CHARACTERISATION OF GRAPHS ON DOMINATION PARAMETERS

Selvam Avadayappan and C.S. Senthilkumar*

Department of Mathematics, V.H.N.S.N. College, Virudhunagar – 626 001, India.
e-mail: selvam_avadayappan@yahoo.co.in

*Department of Mathematics, K.S.R. College of Arts and Science, Tiruchengode – 637 215, India.
e-mail: senthil.graph@gmail.com

ABSTRACT

Let G(V,E) be a graph. A subset S of V is called a dominating set of G if every vertex in V–S is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of G is called a complementary perfect dominating set (cpd-set) if the induced subgraph $\langle V-S \rangle$ has a perfect matching. The complementary perfect domination number, $\gamma_{cp}(G)$, of G is the minimum cardinality taken over all cpd-sets in G.

An induced complementary perfect dominating set of a graph (icpd-set) is a dominating set of G such that the induced subgraph $\langle V-S \rangle$ has only independent edges. That is $\langle V-S \rangle \cong mK_2$, $m \geq$ [illegible]The minimum cardinality taken over all such icpd-sets of G is called the induced complementary perfect domination number of G, and is denoted by $\gamma_{icp}(G)$.

A subset S of V is said to be a complementary connected dominating set(ccd-set) if S is a dominating set and $\langle V-S \rangle$ is connected. The complementary connected domination number of a graph is denoted by $\gamma_{cc}(G)$ and is defined as the minimum number of vertices which form a ccd-set.

It has been proved that $\gamma_{cp}(G) = n = \gamma_{icp}(G)$ and $\gamma_{cc}(G) = n-1$ only if G is a star. And if G is not a star, then γ_{cp}, γ_{icp} and γ_{cc} can be at most n–2. In this paper, we characterise the graphs with $\gamma_{cc} = n-2$, and trees with $\gamma_{icp} = n-2$ and $\gamma_{cp} = n-2$.

Keywords: Dominating set, domination number, complementary perfect domination number, induced complementary perfect domination number, complementary connected domination number.

AMS Subject Classification Code: 05C (primary).

1 Introduction

Throughout this paper, by a graph we mean a finite, simple, connected and undirected graph G(V,E). For notations and terminology, we follow [3]. The number of vertices in G is denoted by n. We denote a cycle on n vertices by C_n and a path of n vertices by P_n. Δ denotes the maximum degree in G. If S is a subset of V, then <S> denotes the vertex induced subgraph of G induced by S. The *eccentricity* of a vertex v is $e(v) = \max\{d(v,w): w \in V\}$ where $d(v,w)$ is the distance between the vertices v and w. The diameter of the graph G is $diam(G) = \max \{e(v): v \in V\}$. Any vertex of degree one is called a pendant vertex and a vertex which is adjacent to a pendant vertex is called a support vertex.

A subset S of V is called *a dominating set* of G if every vertex in V–S is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of all such dominating sets in G. Lot of works has been done on dominating sets and domination numbers. Various types of dominating set and domination number have been introduced and studied by several authors [1],[4],[5],[8],[9],[10],[13],[14],[15],[16] and [17]. Many graph theoretic parameters such as chromatic number χ independence number α and covering number β have been compared and combined with $\gamma(G)$. J.Paulraj Joseph and S.Arumugam have found the relationship between domination number and connectivity in graphs [10], as well as domination number and colouring in graphs [12]. One can get a comprehensive survey of results on various types of domination number of a graph in [6] and [7].

J.Paulraj Joseph, G.Mahadevan and A.Selvam [13] have introduced the concept of complementary perfect domination number. A *complementary perfect dominating set of a graph G (cpd –set)* is a dominating set of G such that the induced subgraph <V–S> has a perfect matching. The minimum cardinality taken over all such cpd-sets of G is called the *complementary perfect domination number of G,* and is denoted by $\gamma_{cp}(G)$. Any cpd-set with γ_{cp} vertices is called a γ_{cp}-set.

