Two kinds of equipackable paths and cycles *

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Abstract

A graph G is called H-equipackable if every maximal H-packing in G is also a maximum H-packing in G. In 2009, P_4-equipackable paths and cycles, M_3-equipackable paths and cycles have been characterized. In this paper, P_k-equipackable paths and cycles, M_k-equipackable paths and cycles are characterized.

Keywords: Equipackable, path, matching.

1 Introduction

The problem that we study stems from research of randomly packable graphs([1],[3]) and equipackable graphs([2],[4]). A graph G has order $|V(G)|$ and size $|E(G)|$. The path and cycle on n vertices are denoted by P_n and C_n, respectively. A matching in the graph G is a set of independent edges in G. A matching with $k(k \geq 1)$ edges is denoted by $M_k(k \geq 1)$.

Let H be a subgraph of G. By $G - H$, we denote the graph left after we delete from G the edges of H and any resulting isolated vertices. A collection of edge disjoint copies of H, say $H_1, H_2, \cdots, H_l$, where each $H_i(i = 1, 2, \cdots, l)$ is a subgraph of G, is called an H-packing in G. A graph G is called H-packable if there exists an H-packing of G. An H-packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximal if $G - \bigcup_{i=1}^{l} E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximum if no more than l edge disjoint copies of H can be packed into G. Let $p(G; H)$ denote the number of H in the maximum H-packing of G. A graph G is called H-decomposable if there exists an H-packing of G which uses all edges in G and G is called randomly

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H-decomposable if every maximal H-packing in G uses all edges in G. A graph G is called H-equipackable if every maximal H-packing in G is also a maximum H-packing in G. In 2006, Zhang and Fan([4]) characterized M_2-equipackable graphs. In 2010, B. Randerath and P. D. Vestergaard([2]) characterized all P_3-equipackable graphs. In 2009, P_4-equipackable paths and cycles, M_3-equipackable paths and cycles have been characterized ([5]).

In this paper, we investigate P_k-equipackable paths and cycles, M_k-equipackable paths and cycles.

The following lemma is important to our work:

Lemma 1. *([5]) Let G be an F-packable graph and H be an F-packable subgraph of G which satisfy: (1) H is not F-equipackable; (2) $G - H$ is F-decomposable. Then G is not F-equipackable.*

2 Main results

2.1 P_k-equipackable paths and cycles($k \geq 3$)

Theorem 2. *A path P_n is P_k-equipackable if and only if $k \leq n \leq 2k-2$ or $n = 3k-3$.*

Proof. It's easy to see that P_n is P_k-equipackable when $k \leq n \leq 2k-2$ or $n = 3k-3$.

Conversely, let P_n be a P_k-equipackable path, then we have five cases:

Case 1: When $n \leq k-1$, since P_n contains no copy of P_k, P_n can't be P_k-equipackable.

Case 2: When $k \leq n \leq 2k-2$, it's easy to see that $p(P_n; P_k)$ is 1. And P_n is P_k-packable, so each maximal P_k-packing is also a maximum P_k-packing. Thus P_n must be P_k-equipackable.

Case 3: When $2k-1 \leq n \leq 3k-4$, it's easy to see $p(P_n; P_k)$ is 2. There exists a P_k-packing H with only one copy of P_k, such that $G - H$ has two components denoted by H_2 and H_3 with $|E(H_2)| = k-2$ and $1 \leq |E(H_3)| \leq k-2$. So H is a maximal P_k-packing which is not maximum. By the definition, P_n is not P_k-equipackable.

Case 4: When $n = 3k-3$, it's easy to see that $p(P_n; P_k)$ is 2. And the number of every maximal P_k-packing of P_n is 2 by the Pigeonhole Principle. So P_n is P_k-equipackable

Case 5: When $n \geq 3k-2$, there are two subcases:

Subcase 1: When $n - (2k-1) \equiv r(mod\ k-1)$ $(r = 0, 1, \cdots, k-3)$, the path $P_{n-2k-r+2} = P_n - P_{2k-1+r}$ has $(k-1)t(t \in Z, t \geq 1)$ edges, so the path $P_n - P_{2k-1+r}$ is P_k-decomposable. From case 3, P_{2k-1+r} is not P_k-equipackable. By Lemma 1, P_n is not P_k-equipackable.

Subcase 2: When $n-(2k-1)\equiv k-2(mod\ k-1)$, we can easily verify that P_{4k-4} is not P_k-equipackable: the number of P_k in the maximal P_k-packing of P_n is 2 or 3. Obviously, the path $P_n - P_{4k-4}$ is P_k-decomposable, so P_n is not P_k-equipackable.

