X_1 and the first n_{m+1} elements of X_2;

(2) B_1 contains 1 as well as the first n_1 elements of $X \backslash ([m] \cup B_2)$;

(3) For $i = 3, 4, \cdots, m$, B_i contains i as well as the last elements of $X \backslash ([m] \cup (\bigcup_{j=1}^{i-1} B_j))$.

Thus, we define a bijection between the set $NC(X,[m])$ and the set of all nonnegative integer solutions of the equation $t_1+t_2+\cdots+t_{m+1}=x_1+x_2$ with $t_2 \le x_1$ and $t_{m+1} \le x_2$.

Since $N[t_1+t_2+\cdots+t_{m+1}=x_1+x_2; t_1 \le x_1, t_2 \le x_2]$
$= \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}$,
we obtain that

$$f_m(x_1,x_2,0,0,\cdots,0) = \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}. \qquad \blacksquare$$

3 $f_m(x_1,x_2,x_3,0,0,\cdots,0)$ and $f_m(x_1,x_2,0,\cdots,0,x_{\rho+3},0,\cdots,0)$

We now turn to find $f_m(x_1,x_2,x_3,0,0,\cdots,0)$.

Theorem 3.1 *For every $m \in \mathbb{N}$, with $m \geqslant 3$*

$$f_m(x_1,x_2,x_3,0,0,\cdots,0) = \binom{x_1+x_2+x_3+m+1}{m+1} - \sum_{1\le i<j\le 3} \binom{x_i+x_j+m}{m+1} + \sum_{i=1}^{3} \binom{x_i+m-1}{m+1} + (x_2+1)\sum_{k=2}^{m-2} \binom{x_1+k-1}{k}\binom{x_3+m-k-1}{m-k}.$$

Proof. We consider the case that $f_m(x_1,x_2,x_3,0,0,\cdots,0) = |NC(X,[m])|$ with $X = [m] \cup \bigcup_{i=1}^{3} X_i$, $X_i = X \cap (i,i+1)$, $x_i = |X \cap (i,i+1)|$ $(i=1,2,3)$.

We partition the set $NC(X,[m])$ into sets $A_{u,v}$ and $T_{u,v,i,j}$ (with $u,v,i,j \in \mathbb{N}$, $u \le x_1$, $v \le x_2$, $i \le x_3-1$, $1 \le j \le x_3-i$), where $A_{u,v}$ and $T_{u,v,i,j}$ are defined as follows:

For every n.c.p. $\pi = B_1/B_2/\cdots/B_m \in NC(X,[m])$, $i \in B_i$ for every $i \in [m]$.

Each set $A_{u,v}$ consists of all $\pi \in NC(X,[m])$ with the property that $|B_2 \cap X_1| = x_1-u$, $|B_2 \cap X_2| = x_2-v$ and $|B_2 \cap X_3| = 0$.

Each set $T_{u,v,i,j}$ consists of all $\pi \in NC(X,[m])$ with the property that $|B_2 \cap X_1| = x_1-u$, $|B_2 \cap X_2| = x_2-v$, $|B_2 \cap X_3| = j$ and $|B_3 \cap X_3| = i$.

Thus $|A_{u,v}| = f_{m-1}(u+v,x_3,0,\cdots,0) = \binom{u+v+x_3+m-1}{m-1} - \binom{u+v+m-2}{m-1} - \binom{x_3+m-2}{m-1}$,

$$|T_{u,v,i,j}| = f_{m-2}(u+x_3-i-j,0,\cdots,0) = \binom{u+x_3-i-j+m-3}{m-3}.$$

Hence, we finally obtain

$f_m(x_1,x_2,x_3,0,0,\cdots,0)$

$$= \sum_{u=0}^{x_1}\sum_{v=0}^{x_2} |A_{u,v}| + \sum_{u=0}^{x_1}\sum_{v=0}^{x_2}\sum_{i=0}^{x_3-1}\sum_{j=0}^{x_3-i} |T_{u,v,i,j}|$$

$$= \sum_{u=0}^{x_1}\sum_{v=0}^{x_2}[\binom{u+v+x_3+m-1}{m-1} - \binom{u+v+m-2}{m-1} - \binom{x_3+m-2}{m-1}]+$$