In [2], Selvam Avadayappan and C.S.Senthilkumar have proved that for any graph G and a vertex v of G, $\gamma_{cp}(G_m(v)) = \gamma_{cp}(G) + m$ or $\gamma_{cp}(G) + m-2$, where $G_m(v)$ is the graph obtained from G by identifying the center vertex of a star $K_{1,m}$ at a vertex v of G. They have also proved that if T is a tree with $\gamma_{cp}(T) = n-2$ and $v \in V(T)$, then $\gamma_{cp}(T_m(v)) = n+m-2$ if and only if i) $e(v) < 5$ in T and ii) The vertex v does not belong to a path P_4 in T such that $V(T_m(v) - P_4)$ is a dominating set in $T_m(v)$. In addition, they have derived a necessary and sufficient condition for the trees with $\gamma_{cp} = n - 4$.

The concept of *induced complementary perfect dominating number of a graph has* been introduced by J.Paulraj Joseph, Selvam Avadayappan and G.Mahadevan[15]. An *induced complementary perfect dominating set* of a graph G (*icpd-set*) is a dominating set of G such that the induced subgraph <V–S> has only independent edges. That is $\langle V-S\rangle \doteq mK_2$, $m \geq 0$. The minimum cardinality taken over all such icpd-sets of G is called the *induced complementary perfect domination number of* G, and is denoted by $\gamma_{icp}(G)$. Any icpd-set with γ_{icp} vertices is called a γ_{icp}-set of G.

T.Tamizhchelvam and B.Jaya Prasad have introduced the complementary connected domination number in [17]. A subset S of V is said to be a *complementary connected dominating set* (ccd- set) if S is a domination set and <V–S> is connected. The minimum cardinality of S is called the *complementary connected domination number* and is denoted by γ_{cc}. V.R.Kulli and B.Janakiram called this parameter as non split domination number in [8].

For example, the graph G shown in Figure 1 has $\gamma_{cp} = 1$, $\gamma_{icp} = 3$ and $\gamma_{cc} = 1$. Here $S = \{v_1\}$ isboth the γ_{cp} – set and the γ_{cc}-set of G. $S_1 = \{v_1,v_2,v_3\}$, $S_2 = \{v_1,v_2,v_5\}$, $S_3 = \{v_1,v_5,v_4\}$, $S_4 = \{v_2,v_3,v_4\}$ and $S_5 = \{v_3,v_4,v_5\}$ are the γ_{icp}-sets of G. For the graph H shown in Figure 1, $\gamma_{cp} = 9$, $\gamma_{icp} = 9$ and $\gamma_{cc} = 8$. $S = \{v_2,v_4,v_5,v_6,v_8,v_{10},v_{12},v_{13}\}$ is a γ_{cc} – set in H and $S = \{v_3,v_4,v_5,v_6,v_8,v_{10},v_{11},v_{12},v_{13}\}$is a γ_{icp} – set as well as a γ_{cp} –set in H. $S = \{v_2,v_4,v_5,v_6,v_8,v_{10},v_{11},v_{12},v_{13}\}$is a γ_{cp} – set but not a γ_{icp} – set