From the above, a path P_n is P_k-equipackable if and only if $k \leq n \leq 2k-2$ or $n = 3k-3$. □

Theorem 3. *A cycle C_n is P_k-equipackable if and only if $k \leq n \leq 3k-4$ or $n = 4k-5$.*

Proof. It's easy to see that C_n is P_k-equipackable when $k \leq n \leq 3k-4$ or $n = 4k-5$.

Conversely, let C_n be a P_k-equipackable cycle, then we have four cases:

Case 1: When $n \leq k-1$, since C_n contains no copy of P_k, C_n can't be P_k-equipackable.

Case 2: When $k \leq n \leq 2k-3$, it's easy to see that $p(C_n; P_k)$ is 1. And C_n is P_k-packable, so each maximal P_k-packing is also a maximum P_k-packing. Thus C_n must be P_k-equipackable.

Case 3: When $2k-2 \leq n \leq 3k-4$, it's easy to see $p(C_n; P_k)$ is 2. And the number of every maximal P_k-packing of P_n is 2 by the Pigeonhole Principle. So C_n is P_k-equipackable.

Case 4: When $n \geq 3k-3$, there are two subcases:

Subcase 1: When $n-(2k-2) \equiv r(mod\ k-1)$ $(r = 0, 1, \cdots, k-3)$, $C_n - P_{2k-1+r}$ has $(k-1)t(t \in Z, t \geq 1)$ edges, so $C_n - P_{2k-1+r}$ is P_k-decomposable. By Theorem 2, P_{2k-1+r} is not P_k-equipackable. By Lemma 1, C_n is not P_k-equipackable.

Subcase 2: When $n-(2k-2) \equiv k-2(mod\ k-1)$, there are two possibilities:

(1) When $n = 4k-5$, it's easy to see $p(C_n; P_k)$ is 3. And the number of every maximal P_k-packing of P_n is 3 by the Pigeonhole Principle. So C_n is P_k-equipackable.

(2) When $n \neq 4k-5$, $C_n - P_{4k-4}$ is P_k-decomposable. By Lemma 1, C_n is not P_k-equipackable.

From the above, a cycle C_n is P_k-equipackable if and only if $k \leq n \leq 3k-4$ or $n = 4k-5$. □

2.2 M_k-equipackable paths and cycles

Theorem 4. *A path P_n is M_k-equipackable if and only if $n = kt(t \in Z, t \geq 2)$.*

Proof. It's easy to see that P_n is M_k-equipackable when $n = kt(t \in Z, t \geq 2)$.

Conversely, let P_n be an M_k-equipackable path, then we have four cases:

Case 1: When $n \leq 2k-1$, P_n can't be M_k-equipackable since P_n contains no M_k.

Case 2: When $2k \leq n \leq 3k$, there are three subcases:

Subcase 1: When $n = 2k$, it's easy to see that $p(P_n; M_k)$ is 1. And P_n is M_k-packable, so each maximal M_k-packing is also a maximum M_k-packing. Thus P_n must be M_k-equipackable.

Subcase 2: When $2k+1 \leq n \leq 3k-1$, it's easy to see $p(P_n; M_k)$ is 2. Denote the edges of P_n by $e_1, e_2, \cdots, e_{n-1}$. In the following, we give a maximal M_k-packing of P_n with one copy of M_k, say $H = \{e_{i_1}, e_{i_2}, \cdots, e_{i_k}\}$, $1 = i_1 < i_2 < \cdots < i_k = n-1$, such that $\forall e_{i_j}$, $2 \leq i_{j+1} - i_j \leq 3$. So H is a maximal M_k-packing of P_n which is not maximum. By the definition, P_n is not M_k-equipackable.

Subcase 3: When $n = 3k$, it's easy to see $p(P_n; M_k)$ is 2. And the number of every maximal P_k-packing of P_n is 2 by the Pigeonhole Principle. So P_n is M_k-equipackable.

Case 3: When $3k+1 \leq n \leq 4k-1$, it's easy to see that $p(P_n; M_k)$ is 3. In the following, we give a maximal M_k-packing of P_n with two copies of M_k denoted by $H = \{H_1, H_2\}$.

If k is even, $H_1 = \{e_1, e_3, \cdots, e_{k-3}, e_{k-1}, e_{n-k+1}, e_{n-k+3}, \cdots, e_{n-3}, e_{n-1}\}$, $H_2 = \{e_2, e_4, \cdots, e_{k-2}, e_k, e_{n-k}, e_{n-k+2}, \cdots, e_{n-4}, e_{n-2}\}$.

