Trees Which Admit No α-labelings

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Abstract

In this paper, we study the existence of α-labelings for trees by means of particular $(0,1)$-matrices called α-labeling matrices. It is shown that each comet $S_{k,q}$ admits no α-labelings whenever $k > 4(q-1)$ and $q \geq 2$. We also give the sufficient conditions for the nonexistence of α-labelings for trees of diameter at most six. This extends a result of Rosa's. As a consequence, we prove that $S_{k,3}$ has an α-labeling if and only if $k \leq 4$.

Keywords: α-labeling matrix; matrix graph; comet

1 Introduction

Throughout this paper only simple finite graphs are considered. For convenient notation, we denote the set of integers $\{i \in \mathbb{N} : s \leq i \leq t\}$ by $[s,t]$ for any two integers $s \leq t$. A *vertex labeling* of a graph G is an assignment f of labels to the vertices of G which induces, for each edge uv, a label depending on the vertex labels $f(u)$ and $f(v)$. A vertex labeling f of a graph G with q edges is called a *β-labeling* if f is an injection from the vertices of G to the set $[0,q]$ such that, when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are distinct. A β-labeling is also known as a *graceful labeling.* In [6], Rosa proved that a graph G with q edges has no β-labeling if q is congruent to 1 or 2 modulo 4 and the degree of each vertex in G is even. However, he believes that every tree is graceful. His conjecture inspires many researchers to focus on the study of β-labelings for trees.

Let f be a graceful labeling of G. If there exists an integer λ so that, for each edge $uv \in E(G)$, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$, then f is called an *α-labeling* of G. It is not difficult to see that a graph that receives an α-labeling must be bipartite. For known results on α-labelings, the readers may refer to [2],

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[3], [4], [5], [7], [8] and [9]. The complete bipartite graph $K_{1,k}$ is called a *star* or a *k-star*. The *comet* $S_{k,q}$ is the graph obtained from the k-star $K_{1,k}$ by replacing each edge with a path of length q, where both k and q are positive integers. Unlike β-labelings, it can not be conjectured that every tree has an α-labeling. Rosa [6] pointed out that trees of diameter four that contain the comet $S_{3,2}$ as a subtree do not admit α-labelings. In the present paper, we shall extend this result.

Let $\mathbf{E} = (e_{ij})$ be an $m \times n$ $(0,1)$-matrix. The *size* of $\mathbf{E}$, denoted by $|\mathbf{E}|$, is the number of 1-entries in $\mathbf{E}$. For a positive integer $k \in [1, m+n-1]$, the set $\{e_{ij} \in \mathbf{E} : j - i + m = k, i \in [1,m] \text{ and } j \in [1,n]\}$ is called the *k-th diagonal* of $\mathbf{E}$, written as $\mathbf{E}[k]$. Let A be a row, column or diagonal of $\mathbf{E}$. $\deg_{\mathbf{E}}(A)$ denotes the degree of A which is the number of 1-entries in A. A row, column or diagonal of zero degree is called a zero row, column or diagnal, respectively. A row or a column is sometimes called a *line* in this paper. In addition, $\mathbf{E}$ is referred to as an *α-labeling matrix* if the degree of each diagonal of $\mathbf{E}$ is equal to one. Therefore, every $m \times n$ α-labeling matrix has size $m + n - 1$ and the transpose of an α-labeling matrix is also an α-labeling matrix.

In the present paper, we study the existence of α-labelings for trees using α-labeling matrices. It is shown that each comet $S_{k,q}$ admits no α-labelings whenever $k > 4(q-1)$ and $q \geq 2$. The sufficient conditions for the nonexistence of α-labelings for trees of diameter at most six are also given. This extends the result of Rosa's [6] mentioned above. As a consequence, we show that $S_{k,3}$ has an α-labeling if and only if $k \leq 4$.