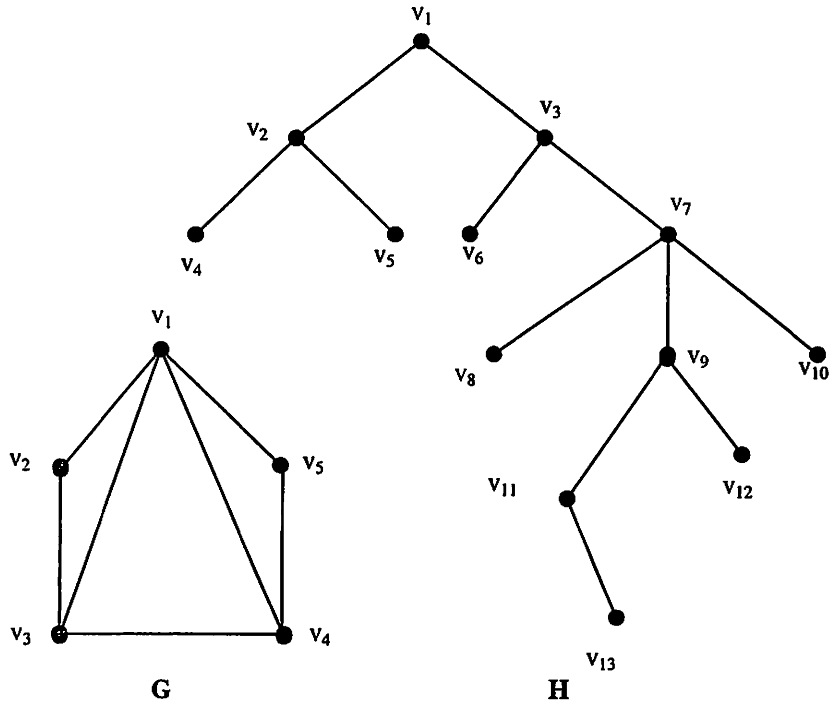


Figure 1

In [15], the following results have been obtained:

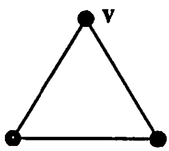
Observation 1 For any connected graph G of order $n \geq 2$, $\gamma_{icp}(G) = n$ if and only if G is a star.

Observation 2 If G is not a star, then $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{icp} \leq n-2$ and the bounds are sharp, where $\lceil x \rceil$ denotes the smallest integer which is at least x.

Observation 3 $\gamma_{icp}(K_n) = n-2$, for any $n \geq 3$.
Observation 4 Any icpd-set of G must contain all the pendant vertices of G.
Observation 5 $\gamma_{icp}(G) = 1$ if and only if G is isomorphic to $K_1 \vee mK_2$, $m \geq 1$.
The following results have been proved in [17].
Observation 6 For $n > 3$, $\gamma_{cc}(P_n) = n-2$.
Observation 7 For $n \geq 3$, $\gamma_{cc}(C_n) = n-2$.
Observation 8 For $m \geq 1$, $\gamma_{cc}(K_{1,m}) = m$.
Observation 9 For any graph G, $\gamma(G) \leq \gamma_{cc}(G)$.
Observation 10 If P_k ($k \geq 4$) is a sub graph of the graph G, then $\gamma_{cc}(G) \leq n-2$.
Observation 11 Let $G_1, G_2 \ldots, G_m$ be the components of a disconnected graph G. Then $\gamma_{cc}(G) = \min_{1 \leq i \leq m} \left\{ \sum_{k=1,,k \neq i}^{m} |V(G_k)| + \gamma_{cc}(G_i) \right\}$ Due to this, here after by a graph G we mean a non trivial connected graph with n vertices.
Observation 12 Let G be a graph with $n \geq 3$. Then there exists a γ_{cc}-set S of G which contains all pendant vertices of G.
Observation 13 Any γ_{cc}-set S of G with $|S| \leq n - 2$ contains all pendant vertices of G.

2. Induced Complementary Perfect Domination Number

Let G be a graph. Fix a vertex v in G. For any $m \geq 0$, let $G_m(v)$ be the graph obtained from G by identifying the centre vertex of a star $K_{1,m}$ at v. Note that $G_0(v) = G$. For example, the graphs G and $G_5(v)$ are shown is Figure 2.



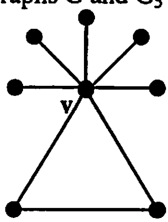


Figure 2 $G_5(v)$

Theorem 1 For any graph G, with n vertices and for any positive integer $m \geq 1$, $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m$ or $\gamma_{icp}(G) + m - 2$.
Proof Let S be a γ_{icp}-set of G and let $u_1, u_2, \ldots, u_m$ be the m pendant vertices of the identified star $K_{1,m}$ in $G_m(v)$.

Let $V_1 = V(G_m(v)) = V(G) \cup \{u_1, u_2, \ldots, u_m\}$. Now $S_1 = S \cup \{u_1, u_2, \ldots, u_m\}$ is an icpd-set of $G_m(v)$. Therefore, $\gamma_{icp}(G_m(v)) \leq \gamma_{icp}(G) + m$.

