CONTINUOUS MAPPINGS AND BOUNDED LINEAR OPERATORS IN FUZZY n-NORMED LINEAR SPACES

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ABSTRACT. The aim of this paper to define different types of continuities of operators and boundedness of linear operators over fuzzy n-normed linear spaces. Also some definitions such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity, strongly fuzzy continuity, weakly fuzzy boundedness and strongly fuzzy boundedness are given in fuzzy n-normed linear spaces. In addition, some theorems related to these definitions are proved.

1. INTRODUCTION

Since the introduction of the concept of fuzzy norm on a linear space by Katsaras [11] in 1984, many authors have studied the fuzzy topological vector spaces. In 1992, Felbin [4] introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated to this fuzzy norm is of Kaleva type [10] fuzzy metric. In 1994, Chang and Moderson [3] introduced another idea of a fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric of it is of Kramosil and Michalek type [13]. Recently Xiao and Zhu [18] redefined the idea of Felbin's [4] definition of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space. In 2003, Bag and Samanta [1] introduced a definition of a fuzzy norm and proved a decomposition theorem of a fuzzy norm into a family of crisp norms.Also in 2005, Bag and Samanta [2] give some properties on fuzzy norms. In [5, 6, 14] were studied various properties of these types of fuzzy normed linear spaces. The concept of 2-norm and n-norm on a linear space has been introduced and developed by Gähler in [7, 8]. Following Misiak [16], Kim and Cho [12] and Malčeski [15] developed the theory of n-normed space. In [9], Gunawan and Mashadi gave a simple way to derive an (n-1)-norm from the n-norm and realized that any n-normed space is an (n-1)-normed space. Also, Narayanan and Vijayabalaji [17] introduced the concept of fuzzy n-normed linear space.

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In the present paper, we define various types of continuities of operators and boundedness of linear operators over fuzzy n-normed linear spaces such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity, strongly fuzzy continuity, weakly fuzzy boundedness and strongly fuzzy boundedness.

2. Preliminaries

Definition 1 ([9]). *Let $n \in \mathbb{N}$ and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $||\cdot, ..., \cdot||$ on $\underbrace{X \times \cdots \times X}_{n}$ satisfying the following four properties,*

(1) $||x_1, x_2, ..., x_n|| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(2) $||x_1, x_2, ..., x_n||$ is invariant under any permutation,
(3) $||x_1, x_2, ..., \alpha x_n|| = |\alpha| \; ||x_1, x_2, ..., x_n||$ for any $\alpha \in \mathbb{R}$,
(4) $||x_1, x_2, ..., x_{n-1}, y+z|| \leq ||x_1, x_2, ..., x_{n-1}, y|| + ||x_1, x_2, ..., x_{n-1}, z||$,

is called an n-norm on X and the pair $(X, ||\cdot, ..., \cdot||)$ is called an n-normed space.

Example 1. *Let $X = \mathbb{R}^n$ and*

$$||x_1, x_2, ..., x_n||_E = abs \left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \right),$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, ..., n$. Then $(X, ||\cdot, ..., \cdot||_E)$ is an n-normed space which is called Euclidean n-normed space.

Definition 2 ([9]). *Let $(X, ||\cdot, ..., \cdot||)$ be an n-normed linear space and $x(k)$ be a sequence in X. Then $x(k)$ is said to be convergent if there exists a $x \in X$ such that*

$$\lim_{k \to \infty} ||x_1, x_2, ..., x_{n-1}, x(k) - x|| = 0$$

for every $x_1, x_2, ..., x_{n-1} \in X$. Then x called limit of the sequence $x(k)$ and denoted by $\lim x(k) = x$ or $x(k) \to x$.

Definition 3 ([9]). *A sequence $x(k)$ in $(X, ||\cdot, ..., \cdot||)$ is called Cauchy sequence, if*

$$\lim_{k,l \to \infty} ||x_1, x_2, ..., x_{n-1}, x(k) - x(l)|| = 0$$

for every $x_1, x_2, ..., x_{n-1} \in X$ and $k, l \in \mathbb{N}$.

Definition 4 ([17]). *Let X be a linear space over a real field F. A fuzzy subset N of $\underbrace{X \times \cdots \times X}_{n} \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy n-norm on X if and only if*

(N1) for all $t \in \mathbb{R}$ with $t \le 0$, $N(x_1, x_2, ..., x_n, t) = 0$,
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(N3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, ..., x_n$,
(N4) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, t/|c|)$, if $c \ne 0$, $c \in F$,
(N5) for all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, ..., x_n + x_n', s + t) \ge \min\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x_n', t)\},$$

(N6) $N(x_1, x_2, ..., x_n, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and

$$\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1.$$

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 1. *From (N3), it follows that in a f-n-NLS,*

(N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, t/|c|),$$

if $c \ne 0$,
(N5) for all $s, t \in \mathbb{R}$,

$$\begin{aligned} & N(x_1, x_2, ..., x_i + x_i', ..., x_n, s + t) \\ \ge\ & \min\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x_i', ..., x_n, t)\}. \end{aligned}$$

Example 2. *Let $(X, \|\cdot, \cdot, ..., \cdot\|)$ be an n-normed space as in Definition 1. Define,*

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t + \|x_1, x_2, ..., x_n\|} & if \quad t > 0,\ t \in \mathbb{R}, \\ 0 & if \quad t \le 0 \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$. Then (X, N) is a f-n-NLS.

Theorem 1 ([17]). *Let (X, N) be a f-n-NLS. Assume further those*

(N7) $N(x_1, x_2, ..., x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent.

Define

$$\|x_1, x_2, ..., x_n\|_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \ge \alpha\},\ \alpha \in (0, 1).$$

Then $\{\|\cdot, \cdot, ..., \cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on X. These n-norms are called α–n-norms on X corresponding to the fuzzy n-norm N on X.

Theorem 2. *Let* $\{||\cdot,\cdot,...,\cdot||_\alpha : \alpha \in (0,1)\}$ *ascending family of n-norms on* X. *Now we define a function* $N' : \underbrace{X \times \cdots \times X}_{n} \times \mathbb{R} \to [0,1]$ *by*

$$N'(x_1, x_2, ..., x_n, t) = \begin{cases} \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \le t\} & \text{otherwise} \\ 0 & \text{if } x_1, x_2, ..., x_n \text{ are linearly dependent and } t = 0 \end{cases}.$$

Then N' *is a fuzzy n-norm on* X.

