The Independence Number for the Generalized Petersen Graphs

Joseph Fox[1], Ralucca Gera[2], Pantelimon Stănică[2]

[1]Salem State College, Department of Mathematics,
Salem, MA 01970; *joseph.fox@salemstate.edu*
[2]Naval Postgraduate School, Department of Applied Mathematics
Monterey, CA 93943; *{rgera,pstanica}@nps.edu*

Abstract

Given a graph G, an independent set $I(G)$ is a subset of the vertices of G such that no two vertices in $I(G)$ are adjacent. The independence number $\alpha(G)$ is the order of a largest set of independent vertices. In this paper, we study the independence number for the Generalized Petersen graphs, finding both sharp bounds and exact results for subclasses of the Generalized Petersen graphs.

Key Words: independence, Petersen Graph.

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1 Introduction and motivation

Given a graph G, an *independent set* $I(G)$ is a subset of the vertices of G such that no two vertices in $I(G)$ are adjacent. The *independence number* $\alpha(G)$ is the cardinality of a largest set of independent vertices. The maximum independent set problem is to find an independent set with the largest number of vertices in a given graph. It is well-known that this problem is NP-hard ([4]). For other graph theory terminology the reader should refer to [8].

The *Generalized Petersen graph* $P(n,k)$ has vertices, and respectively, edges given by

$$\begin{aligned} V(P(n,k)) &= \{a_i, b_i, 0 \le i \le n-1\}, \\ E(P(n,k)) &= \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \,|\, 0 \le i \le n-1\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n ($n \geq 5$). Let $A(n,k)$ (resp., $B(n,k)$) be the subgraph of $P(n,k)$ consisting of the vertices $\{a_i \mid 0 \leq i \leq n-1\}$ (respectively, $\{b_i \mid 0 \leq i \leq n-1\}$) and edges $\{a_i a_{i+1} \mid 0 \leq i \leq n-1\}$ (respectively, $\{b_i b_{i+k} \mid 0 \leq i \leq n-1\}$). We will call $A(n,k)$ (respectively, $B(n,k)$) the *outer* (respectively, *inner*) subgraph of $P(n,k)$.

Albertson, Bollobas and Tucker showed in [1] that every triangle free 3-regular graph with n vertices has an independent set of order at least $\frac{n}{3}$, which was improved by Staton in [7] (with an improved proof and a linear-time algorithm to find such an independent set by C.C. Heckman and R. Thomas in [5]) to $\frac{5n}{14}$ for triangle-free graphs of maximum degree at most three. The bound is sharp since $\alpha(P(7,2)) = 5$ as noted by Fajtlowicz in [3]. However no references regarding the independence number of the Generalized Petersen graphs were found. We propose to study this problem in the present paper.

Since it is going to be (partially) useful later, we mention here a few known results on the Generalized Petersen graphs (see [2, 6]).

Theorem 1.1. *The following are true for the Generalized Petersen graphs* $P(n,k)$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. *Note that we take the "skip"* $k \leq \lfloor \frac{n-1}{2} \rfloor$, *because of the obvious isomorphism* $P(n,k) \cong P(n,n-k)$:

(1) $P(n,k)$ *is a 3-regular graph with* $2n$ *vertices and* $3n$ *edges.*

(2) $P(n,k)$ *is bipartite if and only if* n *is even and* k *is odd.*

(3) *Assume that* n,k,s *are positive integers satisfying* $k \not\equiv \pm s \pmod{n}$. *Then* $P(n,k)$ *is isomorphic to* $P(n,s)$ *if and only if* $ks \equiv \pm 1 \pmod{n}$.

(4) $P(n,k)$ *is vertex-transitive if and only if* $(n,k) = (10,2)$ *or* $k^2 \equiv \pm 1 \pmod{n}$.

(5) $P(n,k)$ *is edge-transitive only in the following seven cases:* $(n,k) = (4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)$.

(6) $P(n,k)$ *is a Cayley graph if and only if* $k^2 \equiv 1 \pmod{n}$.

Throughout this paper, we use the Vinogradov symbols $\gg$, $\ll$ and the Landau symbols O, $\asymp$ with their usual meanings. We recall that $f \ll g$, $g \gg f$ and $f = O(g)$ are all equivalent and mean that $|f(x)| \leq cg(x)$ holds with some positive constant c, and x sufficiently large, while $f \asymp g$ means that both $f \ll g$ and $g \ll f$ hold.