2 Main Results

Through out this section, $\mathbf{E} = (e_{ij})$ is an $m \times n$ $(0,1)$-matrix and r_i and c_j represent the i-th row and the j-th column of $\mathbf{E}$ respectively, for $i \in [1,m]$ and $j \in [1,n]$. We define the *matrix graph* of $\mathbf{E}$ as the bipartite graph G with partite sets the row set $\{r_i : i \in [1,m]\}$ and the column set $\{c_j : j \in [1,n]\}$ of $\mathbf{E}$ and the edge set $E(G) = \{r_i c_j : e_{ij} = 1 \text{ for } i \in [1,m] \text{ and } j \in [1,n]\}$. Clearly, $|E(G)| = |\mathbf{E}|$. Besides, we let $< \mathbf{E} >$ denote the subgraph induced by the edge set of G. Then $< \mathbf{E} >$ has no isolated vertices. Note that the edge sets of $< \mathbf{E} >$ and the matrix graph G of $\mathbf{E}$ are the same , while the vertex sets of them may differ. The vertex set of $< \mathbf{E} >$ can be obtained by removing isolated vertices from the vertex set of G. Therefore, if G has no isolated vertices, that is, $\mathbf{E}$ has no zero lines, the matrix graph G is actually identical with $< \mathbf{E} >$. In the case when $\mathbf{E}$ is an α-labeling matrix and a graph H is isomorphic to $< \mathbf{E} >$, we say that $\mathbf{E}$ is an α-labeling matrix of H.

Theorem 2.1. *Let H be a bipartite graph without isolated vertices. Then H admits an α-labeling if and only if there is an α-labeling matrix of H.*

Proof. For the sufficiency, let $\mathbf{E} = (e_{ij})$ be an α-labeling matrix of H and $G = (U, V)$ be the matrix graph of $\mathbf{E}$ where $U = \{r_i : i \in [1,m]\}$ and $V = \{c_j : j \in [1,n]\}$. Then,

H is isomorphic to $< \mathbf{E} >$ and $|E(G)| = |\mathbf{E}| = m+n-1 = | < \mathbf{E} > | = |E(H)|$. We start by constructing an α-labeling of G as follows. Let f be a mapping from $V(G)$ to $[0, |E(G)|]$ defined by $f(r_i) = i-1$, for $i \in [1,m]$ and $f(c_j) = m+j-1$, for $j \in [1,n]$. Then f is a bijective mapping with $f(r_i) \le m-1 < f(c_j)$. For each $k \in [1, m+n-1]$, let $e_{i_k j_k}$ be the unique 1-entry in the k-th diagonal of $\mathbf{E}$, then $j_k - i_k + m = k$ and the edge $r_{i_k} c_{j_k} \in E(G)$ receives the label $f(c_{j_k}) - f(r_{i_k}) = (m + j_k - 1) - (i_k - 1) = j_k - i_k + m = k$. This implies that the edge labels of G are distinct integers from 1 to $m+n-1$. Since $< \mathbf{E} >$ and G have the same edge set, the function $f|_{<\mathbf{E}>}$ is an α-labeling of $< \mathbf{E} > \cong H$.

For the necessity, we suppose that $H = (U, V)$ has an α-labeling f. Then there exists an integer λ such that $f(u) \le \lambda < f(v)$ or $f(v) \le \lambda < f(u)$ for each edge $uv \in E(H)$. Let $m = \lambda + 1$ and $n = |E(H)| - m + 1$. We define an $m \times n$ matrix $\mathbf{E} = (e_{ij})$ as follows. For $i \in [1,m]$ and $j \in [1,n]$, $e_{ij} = 1$ provided that there exists an edge $uv \in E(H)$ such that $f(u) = i - 1$ and $f(v) = \lambda + j$ and $e_{ij} = 0$ otherwise. Clearly, H is isomorphic to $< \mathbf{E} >$ since H has no isolated vertices. It remains to show that $\mathbf{E}$ is an α-labeling matrix. According to the way we define the matrix $\mathbf{E}$, $|\mathbf{E}| = |E(H)| = m+n-1$, which means that the size of $\mathbf{E}$ is equal to the number of diagonals of $\mathbf{E}$. It suffices to show that each diagonal of $\mathbf{E}$ contains an 1-entry. For each $k \in [1, m+n-1]$, there exists exactly one edge $uv \in E(H)$ such that $f(v) - f(u) = k$. This edge corresponds to an 1-entry $e_{i_k j_k}$ in $\mathbf{E}$ with $i_k = f(u) + 1 \le m$ and $j_k = f(v) - \lambda = f(v) - m + 1 \le n$. These values of i_k and j_k give $j_k - i_k + m = (f(v) - m + 1) - (f(u) + 1) + m = f(v) - f(u) = k$, which means the 1-entry $e_{i_k j_k}$ belongs to the k-th diagonal of $\mathbf{E}$ and we have the proof. □