Proof. (N1) For all $t \in \mathbb{R}$ with $t < 0$ we have

$$N'(x_1, x_2, ..., x_n, t) = \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \le t\} = 0$$

for all $x \in X$. For $t = 0$ and $x_1, x_2, ..., x_n$ are linearly dependent from the definition $N'(x_1, x_2, ..., x_n, t) = 0$.

(N2) Let $N'(x_1, x_2, ..., x_n, t) = 1$ for all $t > 0$. Choose any $\varepsilon \in (0,1)$. Then for any $t > 0$, there exists $\alpha_t \in (\varepsilon, 1]$ such that $||x_1, x_2, ..., x_n||_{\alpha_t} \le t$, and hence $||x_1, x_2, ..., x_n||_\varepsilon \le t$. Since $t > 0$ is arbitrary, this implies that $||x_1, x_2, ..., x_n||_\varepsilon = 0$ then $x_1, x_2, ..., x_n$ are linearly dependent.

If $x_1, x_2, ..., x_n$ are linearly dependent then

$$\begin{aligned} N'(x_1, x_2, ..., x_n, t) &= \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \le t\} \\ &= \vee\{\alpha : \alpha \in (0,1)\} = 1 \end{aligned}$$

for all $t > 0$.

(N3) Since $||x_1, x_2, ..., x_n||_\alpha$ is invariant under any permutation, then $N'(x_1, x_2, ..., x_n, t)$ is invariant under any permutation.

(N4) Let $c \ne 0$, $c \in F$ and $t > 0$. Then

$$\begin{aligned} & N'(x_1, x_2, ..., cx_n, t) \\ = {} & \vee\{\alpha : ||x_1, x_2, ..., cx_n||_\alpha \le t\} = \vee\{\alpha : |c|||x_1, x_2, ..., x_n||_\alpha \le t\} \\ = {} & \vee\left\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \le \frac{t}{|c|}\right\} = N'\left(x_1, x_2, ..., x_n, \frac{t}{|c|}\right) \end{aligned}$$

for all $x_1, x_2, ..., x_n \in X$.

(N5) We have to show that for all $s, t \in \mathbb{R}$,

$$N'(x_1, x_2, ..., x_n + x_n', s + t) \ge \min\left\{ \begin{array}{c} N'(x_1, x_2, ..., x_n, s), \\ N'(x_1, x_2, ..., x_n', t) \end{array} \right\}.$$

If,

(a) $s + t < 0$, (b) $s = t = 0$, (c) $s + t > 0$; $s > 0$, $t < 0$; $s < 0$, $t > 0$, then in these cases the ralation is obvious. If,

(d) $s > 0$, $t > 0$, let $p = N'(x_1, x_2, ..., x_n, s)$, $q = N'(x_1, x_2, ..., x_n', t)$ and $p \le q$.

If $p = 0$, $q = 0$ then obviously (N5) holds.

Let $0 < r < p \leq q$. Then there exists $\alpha > r$ such that $||x_1, x_2, ..., x_n||_\alpha \leq s$ and there exists $\beta > r$ such that $||x_1, x_2, ..., x'_n||_\beta \leq t$. Let $\gamma = \alpha \wedge \beta > r$. Thus

$$\begin{aligned} ||x_1, x_2, ..., x_n||_\gamma &\leq ||x_1, x_2, ..., x_n||_\alpha \leq s \text{ and} \\ ||x_1, x_2, ..., x'_n||_\gamma &\leq ||x_1, x_2, ..., x'_n||_\beta \leq t. \end{aligned}$$

Now

$$||x_1, x_2, ..., x_n + x'_n||_\gamma \leq ||x_1, x_2, ..., x_n||_\gamma + ||x_1, x_2, ..., x'_n||_\gamma \leq s + t.$$

Therefore $N'(x_1, x_2, ..., x_n + x'_n, s+t) \geq \gamma > r$. Since $0 < r < \gamma$ is arbitrary thus

$$N'(x_1, x_2, ..., x_n + x'_n, s + t) \geq p = \min \left\{ \begin{array}{c} N'(x_1, x_2, ..., x_n, s), \\ N'(x_1, x_2, ..., x'_n, t) \end{array} \right\}.$$

Similarly if $p \geq q$, then the relation also holds.

(N6) Let $x_1, x_2, ..., x_n \in X$ and $\alpha \in (0, 1)$. Now $t > ||x_1, x_2, ..., x_n||_\alpha$, then

$$N'(x_1, x_2, ..., x_n, t) = \vee\{\beta : ||x_1, x_2, ..., x_n||_\beta \leq t\} \geq \alpha$$

where $\beta \in (0, 1)$. So $\lim_{t\to\infty} N'(x_1, x_2, ..., x_n, t) = 1$.

Next we show that $N'(x_1, x_2, ..., x_n, \cdot)$ is a nondecreasing function of $\mathbb{R}$. If $t_1 < t_2 \leq 0$, then

$$N'(x_1, x_2, ..., x_n, t_1) = N'(x_1, x_2, ..., x_n, t_2) = 0$$

for all $x_1, x_2, ..., x_n \in X$. If $t_2 > t_1 > 0$ then

$$\begin{aligned} \{\alpha &: ||x_1, x_2, ..., x_n||_\alpha \leq t_1\} \subset \{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t_2\} \\ &\Rightarrow \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t_1\} \leq \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t_2\} \\ &\Rightarrow N'(x_1, x_2, ..., x_n, t_1) \leq N'(x_1, x_2, ..., x_n, t_2). \end{aligned}$$

Thus, $N'(x_1, x_2, ..., x_n, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and hence N' is a fuzzy n-norm on X. □

(N8) We assume that, for $x_1, x_2, ..., x_n$ are linearly independent, and $N(x_1, x_2, ..., x_n, \cdot)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset $\{t : 0 < N(x_1, x_2, ..., x_n, t) < 1\}$ of $\mathbb{R}$.

Lemma 1. *If (X, N) be a fuzzy n-normed linear space satisfying (N7) and (N8) and $\{||\cdot, \cdot, ..., \cdot||_\alpha : \alpha \in (0, 1)\}$ be an ascending family of n-norms on X defined as $||x_1, x_2, ..., x_n||_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$, then for $x_{1_0}, x_{2_0}, ..., x_{n_0}$ in X which are linearly independent,*

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha) \geq \alpha$$

for all $\alpha \in (0, 1)$.