2 Exact Results for Some Generalized Petersen Graphs

First, observe that since there are at most $\lfloor \frac{n}{2} \rfloor$ independent vertices in the outer subgraph and $\lfloor \frac{n}{2} \rfloor$ independent vertices in the inner subgraph, we have the following lemma.

Lemma 2.1. *For any n and k,*

$$\alpha(P(n,k)) \leq \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

The following theorem characterizes all Generalized Petersen graphs $P(n,k)$ attaining the upper bound of n.

Theorem 2.2. *For $P(n,k)$, we have that*

$$\alpha(P(n,k)) = n \text{ if and only if } n \text{ is even and } k \text{ is odd.}$$

Proof. Under the imposed conditions, by Theorem 1.1 $P(n,k)$ is bipartite of equal size partite sets. Taking either of the two partite sets we obtain a maximum independent set of cardinality n.

For the converse, assume to the contrary, that $\alpha(P(n,k)) = n$ and it is not true that n is even and k is odd. By Lemma 2.1, it follows that n is even and k is even. Let I be a maximum independent set. Since $\alpha(P(n,k)) = n$, it follows that exactly $\frac{n}{2}$ vertices of $A(n,k)$ and exactly $\frac{n}{2}$ vertices of $B(n,k)$ must belong to I. Without loss of generality, say that $\{a_{2i} : 0 \leq i \leq \frac{n}{2} - 1\} \subseteq I$. Then no vertex b_{2i} $(0 \leq i \leq \frac{n}{2} - 1)$ can be in I. Therefore $\{b_{2i+1} : 0 \leq i \leq \frac{n}{2} - 1\} \subseteq I$, which is not an independent set, producing a contradiction. □

Next, we present the independence number for Generalized Petersen graphs for small values of k, concentrating on bounds for larger values of k in the next section.

Proposition 2.3. *Let $P(n,1)$ be the Generalized Petersen graph of skip 1. Then*

$$\alpha(P(n,1)) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is easily observed that the set

$$\left\{a_{2i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\} \cup \left\{b_{2j+1} \mid 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

is a maximum independent set and the result is proved. □

Lemma 2.4. *Consider the Generalized Petersen graph $P(n,2)$ with $n \geq 5$, and suppose $n = 5q + r$, where $0 \leq r < 5$. For each $j \in \{0, 5, \ldots, 5(q-1)\}$, define a subset I_j of $V(P(n,2))$ by $I_j = \{a_j, a_{j+1}, a_{j+2}, a_{j+3}, a_{j+4}, b_j, b_{j+1}, b_{j+2}, b_{j+3}, b_{j+4}\}$. Then there are at most 4 independent vertices in each I_j.*

Proof. Observe that for each i $(0 \leq i \leq n-1)$, at most one of a_i and b_i can be in an independent set. Assume, to the contrary, that I_j contains an independent set I'_j with $|I'_j| = 5$. Then either (1) at most 3 of them must be a_i $(j \leq i \leq j+4)$, or (2) at most 3 of them must be b_i $(j \leq i \leq j+4)$. In the first case, $\{a_j, a_{j+2}, a_{j+4}\} \subseteq I'_j$, and so at most one of b_{j+1} and b_{j+3} can belong to I'_j, which is a contradiction. In the second case, we have two options: (a) $\{b_j, b_{j+1}, b_{j+4}\} \subseteq I'_j$, and so at most one of a_{j+2} and a_{j+3} can belong to I'_j (a contradiction), or (b) $\{b_j, b_{j+3}, b_{j+4}\} \subseteq I'_j$, and so at most one of a_{j+1} and a_{j+2} can belong to I'_j, which is also a contradiction.