Example 1. Figure 1 shows an α-labeling matrix and the corresponding α-labeling of $K_{3,3}$.

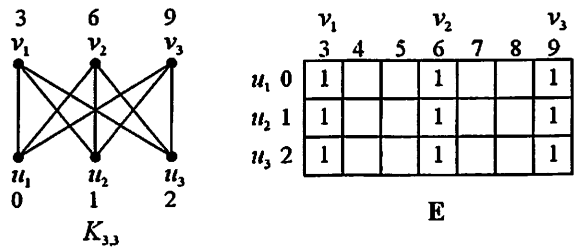


Figure 1: $K_{3,3}$

Now, let's consider the case when H is a tree which admits an α-labeling f. From the construction of the α-labeling matrix $\mathbf{E}$ in the proof of the necessity of Theorem 2.1, we know that $|E(H)| = m+n-1 = |\mathbf{E}|$ and H is isomorphic to $< \mathbf{E} >$. Note that $|V(H)| = |E(H)| + 1 = m+n$, meaning the number of vertices of $< \mathbf{E} >$ is the same as the number of lines of $\mathbf{E}$. This tells us that the tree $< \mathbf{E} >$ and the matrix graph of $\mathbf{E}$ have the same number of vertices. In other words, the matrix graph of $\mathbf{E}$ has no isolated vertices. We can therefore conclude that the graph $< \mathbf{E} >$ and the matrix graph of $\mathbf{E}$ are identical. We state this fact as a remark below for future reference.

Remark 1. In the case when H is a tree, its α-labeling induces an α-labeling matrix $\mathbf{E}$ so that $< \mathbf{E} >$ is isomorphic to H and $< \mathbf{E} >$ is itself the matrix graph of $\mathbf{E}$.

As mentioned in the introduction, Lemma 2.1 is straightforward.

Lemma 2.1. *If* $\mathbf{E}$ *is an* α*-labeling matrix of a graph* H*, then* $\mathbf{E}^t$ *is also an* α*-labeling matrix of* H.

For convenient notation, $\mathbf{E}[s,t]$, where $s,t \in [1, m+n-1]$, denotes the $m \times n$ matrix obtained from $\mathbf{E}$ by replacing the k-th diagonal $\mathbf{E}[k]$ with a zero diagonal for all $k \notin [s,t]$. For a given column A of $\mathbf{E}$, the matrix obtained by replacing A with a zero column in $\mathbf{E}$ is written as $\mathbf{E} - A$. With these notations, the following lemma is easy to see. It will be frequently used later in the text.

Lemma 2.2. *Let* c_l *be the* l*-th column of an* $m \times n$ *matrix* $\mathbf{E}$ *and* $\mathbf{E}' = \mathbf{E} - c_l$, $1 \leq l \leq n$. *Then* $\mathbf{E}'[1, l-1] = \mathbf{E}[1, l-1]$ *when* $l > 1$ *and* $\mathbf{E}'[m+l, m+n-1] = \mathbf{E}[m+l, m+n-1]$ *when* $l < n$.

Before we state our first main result, we need to develop some tools for proving it. Lemma 2.3 plays an important role in the proof of Theorem 2.2.

Lemma 2.3. *Let* $\mathbf{E}$ *be an* $m \times n$ α*-labeling matrix and* $s \in [1, m+n-1]$. *If* $d_1 \neq 0$ *and* $d_i = 0$ *for all* $i \geq 3$ *where* d_i *is the number of lines of degree* i *in* $\mathbf{E}[1,s]$. *Then* $s \leq [2(d_2/d_1) + 1]^2$.