Proof. Let $||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = T$, then $T > 0$. Then there exists a sequence $(t_n)_n$, $t_n > 0$, $n = 1, 2, ...$ such that $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_n) \geq \alpha$ and $t_n \downarrow T$. Therefore,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_n) \geq \alpha \Rightarrow N(x_{1_0}, x_{2_0}, ..., x_{n_0}, \lim_{n\to\infty} t_n) \geq \alpha \text{ by (N8)}$$

which implies $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha) \geq \alpha$ for all $\alpha \in (0, 1)$. □

Lemma 2. *If (X, N) be a fuzzy n-normed linear space satisfying (N7) and (N8) and $\{||\cdot, \cdot, ..., \cdot||_\alpha : \alpha \in (0, 1)\}$ be a family of α-n-norms on X defined by*

$$||x_1, x_2, ..., x_n||_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\},$$

$\alpha \in (0, 1)$ and for all $x \in X$. Then for $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent, $\alpha \in (0, 1)$ and for $t' > 0$,

$$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = t' \textit{ iff } N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha.$$

Proof. Let $\alpha \in (0, 1)$, $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent and

$$t' = ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = \wedge\{s : N(x_{1_0}, x_{2_0}, ..., x_{n_0}, s) \geq \alpha\}.$$

Since $N(x_1, x_2, ..., x_n, \cdot)$ is continuous by (N8), we have

(i) $$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') \geq \alpha.$$

Also

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') \leq N(x_{1_0}, x_{2_0}, ..., x_{n_0}, s)$$

if $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, s) \geq \alpha$. If possible, let $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') > \alpha$, then by the continuity of $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, \cdot)$ at t', there exists $t'' < t'$ such that $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t'') > \alpha$ which is impossible, since $t' = \wedge\{s : N(x_{1_0}, x_{2_0}, ..., x_{n_0}, s) \geq \alpha\}$. Thus

(ii) $$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') \leq \alpha.$$

By combining (i) and (ii) we get $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha$. Thus

(iii) $$t' = ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha \Rightarrow N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha.$$

Next if $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha$, $\alpha \in (0, 1)$, then from definition

(iv) $$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = \wedge\{t : N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t) \geq \alpha\} = t'$$

(Since $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, \cdot)$ is strictly increasing in $\{t : 0 < N(x_1, x_2, ..., x_n, t) < 1\}$). From (iii) and (iv) we have, for $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent, $\alpha \in (0, 1)$ and for $t' > 0$,
$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = t'$ iff $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha$. □

Theorem 3. *Let (X, N) be a fuzzy n-normed linear space satisfying (N7) and (N8). Let*

$$||x_1, x_2, ..., x_n||_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\},$$

$\alpha \in (0,1)$ *and* $N' : \underbrace{X \times \cdots \times X}_{n} \times \mathbb{R} \to [0,1]$ *be a function defined by*

$$N'(x_1, x_2, ..., x_n, t) = \begin{cases} \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t\} & \text{otherwise} \\ 0 & \text{if } x_1, x_2, ..., x_n \text{ are linearly dependent and } t = 0 \end{cases}.$$

Then

(i) $\{||\cdot, \cdot, ..., \cdot||_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-n-norms on X.
(ii) N' is a fuzzy n-norm on X.
(iii) $N' = N$.

Proof. We only prove (iii) and we consider the following cases.

Let $(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) \in \underbrace{X \times \cdots \times X}_{n} \times \mathbb{R}$ and $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = \alpha_0$.

Case I. $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly dependent, $t_0 \leq 0$. Then,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 0.$$

Case II. $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly dependent, $t_0 > 0$. Then,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 1.$$

Case III. $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent, $t_0 \leq 0$. Then,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 0.$$

Case IV. $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent and $t_0 > 0$ such that $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 0$.

For $\alpha \in (0,1)$,

$$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = \wedge\{t : N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t) \geq \alpha\}.$$

By Lemma 1 we have $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha) \geq \alpha$ for all $\alpha \in (0,1)$. Since $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 0 < \alpha$, it follows that $t_0 < ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha$ for all $\alpha \in (0,1)$. So,

$$N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = \vee\{\alpha : ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha \leq t_0\} = \vee\varnothing = 0.$$

Therefore,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 0.$$

Case V. When $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent and $t_0 > 0$ such that $0 < N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) < 1$. Let $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$. Now

$$N'(x_1, x_2, ..., x_n, t) = \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t\}$$

when

(i) $x_1, x_2, ..., x_n$ are linearly independent and $t \neq 0$,

(ii) $$||x_1, x_2, ..., x_n||_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}$$

for all $x_1, x_2, ..., x_n \in X$ and $0 < \alpha < 1$. Since $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = \alpha_0$, we have from (ii)

(iii) $$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_{\alpha_0} \leq t_0.$$

Using (iii), we get from (i)

$$\begin{aligned} N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) \;&\geq\; \alpha_0 \Rightarrow N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) \geq \qquad \text{(iv)} \\ \text{(2.1)} \qquad & N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0). \end{aligned}$$

Now from Lemma 2, we have

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = \alpha_0 \Leftrightarrow ||x_{1_0}, x_{2_0}, ..., x_{n_0}||_{\alpha_0} = t_0.$$

Now for $1 > \alpha > \alpha_0$, let $||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = t'$, then $t' \geq t_0$. Then by Lemma 2, $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha$. So,

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') = \alpha > \alpha_0 = N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0).$$

Since $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, \cdot)$ is strictly monotonically increasing in $S_{x_0} = \{t : 0 < N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t) < 1\}$, $t', t_0 \in S_{x_0}$ and

$$N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t') > N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0),$$

it follows that $t' > t_0$. So for $1 > \alpha > \alpha_0$, $||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha = t' \nleq t_0$. Hence

(v) $$N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) \leq \alpha_0 = N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0).$$

By (iv) and (v) we have $N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0)$.

Case VI. When $x_{1_0}, x_{2_0}, ..., x_{n_0}$ are linearly independent and $t_0 > 0$ such that $N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 1$. Note that,

(i) $$N'(x_1, x_2, ..., x_n, t) = \vee\{\alpha : ||x_1, x_2, ..., x_n||_\alpha \leq t\},$$

for $x_1, x_2, ..., x_n$ are linearly independent and $t \neq 0$,

(ii) $$||x_1, x_2, ..., x_n||_\alpha = \wedge\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}$$

$\alpha \in (0, 1)$, $x_1, x_2, ..., x_n \in X$. It follows that, for all $\alpha \in (0, 1)$,

$$||x_{1_0}, x_{2_0}, ..., x_{n_0}||_\alpha \leq t_0 \text{ by (ii)} \Rightarrow N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 1 \text{ by (i)}.$$

Thus $N'(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = N(x_{1_0}, x_{2_0}, ..., x_{n_0}, t_0) = 1$.