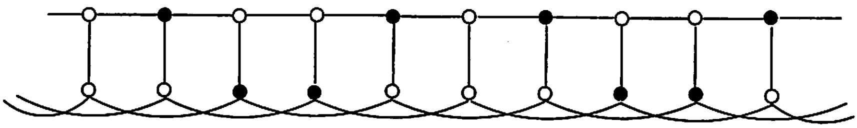

Figure 1: Figure for $P(n,2)$

Thus a maximum independent subset of any set of the form of I_j can have at most 4 vertices. □

Theorem 2.5. *If $n \geq 5$, then $\alpha(P(n,2)) = \lfloor \frac{4n}{5} \rfloor$.*

Proof. Let $G = P(n,2)$. Define a set $S = \{a_{5l-4}, a_{5l-1}, b_{5l-3}, b_{5l-2} \mid 1 \leq l \leq \lfloor \frac{n}{5} \rfloor\}$. As usual, all subscripts are taken modulo n. Then S is an independent set of size $\lfloor \frac{4n}{5} \rfloor$, so $\alpha(G) \geq \lfloor \frac{4n}{5} \rfloor$.

We have 5 cases, according to the value of n modulo 5. Again, we assume that $n = 5q + r$ with $0 \leq r < 5$. If $r = 0$, then $V(G) = \bigcup_{j=0}^{5(q-1)} I_j$, with I_j as defined in Lemma 2.4. Then by that lemma, we have $\alpha(G) \leq 4q = \lfloor \frac{4n}{5} \rfloor$.

Now assume $r = 1$. Then there are two leftover vertices in $V(G) - \bigcup I_j$, say $V(G) - \bigcup I_j = \{a_{n-1}, b_{n-1}\}$. Since at most one of each pair $\{a_i, b_i\}$ $(0 \leq i \leq n-1)$ can belong to an independent set, $\alpha(G) \leq 4q + 1$. Suppose that $\alpha(G) = 4q + 1$, and let T be an independent set in $V(G)$ of size $4q + 1$. Then exactly one vertex of the set $V(G) - \bigcup I_j$ and 4 vertices from each I_j must be in T. If $a_{n-1} \in T$, then $a_0 \notin T$. This forces us to include in T the vertices a_1, a_4, b_3, and one of b_0 and b_2 from I_0, the vertices a_6, a_9, b_8, and one of b_5 or b_7 from I_5, and so on, until we find that we must include $a_{n-5}, a_{n-2}, b_{n-3}$, and one of b_{n-6} or b_{n-4} from $I_{5(q-1)}$. However, a_{n-2} is adjacent to a_{n-1}, which is a contradiction to the independence of T. Thus we cannot include a_{n-1} in T, and a similar argument shows that we cannot include b_{n-1} either. Thus $\alpha(G) \leq 4q = \lfloor \frac{4n}{5} \rfloor$.

If $r = 2$, then $V(G) - \bigcup I_j$ contains 4 vertices, say $a_{n-2}, a_{n-1}, b_{n-2}, b_{n-1}$. At most two of these vertices may be included in an independent set, either a_{n-1} and b_{n-2}, or a_{n-2} and b_{n-1}. As in the $r = 1$ case, each of these choices of pairs forces the choice of independent vertices from each I_j, and this will produce two adjacent vertices in our independent set, a contradiction. Thus we may include at most one vertex from $V(G) - \bigcup I_j$, and hence $\alpha(G) \leq 4q + 1 = \lfloor \frac{4n}{5} \rfloor$.

The remaining two cases are handled the same way: if we try to include 3 (when $r = 3$) or 4 (when $r = 4$) independent vertices from $V(G) - \bigcup I_j$, we get a contradiction, so that $\alpha(G) \leq 4q+2$ when $r = 3$, and $\alpha(G) \leq 4q+3$ when $r = 4$. In any case, $\alpha(G) \leq \lfloor \frac{4n}{5} \rfloor$. □

Proposition 2.6. *If $n > 6$, then* $\alpha(P(n,3)) = \begin{cases} n & \text{if } n \text{ is even} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$

Proof. The case when n is even was already proved in Theorem 2.2. So assume n is odd. By Lemma 2.1 $\alpha(P(n,3)) \leq n-1$. However, we claim that $\alpha(P(n,3))$ cannot be $n-1$. Suppose to the contrary that $\alpha(P(n,3)) = n-1$. Then there must be $\frac{n-1}{2}$ independent vertices in the outer subgraph as well as in the inner subgraph. Without loss of generality, we may assume that the independent set in the outer subgraph is $\{a_0, a_2, \ldots, a_{n-3}\}$. Thus, the independent set from the inner subgraph must be a subset of $\{b_1, b_3, \ldots, b_{n-4}, b_{n-2}, b_{n-1}\}$, a set of size $\frac{n+1}{2}$. However b_{n-2} is adjacent to b_1, and b_{n-1} is adjacent to b_{n-4}. Thus, the inner subgraph can have at most $\frac{n+1}{2} - 2 = \frac{n-3}{2}$ independent vertices, contradicting the fact that $\alpha(P(n,3)) = n-1$. Therefore, $\alpha(P(n,3)) \leq n-2$.