If k is odd, $H_1 = \{e_1, e_3, \cdots, e_{k-4}, e_{k-2}, e_{n-k}, e_{n-k+2}, \cdots, e_{n-3}, e_{n-1}\}$, $H_2 = \{e_2, e_4, \cdots, e_{k-1}, e_{n-k-1}, e_{n-k+1}, \cdots, e_{n-4}, e_{n-2}\}$

Then the number of edges remained is no more than $2k-2$. There exists no M_k in the $2k-2$ edges. So H is a maximal M_k-packing of P_n which is not maximum. By the definition, P_n is not M_k-equipackable.

Case 4: When $n \geq 4k$, there are two subcases:

Subcase 1: When $n \equiv 0(mod\ k)$, it's easy to see $p(P_n; M_k)$ is $\lfloor \frac{n-1}{k} \rfloor$. And the number of every maximal P_k-packing of P_n is $\lfloor \frac{n-1}{k} \rfloor$ by the Pigeonhole Principle. So P_n is M_k-equipackable.

Subcase 2: When $n \equiv r(mod\ k)(r = 1, 2, \cdots, k-1)$, the path $P_n - P_{2k+r}$ has $kt(t \in Z, t \geq 2)$ edges, and it has a maximal M_k-packing which uses all its edges, so it is M_k-decomposable. Since P_{2k+r} is not M_k-equipackable, by Lemma 1, P_n is not M_k-equipackable.

From the above, a path P_n is M_k-equipackable if and only if $n = kt(t \in Z, t \geq 2)$. □

Theorem 5. *A cycle C_n is M_k-equipackable if and only if $2k \leq n \leq 3k-2$, or $n = kt + k - 1(t \in Z, t \geq 2)$.*

Proof. It's easy to see that C_n is M_k-equipackable when $2k \leq n \leq 3k-2$ or $n = kt + k - 1(t \in Z, t \geq 2)$.

Conversely, let C_n be an M_k-equipackable cycle, we have three cases:

Case 1: When $n \leq 2k-1$, C_n can't be M_k-equipackable since C_n contains no M_k.

Case 2: When $2k \leq n \leq 4k-1$, there are three subcases:

Subcase 1: When $2k \leq n \leq 3k-1$, it's easy to see that $p(C_n; M_k)$ is 2. And the number of every maximal P_k-packing of P_n is 2 by the Pigeonhole Principle. So C_n is M_k-equipackable.

Subcase 2: When $3k \leq n \leq 4k-2$, it's easy to see $p(C_n; M_k)$ is 3. Denote the edges of C_n by $e_1, e_2, \cdots, e_n$. In the following, we give a maximal M_k-packing of C_n with two copies of M_k denoted by $H = \{H_1, H_2\}$, where $H_1 = \{e_1, e_3, \cdots, e_{2k-1}\}$, $H_2 = \{e_2, e_4, \cdots, e_{2k}\}$. Then the number of edges remained is no more than $2k-2$. There exists no M_k in the $2k-2$ edges, so H is a maximal M_k-packing which is not maximum. By the definition, C_n is not M_k-equipackable.

Subcase 3: When $n = 4k-1$, it's easy to see $p(C_n; M_k)$ is 3. And the number of every maximal P_k-packing of P_n is 3 by the Pigeonhole Principle. So C_n is M_k-equipackable.

Case 3: When $n \geq 4k$, there are two subcases:

Subcase 1: When $n \equiv r (mod\ k) (r = 0, 1, 2, \cdots, k-2)$, $C_n - P_{2k+1+r}$ has $kt (t \in Z, t \geq 2)$ edges, and it has a maximal M_k-packing which uses all its edges, so it is M_k-decomposable. Since P_{2k+1+r} is not M_k-equipackable, by Lemma 1, C_n is not M_k-equipackable.

Subcase 2: When $n \equiv k-1 (mod\ k)$, it's easy to see $p(C_n; M_k)$ is $\lfloor \frac{n}{k} \rfloor$. And the number of every maximal P_k-packing of P_n is $\lfloor \frac{n}{k} \rfloor$ by the Pigeonhole Principle. So C_n is M_k-equipackable.

From the above, a cycle C_n is M_k-equipackable if and only if $2k \leq n \leq 3k-2$ or $n = kt + k - 1 (t \in Z, t \geq 2)$. □

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