Hence, $N(x_1, x_2, ..., x_n, t) = N'(x_1, x_2, ..., x_n, t)$ for all $x_1, x_2, ..., x_n \in X$ and for all $t \in \mathbb{R}$. □

Definition 5. *Let (X, N) be a f-n-NLS and $x(k)$ be a sequence in X. Then $x(k)$ is said to be convergent if there exists a $x \in X$ such that $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x(k) - x, t) = 1$ for every $x_1, x_2, ..., x_{n-1} \in X$ and for all $t > 0$. Then x called limit of the sequence $x(k)$ and denoted by $\lim x(k) = x$ or $x(k) \to x$.*

Definition 6. *A sequence $x(k)$ in (X, N) is called Cauchy sequence, if*

$$\lim_{k,l\to\infty} N(x_1, x_2, ..., x_{n-1}, x(k) - x(l), t) = 1$$

for every $x_1, x_2, ..., x_{n-1} \in X$ and for all $t > 0$, $k, l \in \mathbb{N}$.

3. Main Results

Throughout this paper (X, N_1) and (Y, N_2) are f-n-NLS over the same field of scalars.

Definition 7. *A mapping T from (X, N_1) to (Y, N_2) is said to be fuzzy continuous at $z \in X$, if for given $\varepsilon > 0$, $\alpha \in (0,1)$, there exists $\delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \in (0,1)$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$,*

$$N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta) > \beta \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) > \alpha.$$

If T is fuzzy continuous at each point of X, then T is said to be fuzzy continuous on X.

Definition 8. *A mapping T from (X, N_1) to (Y, N_2) is said to be strongly fuzzy continuous at $z \in X$, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$,*

$$N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta).$$

If T is strongly fuzzy continuous at each point of X, then T is said to be strongly fuzzy continuous on X.

Definition 9. *A mapping T from (X, N_1) to (Y, N_2) is said to be weakly fuzzy continuous at $z \in X$, if for a given $\varepsilon > 0$, $\alpha \in (0,1)$, there exists $\delta = \delta(\alpha, \varepsilon) > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$,*

$$N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) \geq \alpha.$$

If T is weakly fuzzy continuous at each point of X, then T is said to be weakly fuzzy continuous on X.

Definition 10. *A mapping T from (X, N_1) to (Y, N_2) is said to be sequentially fuzzy continuous at $z \in X$, if for any sequence $x(k)$ in X with $x(k) \to z$ implies $Tx(k) \to Tz$, $k \in \mathbb{N}$. I.e., for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t > 0$,*

$$\begin{aligned} \lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, x(k) - z, t) &= 1 \Rightarrow \\ \lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, t) &= 1. \end{aligned}$$

If T is sequentially fuzzy continuous at each point of X, then T is said to be sequentially fuzzy continuous on X.

Remark 2. *It is easy to see that if a mapping is strongly fuzzy continuous then it is weakly fuzzy continuous.*

Theorem 4. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a mapping. If T is strongly fuzzy continuous then it is sequentially fuzzy continuous..*

Proof. Assume that T is strongly fuzzy continuous at $z \in X$. Then for each $\varepsilon > 0$, there exists $\delta = \delta(z, \varepsilon) > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$,

(i) $\quad N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta)$.

Let $x(k)$ be a sequence in X such that $x(k) \to z$, i.e.,

(ii) $$\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, x(k) - z, t) = 1$$

for all $t > 0$. Now from (i) we have,

$$N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, x(k) - z, \delta)$$

for $k = 1, 2, ...$ Then,

$$\lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) \geq \lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, x(k) - z, \delta),$$

which implies $\lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) = 1$ by (ii). Since $\varepsilon > 0$ is arbitrary, it follows $Tx(k) \to Tz$. □

Theorem 5. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a mapping. Then T is fuzzy continuous iff it is sequentially fuzzy continuous.*

Proof. Assume that T is fuzzy continuous at $z \in X$. Let $x(k)$ be a sequence in X such that $x(k) \to z$. Let $\varepsilon > 0$ be given. Choose $\alpha \in (0, 1)$. Since T is fuzzy continuous at z, then there exists $\delta = \delta(\alpha, \varepsilon) > 0$ and $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$ and $y_1, y_2, ..., y_{n-1} \in Y$,

$$N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta) > \beta \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) > \alpha.$$

Since $x(k) \to z$ in X, there exists positive integer k_0 such that

$$N_1(x_1, x_2, ..., x_{n-1}, x(k) - z, \delta) > \beta$$

for all $k \geq k_0$. Then $N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) > \alpha$ for all $k \geq k_0$. So for a given $\varepsilon > 0$ and for any $\alpha \in (0,1)$, there exists positive integer k_0 such that $N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) > \alpha$ for all $k \geq k_0$. This implies $\lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Tz, \varepsilon) = 1$. Since $\varepsilon > 0$ is arbitrary, thus $Tx(k) \to Tz$ in Y.

Next we suppose that T is sequentially fuzzy continuous at $z \in X$. If possible suppose that T is not fuzzy continuous at z. Thus, there exists $\varepsilon > 0$ and $\alpha > 0$ such that for any $\delta > 0$ and $\beta \in (0,1)$, there exists w (depending on δ, β) such that

$$N_1(x_1, x_2, ..., x_{n-1}, z - w, \delta) > \beta \text{ but } N_2(y_1, y_2, ..., y_{n-1}, Tz - Tw, \varepsilon) \leq \alpha.$$

Thus for $\beta = 1 - \frac{1}{k+1}$, $\delta = \frac{1}{k+1}$, $k = 1, 2, ...$, there exists $w(k)$ such that

$$\begin{aligned} N_1\left(x_1, x_2, ..., x_{n-1}, z - w(k), \frac{1}{k+1}\right) &> 1 - \frac{1}{k+1} \qquad \text{(i)} \\ \text{but } N_2(y_1, y_2, ..., y_{n-1}, Tz - Tw(k), \varepsilon) &\leq \alpha. \end{aligned}$$