From the previous analysis, we also see a way to form an independent subset of $V(P(n,3))$ of size $n-2$: take the vertices $\{a_0, a_2, \ldots, a_{n-3}, b_1, b_3, \ldots, b_{n-4}\}$. Thus, $\alpha(P(n,3)) = n-2$ when n is odd. □

Proposition 2.7. *If $n > 10$, then* $\alpha(P(n,5)) = \begin{cases} n & \text{if } n \text{ is even} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$

Proof. The case when n is even was already proved in Theorem 2.2. So assume n is odd. The set $\{a_0, a_2, \ldots, a_{n-3}, b_1, b_3, \ldots, b_{n-6}\}$ is an independent set of cardinality $\frac{n-1}{2} + \frac{n-5}{2} = n-3$. Thus $\alpha(P(n,5)) \geq n-3$. An argument similar to the one given in the proof of Proposition 2.6 shows that $\alpha(P(n,5)) \leq n-2$.

Suppose that $\alpha(P(n,5)) = n-2$. Then there must be $\frac{n-1}{2}$ independent vertices in one subgraph, and $\frac{n-3}{2}$ independent vertices in the other. Without loss of generality, we may assume that the largest independent set is in the outer subgraph, say $\{a_0, a_2, \ldots, a_{n-3}\}$. Thus, the independent set from the inner subgraph must be a subset of $\{b_1, b_3, \ldots, b_{n-6}, b_{n-4}, b_{n-2}, b_{n-1}\}$, a set of size $\frac{n+1}{2}$. However the vertices $b_{n-4}, b_{n-2}, b_{n-1}$ are adjacent to b_1, b_3, b_{n-6}, respectively. Thus, the inner subgraph can have at most $\frac{n+1}{2} - 3 = \frac{n-5}{2}$ independent vertices, contradicting the fact that $\alpha(P(n,3)) = n-2$. Therefore, $\alpha(P(n,5)) \leq n-3$. □

One may observe the pattern in $P(n,k)$ when n and k are odd. We next show that this is not always equality, but it is a sharp lower bound. We also consider all the other subclasses of the Generalized Petersen graphs.

3 Lower Bounds for Generalized Petersen Graph $P(n,k)$

First, by Theorem 2.2 we have the result in the case that n is even and k is odd. We thus find bounds for the other 3 cases depending on the parities of n and k. We shall first prove the following (sharp, as we shall see) result.

Theorem 3.1. *If n, k are odd integers then $\alpha(P(n,k)) \geq \frac{2n-k-1}{2}$.*

Proof. We first consider the case of coprimes n, k, that is, $\gcd(n,k) = 1$. We can form an independent set by choosing $\frac{n-1}{2}$ vertices of the outer subgraph and showing that we can always get at least $\frac{n-k}{2}$ more independent vertices from the inner subgraph. Since n is odd, without loss of generality, we can choose the vertices a_{2i}, $0 \leq i \leq \frac{n-3}{2}$ to belong to our independent set, and we search for more independent vertices from the inner subgraph. For ease of writing, we shall work on indices of vertices on the inner subgraph. By abuse of notation, we call these indices *adjacent* if they correspond to adjacent vertices. Recall that b_i is adjacent to b_j $(j > i)$ if and only if $j - i \equiv 0 \pmod{k}$ when $i + j \leq n$, and $j - i - n \equiv 0 \pmod{k}$ when $i + j > n$. And so, the sequence of adjacent indices in $B(n,k)$ is

$$0, k, 2k, \ldots, n-k, 0.$$

The independent set cannot contain any vertex of $B(n,k)$ of even index, since it would be adjacent to a vertex that is in the independent set from $A(n,k)$. So we need to concentrate on the odd indices in $B(n,k)$. To that effect, observe that every odd index above $n-k$ will have exactly one other odd index adjacent to it (recall that k, n are odd, and so we cannot have three consecutive odd indices). For instance, in $P(17,7)$, the list of adjacent indices on the inner subgraph is (we underline the odd indices above $n - k = 17 - 7 = 10$)

$$0, 7, 14, 4, \underline{11}, 1, 8, \underline{15}, 5, 12, 2, 9, 16, 6, \underline{13}, 3, 10, 0.$$