$$\sum_{u=0}^{x_1}\sum_{v=0}^{x_2}\sum_{i=0}^{x_3-1}\sum_{j=0}^{x_3-i}\binom{u+x_3-i-j+m-3}{m-3}$$

$$= \binom{x_1+x_2+x_3+m+1}{m+1} - \sum_{1\leq i<j\leq 3}\binom{x_i+x_j+m}{m+1} + \sum_{i=1}^{3}\binom{x_i+m-1}{m+1}+$$

$$(x_2+1)[\binom{x_1+x_3+m-1}{m} - \binom{x_1+m-1}{m} - \binom{x_3+m-2}{m} - x_3\binom{x_1+m-2}{m-1}]-$$

$$(x_1+1)(x_2+1)\binom{x_3+m-2}{m-1}$$

$$= \binom{x_1+x_2+x_3+m+1}{m+1} - \sum_{1\leq i<j\leq 3}\binom{x_i+x_j+m}{m+1} + \sum_{i=1}^{3}\binom{x_i+m-1}{m+1}+$$

$$(x_2+1)\sum_{k=2}^{m-2}\binom{x_1+k-1}{k}\binom{x_3+m-k-1}{m-k}. \qquad \blacksquare$$

Note. (1)We note that if $x_3 = 0$ we obtain Theorem 1.1, whereas if $x_2 = 0$ we obtain Theorem 1.2 for $\rho = 1$.

(2)Using the formula $\binom{a+b+c+m+2}{m} = \sum_{k=0}^{m}\sum_{l=0}^{m-k}\binom{a+l}{a}\binom{b+k}{b}\binom{c+m-k-l}{c}$, we can deduce another expression of Theorem 3.1 as follows:

$$f_m(x_1,x_2,x_3,0,0,\cdots,0) = \binom{x_1+x_2+x_3+m}{m} + \sum_{i=1}^{3}\binom{x_i+m-1}{m}+$$

$$\sum_{\substack{v\geq 1, u\geq 1\\ u+v\leq m}}\binom{x_1+u-1}{u}\binom{x_2+v-1}{v}\binom{x_3+m-u-v}{m+1-u-v}+(x_2+1)\sum_{k=2}^{m-2}\binom{x_1+k-1}{k}\binom{x_3+m-k-1}{m-k}.$$

We now generalize Theorem 1.2 for three variables.

Theorem 3.2 *For every $m \in \mathbb{N}$, with $m \geqslant 5$ and for every $\rho \in \mathbb{N}^*$, with $2\rho \leq m-3$,*

$$f_m(x_1,x_2,0,\cdots,0,x_{\rho+3},0,\cdots,0)$$

$$= \binom{x_1+x_2+x_{\rho+3}+m+1}{m+1} - \sum_{\substack{i<j\\ i,j\in\{1,2,\rho+3\}}}\binom{x_i+x_j+m}{m+1} + \sum_{i\in\{1,2,\rho+3\}}\binom{x_i+m-1}{m+1}+$$

$$\sum_{\delta=2}^{\rho+2}\sum_{v=\delta}^{m-\delta}\binom{x_1+v-1}{v}\binom{x_{\rho+3}+m-v-1}{m-v} + \sum_{\delta=2}^{\rho+1}\sum_{u=\delta}^{m-\delta}\binom{x_2-1+u}{u}\binom{x_{\rho+3}+m-u-1}{m-u}+$$

$$\sum_{\delta=1}^{\rho}\sum_{k=\delta}^{m-2-\delta}\sum_{t=1}^{k+1}\binom{x_1+t-1}{t}\binom{x_2+k+1-t}{k+2-t}\binom{x_{\rho+3}+m-k-2}{m-k-1}+$$

$$\sum_{\substack{u\in[\rho+1]\\ v\in[m-2-\rho]\\ u+v\geq\rho+3}}\binom{x_1+v-1}{v}\binom{x_2+u-1}{u}\binom{x_{\rho+3}+m-u-v}{m+1-u-v}.$$

Proof. We consider the case that $f_m(x_1, x_2, 0, \cdots, 0, x_{\rho+3}, 0, \cdots, 0) = |NC(X, [m])|$ with $X = [m] \cup \bigcup_{i\in\{1,2,\rho+3\}} X_i$, $X_i = X \cap (i, i+1)$, $x_i = |X_i|$ $(i \in \{1, 2, \rho+3\})$.