Taking $\delta > 0$, there exists k_0 such that $\frac{1}{k+1} < \delta$ for all $k \geq k_0$. Then,

$$\begin{aligned} N_1(x_1, x_2, ..., x_{n-1}, z - w(k), \delta) &\geq N_1\left(x_1, x_2, ..., x_{n-1}, z - w(k), \frac{1}{k+1}\right) \\ &> 1 - \frac{1}{k+1} \end{aligned}$$

for all $k \geq k_0$. Therefore $\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, z - w(k), \delta) \geq 1 \Rightarrow w(k) \to z$. But from (i), $N_2(y_1, y_2, ..., y_{n-1}, Tz - Tw(k), \varepsilon) \leq \alpha$ so

$$N_2(y_1, y_2, ..., y_{n-1}, Tz - Tw(k), \varepsilon) \nrightarrow 1$$

as $k \to \infty$. Thus $Tw(k)$ does not convergence to Tz whereas $w(k) \to z$ (w.r.t N_1), which is a contradiction to our assumption. Hence T is fuzzy continuous at z. □

Definition 11. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. T is said to be strongly fuzzy bounded on X iff there exists positive real number M such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $s \in \mathbb{R}$,*

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{s}{M}\right).$$

Example 3. *The zero and identity operators are strongly fuzzy bounded.*

Example 4. *Let $(X, ||\cdot, ..., \cdot||)$ be an n-normed linear space. We define two functions N_1 and N_2 from $\underbrace{X \times \cdots \times X}_{n} \times \mathbb{R}$ to $[0,1]$ by*

$$N_1(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t + \alpha_1 ||x_1, x_2, ..., x_n||} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

and

$$N_2(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t+\alpha_2\|x_1,x_2,...,x_n\|} & if \quad t > 0 \\ 0 & if \quad t \le 0 \end{cases}$$

where α_1 and α_2 are two fixed positive real numbers and $\alpha_1 > \alpha_2$. It is easy to show that N_1 and N_2 are fuzzy n-norms on X. We define an operator $T : (X, N_1) \to (X, N_2)$ by $Ty = ry$ where $r \in \mathbb{R}\backslash\{0\}$ is fixed. Clearly T is a linear operator. If we choose a positive number M such that $M \ge |r|$, then it can be shown that

(ii) $$N_2(x_1, x_2, ..., x_{n-1}, Ty, t) \ge N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M}\right)$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, and for all $t \in \mathbb{R}$. For all $x_1, x_2, ..., x_{n-1}, y \in X$, $M \ge |r|$ we have $\alpha_1 M \ge \alpha_2|r|$, for all $t > 0$,

$$\begin{aligned}
&\Rightarrow \quad \alpha_1 M\|x_1, x_2, ..., x_{n-1}, y\| \ge \alpha_2|r|\|x_1, x_2, ..., x_{n-1}, y\| \\
&\Rightarrow \quad t + \alpha_1 M\|x_1, x_2, ..., x_{n-1}, y\| \ge t + \alpha_2|r|\|x_1, x_2, ..., x_{n-1}, y\| \\
&\Rightarrow \quad \frac{t}{t + \alpha_2|r|\|x_1, x_2, ..., x_{n-1}, y\|} \ge \frac{t}{t + \alpha_1 M\|x_1, x_2, ..., x_{n-1}, y\|}, \\
&\Rightarrow \quad \frac{t}{t + \alpha_2\|x_1, x_2, ..., x_{n-1}, ry\|} \ge \frac{\frac{t}{M}}{\frac{t}{M} + \alpha_1\|x_1, x_2, ..., x_{n-1}, y\|} \\
&\Rightarrow \quad N_2(x_1, x_2, ..., x_{n-1}, Ty, t) \ge N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M}\right).
\end{aligned}$$

If $t \le 0$ then above relation holds for all $x_1, x_2, ..., x_{n-1}, y \in X$. Hence T is a strongly fuzzy bounded linear operator.

Definition 12. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. T is said to be weakly fuzzy bounded on X if for any $\alpha \in (0, 1)$, there exists $M_\alpha > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t \in \mathbb{R}$,*

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \ge \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \ge \alpha.$$

Theorem 6. *Let (X, N_1) and (Y, N_2) be f-n-NLS and $T : X \to Y$ be a linear operator. If T is strongly fuzzy bounded then it is weakly fuzzy bounded but not conversely.*

Proof. We suppose that T is strongly fuzzy bounded. Thus there exists $M > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t \in \mathbb{R}$ we have

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \ge N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M}\right).$$

Thus for any $\alpha \in (0,1)$, there exists $M_\alpha (= M) > 0$ such that

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t \in \mathbb{R}$. This implies that T is weakly fuzzy bounded. □

For the converse result we consider the following example.

Example 5. *Let $(X, ||\cdot, ..., \cdot||)$ be an n-normed linear space. We define two functions N_1 and N_2 from $\underbrace{X \times \cdots \times X}_{n} \times \mathbb{R}$ to $[0,1]$ by*

$$N_1(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t^2 - ||x_1, x_2, ..., x_n||^2}{t^2 + ||x_1, x_2, ..., x_n||^2} & if \quad t > ||x_1, x_2, ..., x_n|| \\ 0 & if \quad t \leq ||x_1, x_2, ..., x_n|| \end{cases}$$

and

$$N_2(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t + ||x_1, x_2, ..., x_n||} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$. It can be shown that N_1 is a fuzzy n-norm on X. We define a linear operator $T : X \to X$ by $Ty = y$. If we choose $M_\alpha = \frac{1}{1-\alpha}$ for all $\alpha \in (0,1)$, we get

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(x_1, x_2, ..., x_{n-1}, Ty, t) \geq \alpha.$$

Hence T is weakly fuzzy bounded. But it can be proved that T is not strongly fuzzy bounded.

Definition 13. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. T is said to be uniformly bounded if there exists $M > 0$ such that for all $\alpha \in (0,1)$,*

$$||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq M||x_1, x_2, ..., x_{n-1}, y||_\alpha^1$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$, where $||\cdot, \cdot, ..., , \cdot||_\alpha^1$ and $||\cdot, \cdot, ..., , \cdot||_\alpha^2$ are α-n-norms of N_1 and N_2 respectively.

Theorem 7. *Let (X, N_1) and (Y, N_2) be two f-n-NLS satisfying (N7) and (N8). Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator. Then T is strongly fuzzy bounded iff it is uniformly bounded with respect to α-n-norms of N_1 and N_2.*

Proof. Let $||\cdot, \cdot, ..., , \cdot||_\alpha^1$ and $||\cdot, \cdot, ..., , \cdot||_\alpha^2$ be α-n-norms of N_1 and N_2 respectively.