The odd indices below $n-k$ have only even indices or indices above $n-k$ as adjacent indices.

Therefore, we disregard all odd indices above $n-k$ and choose *all* odd indices below $n-k$ as indices of independent vertices on the inner subgraph. Since there are $\frac{n-1}{2}$ odd indices altogether, and $\frac{k-1}{2}$ of these are above $n-k$, then our independent set contains now

$$\frac{n-1}{2}+\left(\frac{n-1}{2}-\frac{k-1}{2}\right)=\frac{2n-k-1}{2}$$

vertices, and the theorem is proved in this case.

If $\gcd(n,k)=d$, then the inner subgraph has exactly d cycles, each containing $\frac{n}{d}$ vertices. First, choose $\frac{n-1}{2}$ even-indexed independent vertices on the outer subgraph. Again, as before, on the inner subgraph, two odd indices are adjacent if and only if one of them is greater than $n-k$. Excluding all odd indices greater than $n-k$, we can pick the rest of the $\frac{n-k}{2}$ odd indexed vertices of $B(n,k)$ as independent vertices, for a total of $\frac{2n-k-1}{2}$ independent vertices, and the theorem is proved. □

The lower bound in the above theorem is sharp as we show below.

Proposition 3.2. *If n and k are odd and k divides n, then*

$$\alpha(P(n,k))=\frac{2n-k-1}{2}.$$

Proof. First, notice that the inner subgraph will be a union of k $\frac{n}{k}$-cycles. Thus there are at most $k\cdot\left\lfloor\frac{n/k}{2}\right\rfloor=k\left(\frac{n/k-1}{2}\right)=\frac{n-k}{2}$ independent vertices in the inner subgraph. Since there are at most $\frac{n-1}{2}$ independent vertices in the outer subgraph, we have that $\alpha(P(n,k))\le\frac{2n-k-1}{2}$.

For the lower bound, we can use Theorem 3.1, or we can show it directly. We claim that the set consisting of the vertices $a_0,a_2,\ldots,a_{n-3},b_1,b_3,\ldots,$ b_{n-k-1} is independent. None of the a_i are adjacent since the distance between any two a_i is at least 2. Also, none of the a_i are adjacent to any of the b_j since the indices of the a_i are all even, while those of the b_j are all odd. Finally, for any b_j in the set, neither b_{j+k} nor b_{j-k} is in the set. Thus none of the b_j are adjacent. Since there are $\frac{2n-k-1}{2}$ vertices in this set, we have $\alpha(P(n,k))\ge\frac{2n-k-1}{2}$. Therefore, $\alpha(P(n,k))=\frac{2n-k-1}{2}$. □

Remark 3.3. *One would perhaps venture to conjecture that the bound in Theorem 3.1 is attained in all cases $P(n,k)$ with n and k odd. That turns out to be false. With the help of Mathematica*[1]*, we found the smallest example (with respect to n) for which this does not happen, namely, $\alpha(P(15,7))=\frac{2\cdot 15-7-1}{2}+1=12$.*

[1] A trademark of Wolfram Research

We next present a table containing the independence numbers $\alpha(P(n,k))$ for $n \leq 20$, $k \leq \lfloor \frac{n-1}{2} \rfloor$ (the calculations were performed on a Core 2 Duo 2Ghz PC with 2Gb of RAM in over 45 hours, using a Mathematica program).