As <V–S> and <V_1–S_1> has only independent edges, both $n - \gamma_{icp}(G)$ and $(n+m) - \gamma_{icp}(G_m(v))$ are even numbers. This implies that $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m-2i$, for some $i \geq 0$.
Claim $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m-2i$ where $i = 0$ (or) 1.

On the contrary, assume that $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m-2i$ where $i \geq 2$. Let S_1 be a γ_{icp}-set of $G_m(v)$. If v belongs to S_1, then $S = S_1 \setminus \{u_1,u_2,\ldots,u_m\}$ is an icpd-set of G with $\gamma_{icp}(G) - 2i$ elements, a contradiction to the definition of $\gamma_{icp}(G)$. On the other hand, if v belongs to $V_1 - S_1$, then v is an end vertex of the edge uv in $<V_1 - S_1>$ and therefore, $S = S_1 \cup \{u,v\} \setminus \{u_1,u_2,\ldots,u_m\}$ is a γ_{icp}-set of G with $\gamma_{icp}(G) - 2(i-1)$ elements. This is a contradiction to the definition of $\gamma_{icp}(G)$, since $i \geq 2$.

Thus $i = 0$ or 1 and hence, $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m$ (or) $\gamma_{icp}(G) + m - 2$. ■

Corollary 1.1 If v belongs to a γ_{icp}-set of $G_m(v)$, then $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m$.

Proof By Theorem 1, for any graph G, $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m$ (or) $\gamma_{icp}(G) + m-2$. Suppose that $\gamma_{icp}(G_m(v)) = \gamma_{icp}(G) + m-2$. If v belongs to a γ_{icp}-set S_1, of $G_m(v)$, then $S = S_1 \setminus \{u_1,u_2,\ldots,u_m\}$ is an icpd-set of G with $\gamma_{icp}(G) - 2$ elements, which is a contradiction. Hence the proof follows. ■

Theorem 2 Let T be a tree. Then $\gamma_{icp}(T) = n-2$ if and only if $3 \leq \text{diam}(T) \leq 5$.

Proof If $\gamma_{icp}(T) = n-2$, then $<V-S>$ has an edge uv, such that u and v are adjacent to at least one vertex in S. As T is acyclic, S has at least two vertices u_1 and v_1 which are adjacent to u and v respectively. Thus T contains an induced subgraph P_4: u_1uvv_1 and hence diam (T) ≥ 3. If diam(T) > 5, then T has a path of length at least 6, that is, T has a path $u_1u_2\ldots u_7$. Now $S = V(T) - \{u_2,u_3,u_5,u_6\}$ is a γ_{icp}-set with $n-4$ vertices, which is a contradiction. This completes the proof. ■

Corollary 2.1 Let T be a tree on n vertices with $\gamma_{icp}(T) = n-2$. Then $\gamma_{icp}(T_m(v)) = n(T_m(v)) - 2$ if and only if $e(v) < 5$ in T.

Proof If $\gamma_{icp}(T_m(v)) = n(T_m(v)) - 2$, then $\text{diam}(T_m(v)) \leq 5$. This forces that $e(v) < 5$ in T. conversely, let $\gamma_{icp}(T) = n-2$. This implies that $3 \leq \text{diam}(T) \leq 5$. Let v be a vertex of T with $e(v) < 5$. Then $3 \leq \text{diam}(T_m(v)) \leq 5$, and hence $\gamma_{icp}(T_m(v)) = n(T_m(v)) - 2$. ■

Let the bistar B(m,n) be the graph obtained from the stars $K_{1,m}$ and $K_{1,n}$ by joining their centres by an edge.

B*(m,n) is the graph obtained by identifying centre of a star of arbitrary order at all pendant vertices of a bistar B(m,n). A structure of B*(m,n) is illustrated in Figure 3.