First we suppose that T is strongly fuzzy bounded. Thus there exists $M > 0$ such that

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{s}{M}\right)$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $s \in \mathbb{R}$. This implies

(i) $$N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq N_1 (x_1, x_2, ..., x_{n-1}, My, s).$$

Now $||x_1, x_2, ..., x_{n-1}, My||^1_\alpha < t$

$$\Rightarrow \wedge\{s : N_1 (x_1, x_2, ..., x_{n-1}, My, s) \geq \alpha\} < t$$

$$\Rightarrow \exists s_0 < t \text{ such that } N_1 (x_1, x_2, ..., x_{n-1}, My, s_0) \geq \alpha$$

$$\Rightarrow \exists s_0 < t \text{ such that } N_2(y_1, y_2, ..., y_{n-1}, Ty, s_0) \geq \alpha \text{ by (i).}$$

$$\Rightarrow ||y_1, y_2, ..., y_{n-1}, Ty||^2_\alpha \leq s_0 < t.$$

Hence

$$||y_1, y_2, ..., y_{n-1}, Ty||^2_\alpha \leq ||x_1, x_2, ..., x_{n-1}, My||^1_\alpha = M||x_1, x_2, ..., x_{n-1}, y||^1_\alpha$$

for all $\alpha \in (0, 1)$. This implies that T is uniformly bounded w.r.t. α-n-norms, $\alpha \in (0, 1)$.

Conversely suppose that, there exists $M > 0$ such that

(ii) $$||y_1, y_2, ..., y_{n-1}, Ty||^2_\alpha \leq M||x_1, x_2, ..., x_{n-1}, y||^1_\alpha$$

holds for all $\alpha \in (0, 1)$ and for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$. Now $r < N_1 (x_1, x_2, ..., x_{n-1}, My, s)$ then

$r < \vee\{\alpha \in (0, 1) : ||x_1, x_2, ..., x_{n-1}, My||^1_\alpha \leq s\}$ by Theorem 3. Then there exists an $\alpha_0 \in (0, 1)$ such that $r < \alpha_0$ and $||x_1, x_2, ..., x_{n-1}, My||^1_{\alpha_0} \leq s$. Therefore $||y_1, y_2, ..., y_{n-1}, Ty||^2_{\alpha_0} \leq s$ by (ii) which implies

$N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq \alpha_0 > r$. Hence

$$\begin{aligned} N_2(y_1, y_2, ..., y_{n-1}, Ty, s) &\geq N_1 (x_1, x_2, ..., x_{n-1}, My, s) \\ &= N_1 \left(x_1, x_2, ..., x_{n-1}, y, \frac{s}{M}\right) \end{aligned}$$

which shows T is strongly fuzzy bounded. □

Theorem 8. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. Then*

(i) T is strongly fuzzy continuous everywhere on X if T is strongly fuzzy continuous at a point $z \in X$.

(ii) T is strongly fuzzy continuous iff T is strongly fuzzy bounded.

Proof. (i) Since T is strongly fuzzy continuous at a point $z \in X$, thus for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ we have

$$N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta).$$

Taking any $w \in X$ and replacing y by $y + z - w$ we get

$N_2(y_1, y_2, ..., y_{n-1}, T(y + z - w) - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y + z - w - z, \delta)$

$\Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty + Tz - Tw - Tz, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y - w, \delta)$

$\Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - Tw, \varepsilon) \geq N_1(x_1, x_2, ..., x_{n-1}, y - w, \delta)$. Since w is arbitrary, it follows that T is strongly fuzzy continuous on X.

(ii) First we suppose that T is strongly fuzzy bounded. Thus there exists $M > 0$ such that

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, \varepsilon) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{\varepsilon}{M}\right)$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $\varepsilon > 0$, i.e.,

$$N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), \varepsilon) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \frac{\varepsilon}{M}\right)$$

which implies

$$N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), \varepsilon) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \delta\right)$$

where $\delta = \frac{\varepsilon}{M}$. This shows that T is strongly fuzzy continuous at $\underline{0}$ and hence it is strongly fuzzy continuous on X.

Conversely, assume that T is strongly fuzzy continuous on X. Using continuity of T at $y = \underline{0}$, for $\varepsilon = 1$, there exists $\delta > 0$ such that

$$N_2(x_1, x_2, ..., x_{n-1}, Ty - T(\underline{0}), 1) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \delta\right)$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$. Suppose that $y \neq \underline{0}$ and $t > 0$. Putting $u = y/t$ then

$$\begin{aligned} N_2(y_1, y_2, ..., y_{n-1}, Ty, t) &= N_2(y_1, y_2, ..., y_{n-1}, tTu, t) \\ &= N_2(y_1, y_2, ..., y_{n-1}, Tu, 1) \\ &\geq N_1\left(x_1, x_2, ..., x_{n-1}, u, \delta\right) \\ &= N_1\left(x_1, x_2, ..., x_{n-1}, y/t, \delta\right) \\ &= N_1\left(x_1, x_2, ..., x_{n-1}, y, t/M\right) \end{aligned}$$

where $M = 1/\delta$. So

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq N_1\left(x_1, x_2, ..., x_{n-1}, y, t/M\right)$$

If $y \neq \underline{0}$ and $t \leq 0$ then

$$N_2(y_1, y_2, ..., y_{n-1}, Ty, t) = 0 = N_1\left(x_1, x_2, ..., x_{n-1}, y, t/M\right).$$

If $y = \underline{0}$ and $t \in \mathbb{R}$ then $T(\underline{0}_X) = \underline{0}_Y$ and

$$\begin{aligned} N_2(y_1, y_2, ..., y_{n-1}, \underline{0}_Y, t) &= N_1\left(x_1, x_2, ..., x_{n-1}, \underline{0}_X, t/M\right) = 1 \text{ if } t > 0, \\ N_2(y_1, y_2, ..., y_{n-1}, \underline{0}_Y, t) &= N_1\left(x_1, x_2, ..., x_{n-1}, \underline{0}_X, t/M\right) = 0 \text{ if } t \leq 0. \end{aligned}$$

From the above discussion it follows that T is strongly fuzzy bounded. □

Remark 3. *If T is strongly fuzzy bounded then it is sequentially fuzzy continuous on X.*

Theorem 9. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. If T is sequentially fuzzy continuous at a point then it is sequentially fuzzy continuous on X.*

Proof. Suppose that T is sequentially continuous at $y_0 \in X$. Let $y \in X$ be an arbitrary point and let $x(k)$ be a sequence in X such that $x(k) \to y$. Then for all $x_1, x_2, ..., x_{n-1}, y \in X$,

$$\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, x(k) - y, t) = 1 \text{ for all } t > 0,$$

i.e.,

$$\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, (x(k) - y + y_0) - y_0, t) = 1 \text{ for all } t > 0.$$

Since T is continuous at $y_0 \in X$, then from Theorem 5 we have, for all $y_1, y_2, ..., y_{n-1} \in Y$,

$$\lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, T(x(k) - y + y_0) - Ty_0, t) = 1 \text{ for all } t > 0$$

$\Rightarrow \lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Ty + Ty_0 - Ty_0, t) = 1$ for all $t > 0$
$\Rightarrow \lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Ty, t) = 1$ for all $t > 0$.