We partition the set $NC(X, [m])$ into sets $A_{u,v}$ and $T_{u,v,k,l}$ (with $u, v, k, l \in \mathbb{N}$, $u \le x_1$, $v \le x_2$, $l \le k \le x_{\rho+3} - 1$), where $A_{u,v}$ and $T_{u,v,k,l}$ are defined as follows:

For every n.c.p. $\pi = B_1/B_2/\cdots/B_m \in NC(X, [m])$, $i \in B_i$ for every $i \in [m]$.

Each set $A_{u,v}$ consists of all $\pi \in NC(X, [m])$ with the property that $|B_2 \cap X_1| = x_1 - u$, $|B_2 \cap X_2| = x_2 - v$ and $|B_2 \cap X_{\rho+3}| = 0$.

Each set $T_{u,v,k,l}$ consists of all $\pi \in NC(X, [m])$ with the property that $|B_2 \cap X_1| = x_1 - u$, $|B_2 \cap X_2| = x_2 - v$, $|B_2 \cap X_{\rho+3}| = x_{\rho+3} - k$ and $|\{y \in X_{\rho+3} | y > max\{x \in B_2\}\}| = l$.

Thus $|A_{u,v}| = f_{m-1}(u + v, 0, 0, \cdots, 0, x_{\rho+3}, \cdots, 0)$,

$$|T_{u,v,k,l}| = f_{\rho+1}(v + k - l, 0, \cdots, 0) f_{m-\rho-2}(u + l, 0, \cdots, 0).$$

Hence, we finally obtain

$$
\begin{aligned}
& f_m(x_1, x_2, 0, \cdots, 0, x_{\rho+3}, 0, \cdots, 0) \\
&= \sum_{u=0}^{x_1} \sum_{v=0}^{x_2} |A_{u,v}| + \sum_{u=0}^{x_1} \sum_{v=0}^{x_2} \sum_{k=0}^{x_{\rho+3}-1} \sum_{l=0}^{k} |T_{u,v,k,l}| \\
&= \sum_{u=0}^{x_1} \sum_{v=0}^{x_2} [\binom{u+v+x_{\rho+3}+m-1}{m-1} - \binom{u+v+m-2}{m-1} - \binom{x_{\rho+3}+m-2}{m-1} + \\
&\quad \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-1-\delta} \binom{u+v+k-1}{k} \binom{x_{\rho+3}+m-k-2}{m-k-1}] + \\
&\quad \sum_{u=0}^{x_1} \sum_{v=0}^{x_2} \sum_{k=0}^{x_{\rho+3}-1} \sum_{l=0}^{k} \binom{v+k+\rho-l}{\rho} \binom{u+l+m-\rho-3}{m-\rho-3} \\
&= \binom{x_1+x_2+x_{\rho+3}+m+1}{m+1} - \sum_{\substack{i<j \\ i,j\in\{1,2,\rho+3\}}} \binom{x_i+x_j+m}{m+1} + \sum_{i\in\{1,2,\rho+3\}} \binom{x_i+m-1}{m+1} \\
&\quad + \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-1-\delta} [\binom{x_1+x_2+k+1}{k+2} - \binom{x_1+k}{k+2} - \binom{x_2+k}{k+2}] \binom{x_{\rho+3}+m-k-2}{m-k-1} \\
&\quad + \sum_{u=1}^{\rho+1} \sum_{v=1}^{m-2-\rho} \binom{x_1-1+v}{v} \binom{x_2-1+u}{u} \binom{x_{\rho+3}+m-u-v}{m+1-u-v} - x_1 x_2 \binom{x_{\rho+3}+m-2}{m-1} \\
&\quad + \sum_{u=2}^{\rho+1} \binom{x_2-1+u}{u} \binom{x_{\rho+3}-1+m-u}{m-u} + \sum_{v=2}^{m-2-\rho} \binom{x_1-1+v}{v} \binom{x_{\rho+3}-1+m-v}{m-v} \\
&= \binom{x_1+x_2+x_{\rho+3}+m+1}{m+1} - \sum_{\substack{i<j \\ i,j\in\{1,2,\rho+3\}}} \binom{x_i+x_j+m}{m+1} + \sum_{i\in\{1,2,\rho+3\}} \binom{x_i+m-1}{m+1} \\
&\quad + \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-1-\delta} [\sum_{t=1}^{k+1} \binom{x_1+t-1}{t} \binom{x_2+k+1-t}{k+2-t} + \binom{x_1+k}{k+1} + \binom{x_2+k}{k+1}] \binom{x_{\rho+3}+m-k-2}{m-k-1}
\end{aligned}
$$