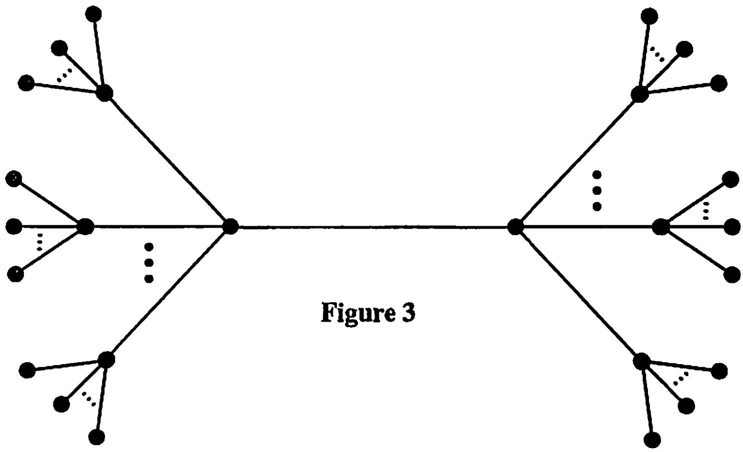


Figure 3

Corollary 2.2 Let T be a tree. Then $\gamma_{icp}(T) = n - 2$ if and only if T is isomorphic B*(m, n).

Proof Let $\gamma_{icp}(T) = n-2$. Then $n \geq 4$ and equality possible only when $T \cong P_4$. Now, by Corollary 2.1 we can identify centre of a star at a vertex v of T with e (v) < 5 to get a new tree T with $\gamma_{icp}(T) = n(T) -2$. Further if we proceed the same process, finally we get a B*(m,n).

The converse is obvious. ■

3 Complementary Connected Domination Number

Let $P_k(m_1,m_2)$ where $k \geq 2$, $m_1,m_2 \geq 1$ be the graph obtained by identifying centres of the stars K_{1,m_1} and K_{1,m_2} at the ends of path P_k respectively. The graph C_3 $(m_1, m_2, 0)$ is obtained from C_3 by identifying the centres of stars K_{1,m_1} and K_{1,m_2} at any two vertices of C_3 respectively.

Lemma 3 If T is a tree, then γ_{cc} (T) = n–2 if and only if $T \cong P_k(m_1,m_2)$, $k \geq 2$, $m_1, m_2 \geq 1$.

Proof As γ_{cc} (T) = n – 2, T is not a star and T has an edge such that both of its end vertices have degree at least 2.

Claim 1 For any vertex v of T with d (v) ≥3, N (v) contains exactly one nonpendant vertex. If any vertex v of the tree with d (v) ≥ 3 has at least two nonpendant neighbours, say u_1 and u_2, then $S = V - \{v,u_1,u_2\}$ is a ccd-set with n – 3 elements, which is a contradiction. Also since T is not a star all the neighbours of v can not be the pendant vertices. Hence the claim.

Claim 2 T contains at most two vertices of degree 3 or more.

On the contrary, assume that there are three vertices v_1, v_2 and v_3 in T with degree 3 or more. As T is connected there is a (v_1,v_2) - path P_1, (v_2,v_3) - path P_2 and a (v_1, v_3) - path P_3. But since T is acyclic any two vertices of T are connected by a unique path. Therefore, P_3 is a subgraph of $P_1 \cup P_2$.

This forces that either v_2 is contained in P_3 or there exists a vertex u which is common to all the three paths P_1,P_2 and P_3. In both the cases, there exists a vertex (either v_2 or u) of degree 3 or more, which has at least 2 nonpendant neighbours, which is a contradiction to our Claim 1. This means that, T contains at most two vertices of degree 3 or more.

Now if T has no vertex of degree 3 or more, then T is isomorphic to a path and thus $T \cong P_k(1,1)$. Suppose T has two vertices of degree 3 or more, say u and v. Since T is connected, there is a (u,v)-path P in G. But u and v can have only one nonpendant neighbour. This forces that $N(\{u, v\})-P$ has only pendant vertices. Thus it is clear that T is isomorphic to a path with a star attached on both of its ends. That is, $T \cong P_k(m_1, m_2)$ with $k \geq 2$. $m_1, m_2 > 1$.