Thus $\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, x(k) - y, t) = 1$ for all $t > 0$ which implies $\lim_{k\to\infty} N_2(y_1, y_2, ..., y_{n-1}, Tx(k) - Ty, t) = 1$ for all $t > 0$. This shows that T is sequentially fuzzy continuous on X. □

Theorem 10. *Let (X, N_1) and (Y, N_2) be two f-n-NLS and $T : X \to Y$ be a linear operator. Then*

(i) T is weakly fuzzy continuous everywhere on X if T is weakly fuzzy continuous at a point $y_0 \in X$.
(ii) T is weakly fuzzy continuous iff T is weakly fuzzy bounded.

Proof. (i) Since T is weakly fuzzy continuous at a point $y_0 \in X$, thus for each $\varepsilon > 0$ and $\alpha \in (0, 1)$ there exists $\delta(\alpha, \varepsilon) > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ we have

$$N_1(x_1, x_2, ..., x_{n-1}, y - y_0, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - Ty_0, \varepsilon) \geq \alpha.$$

Taking any $z \in X$ and replacing y by $y + y_0 - z$ we get

$$\begin{aligned} N_1(x_1, x_2, ..., x_{n-1}, y + y_0 - z - y_0, \delta) &\geq \alpha \Rightarrow \\ N_2(y_1, y_2, ..., y_{n-1}, T(y + y_0 - z) - Ty_0, \varepsilon) &\geq \alpha, \text{ i.e.,} \\ N_1(x_1, x_2, ..., x_{n-1}, y - z, \delta) &\geq \alpha \Rightarrow \\ N_2(y_1, y_2, ..., y_{n-1}, Ty - Tz, \varepsilon) &\geq \alpha. \end{aligned}$$

Since z is arbitrary, it follows that T is weakly fuzzy continuous on X.

(ii) First we suppose that T is weakly fuzzy bounded. Thus for any $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t \in \mathbb{R}$, i.e.,
$N_1\left(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), t) \geq \alpha$, i.e.,
$N_1\left(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \frac{\varepsilon}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), \varepsilon) \geq \alpha$ for $\varepsilon > 0$, i.e.,
$N_1(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), \varepsilon) \geq \alpha$ where $\delta = \frac{\varepsilon}{M_\alpha}$.

This implies that T is weakly fuzzy continuous at $x = \underline{0}$ and hence weakly fuzzy continuous on X.

Conversely, assume that T is weakly fuzzy continuous on X. Using weakly fuzzy continuity of T at $y = \underline{0}$, for $\varepsilon = 1$ we have for all $\alpha \in (0,1)$, there exists $\delta(\alpha, 1) > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$,

$$N_1(x_1, x_2, ..., x_{n-1}, y - \underline{0}, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty - T(\underline{0}), 1) \geq \alpha$$

i.e. $N_1(x_1, x_2, ..., x_{n-1}, y, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, 1) \geq \alpha$.

Suppose that $y \neq \underline{0}$ and $t > 0$. Putting $y = u/t$ then

$$N_1(x_1, x_2, ..., x_{n-1}, u/t, \delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, T(u/t), 1) \geq \alpha$$

i.e. $N_1(x_1, x_2, ..., x_{n-1}, u, t\delta) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Tu, t) \geq \alpha$, i.e.,
$N_1\left(x_1, x_2, ..., x_{n-1}, u, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Tu, t) \geq \alpha$ where $M_\alpha = \frac{1}{\delta(\alpha,1)}$. This shows that T is weakly fuzzy bounded.

If $y \neq \underline{0}$ and $t \leq 0$ then

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) = N_2(y_1, y_2, ..., y_{n-1}, Ty, t)$$

for any $M_\alpha > 0$.

If $y = \underline{0}$ then for $M_\alpha > 0$,

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) = N_2(y_1, y_2, ..., y_{n-1}, Ty, t) = 1 \text{ if } t > 0,$$

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) = N_2(y_1, y_2, ..., y_{n-1}, Ty, t) = 0 \text{ if } t \leq 0.$$

From the above three cases it folows that for $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that

$$N_1\left(x_1, x_2, ..., x_{n-1}, u, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Tu, t) \geq \alpha$$

for all $x_1, x_2, ..., x_{n-1}, u \in X$, $y_1, y_2, ..., y_{n-1} \in Y$ and for all $t \in \mathbb{R}$. Hence T is weakly fuzzy bounded. □

Theorem 11. *Let (X, N_1) and (Y, N_2) be two f-n-NLS satisfying (N7) and (N8). Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator. Then T is weakly fuzzy bounded iff T be bounded w.r.t. α-n-norms of N_1 and N_2, $\alpha \in (0,1)$.*

Proof. First we suppose that T is weakly fuzzy bounded. Thus for all $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$, for all $t \in \mathbb{R}$ we have

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha,$$

i.e., $N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, t) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha$, i.e.,

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha \Rightarrow \\ \vee\{\beta &\in (0,1) : ||y_1, y_2, ..., y_{n-1}, Ty||_\beta^2 \leq t\} \geq \alpha. \end{aligned} \quad \text{(i)}$$

Now we show that,

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha \\ &\Leftrightarrow ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t. \end{aligned}$$

If $y = \underline{0}$ then the relation is obvious. Suppose $y \neq \underline{0}$. Now if

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} > \alpha \\ &\Rightarrow ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t. \end{aligned} \quad \text{(ii)}$$

If $\vee\{\beta \in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} = \alpha$, then there exists an increasing sequence $(\alpha_n)_n$ in $(0,1)$ such that $\alpha_n \uparrow \alpha$ and $||x_1, x_2, ..., x_{n-1}, M_\alpha y||_{\alpha_n}^1 \leq t$. Then, we have

$$\text{(iii)} \qquad ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t.$$

Thus from (ii) and (iii) we get,

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha \\ &\Rightarrow ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t. \end{aligned} \quad \text{(iv)}$$

Next we suppose that $||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t$.