$$+\sum_{u=1}^{\rho+1}\sum_{v=1}^{m-2-\rho}\binom{x_1-1+v}{v}\binom{x_2-1+u}{u}\binom{x_{\rho+3}+m-u-v}{m+1-u-v}-x_1x_2\binom{x_{\rho+3}+m-2}{m-1}$$

$$+\sum_{u=2}^{\rho+1}\binom{x_2-1+u}{u}\binom{x_{\rho+3}-1+m-u}{m-u}+\sum_{v=2}^{m-2-\rho}\binom{x_1-1+v}{v}\binom{x_{\rho+3}-1+m-v}{m-v}$$

$$=\binom{x_1+x_2+x_{\rho+3}+m+1}{m+1}-\sum_{\substack{i<j\\ i,j\in\{1,2,\rho+3\}}}\binom{x_i+x_j+m}{m+1}+\sum_{i\in\{1,2,\rho+3\}}\binom{x_i+m-1}{m+1}$$

$$\sum_{\delta=2}^{\rho+2}\sum_{v=\delta}^{m-\delta}\binom{x_1+v-1}{v}\binom{x_{\rho+3}+m-v-1}{m-v}+\sum_{\delta=2}^{\rho+1}\sum_{u=\delta}^{m-\delta}\binom{x_2-1+u}{u}\binom{x_{\rho+3}+m-u-1}{m-u}+$$

$$\sum_{\delta=1}^{\rho}\sum_{k=\delta}^{m-2-\delta}\sum_{t=1}^{k+1}\binom{x_1+t-1}{t}\binom{x_2+k+1-t}{k+2-t}\binom{x_{\rho+3}+m-k-2}{m-k-1}+$$

$$\sum_{\substack{u\in[\rho+1]\\ v\in[m-2-\rho]\\ u+v\geq\rho+3}}\binom{x_1+v-1}{v}\binom{x_2+u-1}{u}\binom{x_{\rho+3}+m-u-v}{m+1-u-v}. \qquad \blacksquare$$

Note. (1)From Theorem 3.2 we can easily obtain Theorem 3.1 (for $\rho=0$), as well as Theorem 1.2 (for $x_2=0$).

(2)we can easy obtain that

$$f_m(x_1,x_2,0,\cdots,0,x_{\rho+3},0,\cdots,0)$$
$$=f_m(x_1,x_2,x_{\rho+3},0,0,\cdots,0)+\sum_{\substack{u\in[\rho+1]\\ v\in[m-2-\rho]\\ u+v\geq\rho+3}}\binom{x_1+v-1}{v}\binom{x_2+u-1}{u}\binom{x_{\rho+3}+m-u-v}{m+1-u-v}+$$

$$\sum_{\delta=3}^{\rho+2}\sum_{v=\delta}^{m-\delta}\binom{x_1+v-1}{v}\binom{x_{\rho+3}+m-v-1}{m-v}+\sum_{\delta=2}^{\rho+1}\sum_{u=\delta}^{m-\delta}\binom{x_2+u-1}{u}\binom{x_{\rho+3}+m-u-1}{m-u}+$$

$$\sum_{k=1}^{m-3}\sum_{t=1}^{k}\binom{x_1+t-1}{t}\binom{x_2+k+1-t}{k+2-t}\binom{x_{\rho+3}+m-k-2}{m-k-1}+$$

$$\sum_{\delta=2}^{\rho}\sum_{k=\delta}^{m-2-\delta}\sum_{t=1}^{k+1}\binom{x_1+t-1}{t}\binom{x_2+k+1-t}{k+2-t}\binom{x_{\rho+3}+m-k-2}{m-k-1}$$

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