Finally, if T has only one vertex w of degree 3 or more, then N (w) has only one nonpendant vertex, say u, and it has degree 2. Therefore, u has exactly one more neighbour, say u_1. If $u_1$is a pendant vertex, then $T \cong P_2(m, 1)$ $m > 1$. Otherwise, u_1 has degree 2 and hence $T \cong P_k(m, 1)$, $m > 1$ and $k > 2$.

Conversely, if $T \cong P_k(m_1, m_2)$ where $k \geq 2$, $m_1, m_2 \geq 1$, then obviously $\gamma_{cc}(G) = n-2$. ■

Lemma 4 If G contains a triangle C_3 then $\gamma_{cc}(G) = n-2$, if and only if $G \cong C_3(m_1,m_2,0)$ where $m_1, m_2 \geq 0$.

Proof Let G contain a triangle with vertices u,v and w. If all the three vertices u,v and w are of degree 3 or more, then $V - \{u,v,w\}$ will be a ccd-set with $n-3$ elements, which is a contradiction. This implies that, at most two vertices of the u,v and w can have degree 3 or more. Without loss of generality, let them be u and v. Clearly, $N(\{u,v\}) - \{u,v,w\}$contains only pendant vertices. Otherwise, if u_1 is a nonpendant neighbour of u, then $V- \{u_1,u,v\}$ is a ccd-set with n–3 elements, which is a contradiction. Thus, $G \cong C_3(m_1,m_2,0)$.

Converse is obvious. ■

Theorem 5 For any graph G, $\gamma_{cc}(G) = n-2$ if and only if G is isomorphic to any one of the following graphs:

(i) $P_k(m_1,m_2)$ where $k \geq 2$, $m_1,m_2 \geq 1$,

(ii) C_n, $n \geq 3$,

and (iii) $C_3(m_1,m_2, 0)$ where $m_1,m_2 \geq 0$.

Proof If G contains no cycle, then G is a tree and by Lemma 3 $G \cong P_k(m_1,m_2)$, where $k \geq 2$, m_1, $m_2 \geq 1$. If G contains a cycle C_3 then by Lemma 4 , $G \cong C_3(m_1,m_2,0)$ where m_1, $m_2 \geq 0$. Therefore, we assume that G contains a cycle C_k, $k \geq 4$ with vertices $u_1,u_2,\ldots,u_k$.

If u_i is a vertex of C_k with degree more than 2, then $V-\{u_{i-1},u_i,u_{i+1}\}$ is a ccd-set with n–3 elements, a contradiction. Therefore, every vertex of C_k has degree 2 in G, and hence $G \cong C_k$, where $k \geq 4$. This completes the proof. ■

4. Complementary Perfect Domination Number

Recall that it has been proved in [2] that:

If T is a tree with $\gamma_{cp}(T) = n - 2$ and let $v \in V(T)$, then $\gamma_{cp}(T_m(v)) = n + m - 2$ if and only if $e(v) < 5$ in T and v does not belong to a P_4 in T such that $V(T_m(v) - P_4)$ is a dominating set in $T_m(v)$.

Using this result, in the next theorem, we classify the trees with $\gamma_{cp} = n-2$.

Theorem 6 Let T be a tree. Then $\gamma_{cp}(T) = n-2$ if and only if T is isomorphic to any one of the structures given in Figure 4.

Proof Let $\gamma_{cp}(T) = n-2$ then $n \geq 4$, and the equality is possible only if $T \cong P_4$. Now, we can extend T by identifying centre of a star $K_{1,m}$ at any vertex v of T such that $e(v) < 5$ in T and v does not belong to P_4 in T such that $V(T_m(v) - P_4)$ is a dominating set in $T_m(v)$ to get a new tree T_1 with $\gamma_{cp}(T_1) = n(T_1) - 2$. Further, if we proceed the same process, eventually we will get a tree with any one of the following structures.