If $||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 < t$ then

$$\text{(v)} \qquad \vee\{\beta \in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha.$$

If $||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 = t$ i.e. $\wedge\{s : N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, s) \geq \alpha\} = t$, then there exists a sequence $(s_k)_k$ in $\mathbb{R}$ such that $s_k \downarrow t$ and $N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, s_k) \geq \alpha$.

This implies $\lim_{k\to\infty} N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, s_k) \geq \alpha$,
then $N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, \lim_{k\to\infty} s_k) \geq \alpha$ by (N8).

Therefore $N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, t) \geq \alpha$, then

(vi) $\vee\{\beta \in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha.$

From (v) and (vi) it follows that,

(vii) $\vee\{\beta \in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha.$

Hence,

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\beta^1 \leq t\} \geq \alpha \qquad \text{(viii)} \\ &\Leftrightarrow ||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t. \end{aligned}$$

In a similar way we can show that,
(ix)
$\vee\{\beta \in (0,1) : ||y_1, y_2, ..., y_{n-1}, Ty||_\beta^2 \leq t\} \geq \alpha \Leftrightarrow ||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq t.$

Therefore from (viii) and (ix) we have if

$$N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, t) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha$$

holds. Then

$$||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t \Rightarrow ||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq t$$

holds. Then weakly fuzzy boundedness of T implies

$$||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq M_\alpha ||x_1, x_2, ..., x_{n-1}, y||_\alpha^1$$

for all $\alpha \in (0,1)$.

Conversely suppose that for all $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that

$$||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq M_\alpha ||x_1, x_2, ..., x_{n-1}, y||_\alpha^1$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$.

Then for $y \neq \underline{0}$,

$$||x_1, x_2, ..., x_{n-1}, M_\alpha y||_\alpha^1 \leq t \Rightarrow ||y_1, y_2, ..., y_{n-1}, Ty||_\alpha^2 \leq t$$

for all $t > 0$, i.e.,

$$\begin{aligned} \wedge\{s &: N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, s) \geq \alpha\} \leq t \\ &\Rightarrow \wedge\{s : N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq \alpha\} \leq t. \end{aligned}$$

In a similar way as above we can show that

$$\begin{aligned} \wedge\{s &: N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, s) \geq \alpha\} \leq t \\ &\Leftrightarrow N_1(x_1, x_2, ..., x_{n-1}, M_\alpha y, t) \geq \alpha \end{aligned}$$

and

$\wedge\{s : N_2(y_1, y_2, ..., y_{n-1}, Ty, s) \geq \alpha\} \leq t \Leftrightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha.$

Thus we have

$$N_1\left(x_1, x_2, ..., x_{n-1}, y, \frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(y_1, y_2, ..., y_{n-1}, Ty, t) \geq \alpha$$

for all $x_1, x_2, ..., x_{n-1}, y \in X$, $y_1, y_2, ..., y_{n-1} \in Y$.

If $y \neq \underline{0}$, $t \leq 0$ and $y = \underline{0}$, $t > 0$ then the above relation is obvious. Hence the theorem follows. □

Theorem 12. *Let (X, N_1) and (Y, N_2) be two f-n-NLS satisfying (N7) and (N8). Let $T : X \to Y$ be a linear operator. If X is of finite dimension then T is weakly fuzzy bounded.*

Proof. Since (X, N_1) and (Y, N_2) satisfying (N7), we may suppose that $||\cdot,\cdot,...,\cdot||_\alpha^1$ and $||\cdot,\cdot,...,\cdot||_\alpha^2$ are the α-n-norms of N_1 and N_2 respectively. Since X is of finite dimension, thus $T : (X, ||\cdot,\cdot,...,\cdot||_\alpha^1) \to (Y, ||\cdot,\cdot,...,\cdot||_\alpha^2)$ is a bounded linear operator for each $\alpha \in (0,1)$. Thus by Theorem 11, it follows that T is weakly fuzzy bounded. □

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References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003), 687 – 705.

[2] T. Bag and S. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems, 151 (2005), 513 – 547.

[3] S. C. Chang and J. N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. 86 (5) (1994), 429 – 436.

[4] C. Felbin, Finite-dimensional fuzzy normed linear space, Fuzzy Sets and Systems 48 (2) (1992), 239 – 248.

[5] C. Felbin, The completion of a fuzzy normed linear space, J. Math. Anal. Appl. 174 (2) (1993), 428 – 440.

[6] C. Felbin, Finite dimensional fuzzy normed linear space. II, J. Anal. 7 (1999), 117 – 131.

[7] S. Gähler, Lineare 2-normierte Räume, Math.Nachr. 28 (1964), 1 – 43.

[8] S. Gähler, Untersuchungen über verallgemeinerte m-metrische Räume, I, Math.Nachr. 40 (1969), 165 – 189.

[9] H. Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (10) (2001), 631 – 639.

[10] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Set and Systems 12 (1984), 215 – 229.

[11] A. K. Katsaras, Fuzzy topological vector spaces. II, Fuzzy Sets and Systems 12 (2) (1984), 143 – 154.

[12] S. S. Kim and Y. J. Cho, Strict convexity in linear n-normed spaces, Demonstratio Math. 29 (4) (1996), 739 – 744.

[13] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326 – 334.

[14] S. V. Krishna and K. K. M. Sarma, Separation of fuzzy normed linear spaces, Fuzzy Sets and Systems 63 (2) (1994), 207 – 217.

[15] R. Malčeski, Strong n-convex n-normed spaces, Mat. Bilten 21 (47) (1997), 81 – 102.

[16] A. Misiak, n-inner product spaces, Math.Nachr. 140(1989), 299 – 319.

[17] Al. Narayanan and S. Vijayabalaji, Fuzzy n-normed linear space, Int. J. Math. Math. Sci. 24(2005), 3963 – 3977.

[18] J.-z. Xiao and X.-h. Zhu, Fuzzy normed spaces of operators and its completeness, Fuzzy Sets and Systems 133 (3) (2003) 389 – 399.

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