$n \backslash k$	1	2	3	4	5	6	7	8	9
5	4	4							
6	6	4							
7	6	5	5						
8	8	6	8						
9	8	7	7	7					
10	10	8	10	8					
11	10	8	9	9	8				
12	12	9	12	9	12				
13	12	10	11	11	10	10			
14	14	11	14	11	14	12			
15	14	12	13	12	12	12	12		
16	16	12	16	14	16	13	16		
17	16	13	15	14	14	15	14	13	
18	18	14	18	15	18	14	18	16	
19	18	15	17	16	16	17	15	15	15
20	20	16	20	16	20	16	20	17	20

We next consider the case when n is odd and k is even.

Theorem 3.4. *If n, k are integers with n odd and k even, then*

$$\begin{aligned}\alpha(P(n,k)) \geq & \frac{n-1}{2} + \left\lfloor \frac{\lceil \frac{n}{k} \rceil + 1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2d\lceil \frac{n}{k} \rceil} \right\rfloor \\ & + \frac{d-1}{2} \cdot \left\lfloor \frac{1}{2} \left(\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \right\rfloor \\ \asymp & \frac{(2d+1)n}{4d} + \frac{d-1}{2},\end{aligned} \tag{1}$$

where $d = \gcd(n,k)$.

Proof. Note that d is odd. The inner subgraph of the Generalized Petersen $P(n,k)$ will consist of d cycles each of length $\frac{n}{d}$. We start by choosing in our independent set all vertices of even index of $A(n,k)$: $a_0, a_2, \ldots, a_{n-3}$. There are $\frac{n-1}{2}$ such vertices. Next, as in the proof of Theorem 3.1, we list all indices of vertices in each cycle, starting with $0, k, 2k, \ldots, n-k$. (It is understood that in such a cycle, the first and the last indices are adjacent.) Since n is odd and k even, in each cycle there will be blocks of length $\lceil \frac{n}{k} \rceil$

of consecutive indices of the same parity, except for the last block (we call such a block an *incomplete block*), which will have length

$$\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil}.$$

(The indices of independent vertices will be called independent indices, to avoid overuse of "indices of independent vertices.") In each of the odd numbered cycles, we choose

$$\left\lfloor \frac{\lceil \frac{n}{k} \rceil + 1}{2} \right\rfloor$$

independent indices of every block of consecutive odd integers, and in the even numbered cycles we chose

$$\left\lfloor \frac{1}{2} \left(\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \right\rfloor \text{ (least residue)}$$

independent indices of the last block of odd vertices.

As an example, consider $P(55, 10)$. There are $d = 5$ cycles of length $\frac{n}{d} = 11$. We underline the chosen independent indices, of count at least $\left\lfloor \frac{1}{2} \left(\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \right\rfloor = \lfloor \frac{1}{2}(11 \pmod 6) \rfloor = 2$ (in this case, we can choose 3 independent indices, that is, $\left\lfloor \frac{1}{2} \left(\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \right\rfloor + 1$, but that may not be possible in general, as the first index of the cycle may be an odd index)

$$
\begin{gathered}
0, 10, 20, 30, 40, 50, \underline{5}, 15, \underline{25}, 35, \underline{45} \\
\underline{1}, 11, \underline{21}, 31, \underline{41}, 51, 6, 16, 26, 36, 46 \\
2, 12, 22, 32, 42, 52, \underline{7}, 17, \underline{27}, 37, \underline{47} \\
\underline{3}, 13, \underline{23}, 33, \underline{43}, 53, 8, 18, 28, 38, 48 \\
4, 14, 24, 34, 44, 54, \underline{9}, 19, \underline{29}, 39, \underline{49}
\end{gathered}
$$

We then have blocks of $\lceil \frac{n}{k} \rceil = \lceil \frac{55}{10} \rceil = 6$ same parity indices, except for a last (incomplete) block of length $\frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} = 11 \pmod 6 = 5$.

One of the cycles will contain the index $n-1$, followed by at least two odd indices. Thus, this cycle contains the block of indices $n-1, n-1+k-n = k-1, 2k-1$, so, of course, one can choose $n-1$, $2k-1, \ldots$ as independent indices (note that $n-1$ is an even index whose corresponding vertex is not adjacent to any vertex of the independent set. For instance, for $P(55, 10)$ we would choose $54, 9$ as independent indices, in the last cycle, improving the count by 1. However, this improves the lower bound just by 1 in some cases, so we shall disregard this (practical) trick.