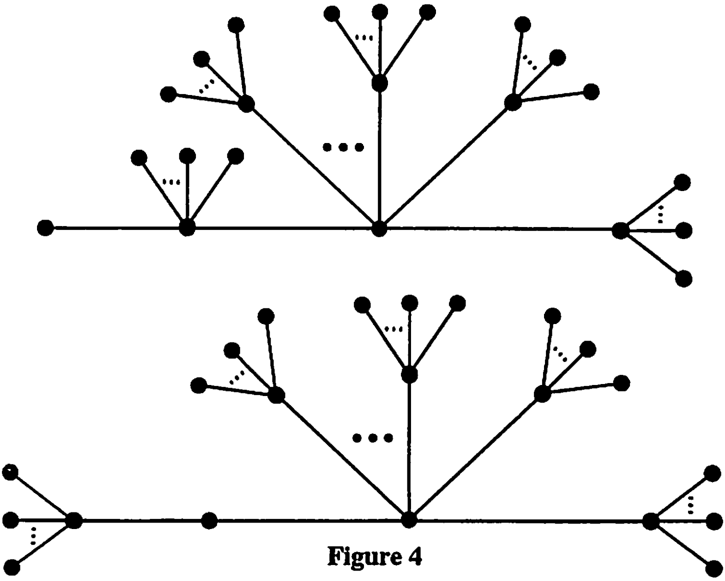


Figure 4

■

The converse part can easily be verified.

5 References

1. S.Arumugam and A.Thuraiswamy *Total domination in graphs,* Ars Combinatoria,43 (1996), 89-92.

2. Selvam Avadayappan and C.S.Senthilkumar *Some results on complementary perfect domination number of a graph,* Graphs, Combinatorics and Algorithms (2004)145-149.
3. R.Balakrishnan and K.Renganathan, *A text book of graph theory* Springer-verlag, New York, Inc(1999).
4. E.J.Cockayne and S.T.Hedetniemi *Towards a theory of domination in graphs,* Networks 7 (1977), 247-261.
5. Teresa W. Haynes, *Paired domination in graphs,* Congr. Numer 150(2001).
6. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater *Fundamentals of domination in graphs,* Marcel deckker, New York (1998).
7. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater *Domination in graphs, Advanced topics,* Marcel deckker, New York (1998).
8. V.R.Kulli and B.Janakiraman *The nonsplit domination number of a graph,* Indian J. pure appl. Math.,31(5): 545-550(2000).
9. G. Mahadevan, J. Paulraj Joseph and A. Selvam, *Complementary perfect domination number and chromatic number of a graph* Oriental Journal of Mathematical Sciences, Vol 1, No. 2, December 2007, 129 – 144.
10. J.Paulraj Joseph and S.Arumugam. *Domination and connectivity in graphs,* International Journal of Management and Systems. 8 No.3 (1992), 233-236.
11. J.Paulraj Joseph and S.Arumugam. *On graphs with equal domination and connected domination numbers, Disc.math 206 (1999), 45-49.*
12. J.Paulraj Joseph and S.Arumugam *Domination and colouring in graphs,* International Journal of Management and Systems.11 (1999), 177-182.
13. J.Paulraj Joseph, G.Mahadevan and A.Selvam on *Complementary perfect domination number of a graph* Acta Ciencia Indica Vol. XXXI M, No.2, 847 – 854 (2006).
14. J.Paulraj Joseph and G.Mahadevan *On domination theory and related concepts in graph theory,* Research thesis, Manonmaniam Sundaranar University (2004).
15. J.Paulraj Joseph. Selvam Avadayappan and G.Mahadevan *Induced complementary perfect dominating number of a graph, Mathematical* and Computational models, Dec 2005 (425-435).
16. E.Sampathkumar and H.B.Walikar *The connected domination number of a graph,* J.Math.Phy.Sci.13 (1979), No.6, 607-613.
17. T.Tamizh Chelvam and B.Jaya Prasad *Complementary connected domination number of a graph,* International Journal of Management and Systems. 18 (2002), No.22,147-153.