Next, we need to count the number of groups of consecutive odd indices in each cycle. We note that in the odd numbered cycles there are $\left\lfloor \frac{n}{2d\lceil \frac{n}{k} \rceil} \right\rfloor$

blocks of consecutive odd indices, and in the even numbered cycles there are $\left\lceil \frac{n}{2d\lceil \frac{n}{k}\rceil} \right\rceil$ blocks of consecutive odd indices. (We will deal later with the incomplete block of odd indices, which can be found every two cycles.) Therefore, regardless of the cycle, there are at least $\left\lfloor \frac{n}{2d\lceil \frac{n}{k}\rceil} \right\rfloor$ odd-index blocks in every cycle.

Putting all this together, and noting that there are at least $\frac{d-1}{2}$ incomplete blocks of odd indices, we can bound from below the number of independent vertices that we can choose from the inner subgraph by

$$\left\lfloor \frac{\lceil \frac{n}{k}\rceil + 1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2d\lceil \frac{n}{k}\rceil} \right\rfloor + \frac{d-1}{2} \cdot \left\lfloor \frac{1}{2}\left(\frac{n}{d} \pmod{\lceil \frac{n}{k}\rceil}\right) \right\rfloor$$

independent indices. We obtain that

$$\alpha(P(n,k)) \geq \frac{n-1}{2} + \left\lfloor \frac{\lceil \frac{n}{k}\rceil + 1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2d\lceil \frac{n}{k}\rceil} \right\rfloor + \frac{d-1}{2} \cdot \left\lfloor \frac{1}{2}\left(\frac{n}{d} \pmod{\lceil \frac{n}{k}\rceil}\right) \right\rfloor .$$

The approximation follows by observing that

$$\alpha(P(n,k)) \asymp \frac{n}{2} + \frac{n}{2k} \cdot \frac{nk}{2dn} + \frac{d-1}{2} = \frac{(2d+1)n}{4d} + \frac{d-1}{2},$$

and the theorem is proved. □

The lower bound in the above theorem is attained for $P(13,6)$ and $P(19,8)$ among other examples.

Theorem 3.5. *If n,k are even integers, then*

$$\alpha(P(n,k)) \geq \frac{n}{2} + \frac{d}{2}\left\lfloor \frac{n}{2d} \right\rfloor \asymp \frac{3n}{4}, \tag{2}$$

where $d = \gcd(n,k)$.

Proof. We shall use again the method of Theorem 3.1. Let $d = \gcd(n,k)$, which is now an even integer. On the outer subgraph of $P(n,k)$ we pick all even indexed vertices as independent vertices. Next, in the inner subgraph, there are d cycles of length $\frac{n}{d}$. Since both of n,k are even, we see that in each cycle we have indices of the same parity (the 1st cycle contains even indices, the 2nd contains odd indices, etc.). Thus, in each even numbered cycle we choose $\lfloor \frac{n}{2d} \rfloor$ independent indices. Since there are exactly $\frac{d}{2}$ even numbered cycles, we get a total of

$$\frac{d}{2}\left\lfloor \frac{n}{2d} \right\rfloor$$

inner independent indices. This will give us

$$\alpha(P(n,k)) \geq \frac{n}{2} + \frac{d}{2}\left\lfloor \frac{n}{2d} \right\rfloor.$$

The asymptotic follows immediately and the theorem is proved. □

The lower bound in the above theorem is attained for $P(16,2)$ among other examples.

Using the theorems we proved above, and the isomorphism mentioned in Theorem 1.1 we get some bounds "for free", moreover even exact results for general values of k. We mention here one such result.

Corollary 3.6. *If m, l are positive integers with m odd, then*

$$\alpha(P(2m\ell \pm 1, 2\ell)) \geq \frac{2(2m\ell \pm 1) - m - 1}{2}.$$

Furthermore,

$$\alpha(P(2^p\ell \pm 1, 2^{p-1}\ell)) = \left\lfloor \frac{4(2^p\ell \pm 1)}{5} \right\rfloor,$$

for any integer $p \geq 1$.

Proof. Using Theorem 1.1, since $2\ell \cdot m \equiv \pm 1 \pmod{2m\ell \pm 1}$, we obtain $P(2m\ell \pm 1, 2\ell) \cong P(2m\ell \pm 1, m)$, where m is odd. Now, by Theorem 3.1, $P(2m\ell \pm 1, m) \geq \frac{2(2m\ell \pm 1) - m - 1}{2}$, which proves the first claim.

For the second claim, we use a similar approach. Thus, by Theorem 1.1 $P(2^p\ell \pm 1, 2^{p-1}\ell) \cong P(2^p\ell \pm 1, 2)$. By Theorem 2.5 and the previous isomorphism we get $\alpha(P(2^p\ell \pm 1, 2^{p-1}\ell)) = \left\lfloor \frac{4\cdot(2^p\ell \pm 1)}{5} \right\rfloor$. □

Remark 3.7. *The reason we include the case of $P(2^p\ell - 1, 2^{p-1}\ell)$ (recall that one can assume $k \leq \frac{n-1}{2}$ in $P(n,k)$) in the previous theorem is to obtain immediately $\alpha(P(2^p\ell - 1, 2^{p-1}\ell)) = \alpha(P(2^p\ell - 1, 2^{p-1}\ell - 1)) = \left\lfloor \frac{4\cdot(2^p\ell - 1)}{5} \right\rfloor$.*

Considering the way we obtained our lower bounds, one might think that the best way to find a maximal independent set of a Generalized Petersen graph is to use $\lfloor \frac{n}{2} \rfloor$ vertices of one subgraph (inner or outer) and then as many more vertices as possible from the other. This "greedy algorithm" does not always produce a maximal independent set as we next show for a subclass of Generalized Petersen graphs (when k is even).

Proposition 3.8. *For any integer $k \geq 1$, we have that*

$$\alpha(P(3k,k)) = \left\lceil \frac{5k-2}{2} \right\rceil.$$

Proof. When k is odd, this was proven in Proposition 3.2. So we assume that k is even. Let $k = 2t$ for some positive integer t. For the lower bound, consider the set $\{a_0, a_2, \ldots, a_{2t-2}, a_{2t+1}, a_{2t+3}, \ldots, a_{6t-3}\} \cup \{b_1, b_3, \ldots, b_{2t-1}, b_{2t}, b_{2t+2}, \ldots, b_{4t-2}\}$, of cardinality $5t - 1 = \frac{5k-2}{2}$. This is an independent set, so $\alpha(P(3k, k)) \geq \lceil \frac{5k-2}{2} \rceil$.

Now we show $\lceil \frac{5k-2}{2} \rceil$ is also an upper bound for $\alpha(P(3k, k))$. Again, let $k = 2t$. One can choose at most $3t$ independent vertices from the outer subgraph, and we can choose at most $2t$ independent vertices from the inner subgraph since it is the union of $2t$ triangles. Thus $\alpha(P(6t, 2t)) \leq 5t$. Suppose there exists an independent set of size $5t$. We may assume that the independent vertices from the outer subgraph form the set $I = \{a_{2i} : 0 \leq i \leq 6t - 2\}$. However, this means we cannot include in our independent set any of the vertices b_0, b_{2t}, b_{4t} because each of these is adjacent to a vertex in I. Therefore the triangle formed by b_0, b_{2t}, b_{4t} cannot contribute any independent vertices, contradicting the statement that there exists an independent set of size $5t$. Thus $\alpha(P(3k, k)) \leq 5t - 1 = \frac{5k-2}{2} = \lceil \frac{5k-2}{2} \rceil$. □

The above result and a weaker version of Theorem 2.5 are proposed as supplementary problems to [8], and can be found on the following web site: http://www.math.uiuc.edu/~west/igt/newprob.html.

The Generalized Petersen graphs are particular cases of the *I-graphs* (see [2]). The I-graph $I(n, j, k)$ is a graph with vertex and and edge set

$$V(I(n, j, k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

$$E(I(n, j, k)) = \{u_i u_{i+j}, u_i v_i, v_i v_{i+k} \mid i = 0, \ldots, n - 1\}.$$

Since $I(n, j, k) = I(n, k, j)$ we will usually assume that $j \leq k$. Clearly $P(n, k) = I(n, 1, k)$. It could be an interesting project to investigate the independence number for this class of graphs, as well, and we propose that to the interested reader.

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