Proof. Let $\mathbf{D} = \mathbf{E}[1,s]$ and $d_2/d_1 = r$, then $|\mathbf{D}| = s$ and $d_2 = rd_1$. Since the sum of the degrees of all lines of $\mathbf{D}$ equals to $2|\mathbf{D}| = 2s$ and $d_i = 0$ for all $i \geq 3$, we have

$$d_1 + 2d_2 = d_1 + 2rd_1 = 2s. \tag{1}$$

This gives

$$d_1 = 2s/(2r+1) \qquad \text{and} \qquad d_2 = rd_1 = 2rs/(2r+1). \tag{2}$$

Now, we define the weight of an entry e_{ij}, written as $w(e_{ij})$, by $w(e_{ij}) = (m + 1 - i) + j$ if $e_{ij} = 1$ and $w(e_{ij}) = 0$ if $e_{ij} = 0$. For each $k \in [1, m+n-1]$, let $e_{i_k j_k}$ be the unique 1-entry of the diagonal $\mathbf{E}[k]$. Then $j_k + m - i_k = k$ and $\sum_{e \in \mathbf{E}[k]} w(e) = w(e_{i_k j_k}) = (m+1-i_k) + j_k = k+1$. Since the diagonal $\mathbf{D}[k]$ is the same as $\mathbf{E}[k]$ when $k \in [1,s]$ and is a zero diagonal otherwise, the sum of the weight of every enrty in $\mathbf{D}$ is

$$\sum_{e \in \mathbf{D}} w(e) = \sum_{k=1}^{m+n-1} \sum_{e \in \mathbf{D}[k]} w(e) = \sum_{k=1}^{s} (k+1) = s(s+3)/2.$$

Let r'_i and c'_j be the i-th row and j-th column of $\mathbf{D}$ respectively, for $i \in [1, m]$ and $j \in [1, n]$. By the two-way counting,

$$\begin{aligned} s(s+3)/2 &= \textstyle\sum_{e \in \mathbf{D}} w(e) \\ &= \textstyle\sum_{i=1}^{m} (m+1-i) \deg(r'_i) + \sum_{j=1}^{n} j \deg(c'_j) \\ &= \textstyle\sum_{k=1}^{m} k \deg(r'_{m+1-k}) + \sum_{k=1}^{n} k \deg(c'_k) \end{aligned} \tag{3}$$

In order to complete our proof, we need the following fact.

Fact(*): Let $S = \{a_1, a_2, ..., a_k\}$ be a set of real numbers, and let π be a permutation such that $\pi(a_i) \geq \pi(a_j)$ whenever $1 \leq i < j \leq k$. Then $\sum_{i=1}^{k} i a_i \geq \sum_{i=1}^{k} i \pi(a_i)$.

Let d_i^r and d_i^c be the number of rows and columns of degree i in $\mathbf{D}$ respectively. Then $d_i^r + d_i^c = d_i$. Since $d_i = 0$ for $i \geq 3$, we have $d_i^r = d_i^c = 0$ for $i \geq 3$ and $0 \leq \deg(r'_i), \deg(c'_j) \leq 2$ for all $i \in [1, m]$ and $j \in [1, n]$. Now applying Fact (*), Equation (3) becomes

$$\begin{aligned} s(s+3)/2 &= \textstyle\sum_{k=1}^{m} k \deg(r'_{m+1-k}) + \sum_{k=1}^{n} k \deg(c'_k) \\ &\geq \textstyle(\sum_{k=1}^{d_2^r} 2k + \sum_{k=d_2^r+1}^{d_2^r+d_1^r} k) + (\sum_{k=1}^{d_2^c} 2k + \sum_{k=d_2^c+1}^{d_2^c+d_1^c} k) \end{aligned} \tag{4}$$

Note that Equation (1) implies that d_1 is even. We split the discussion into the following two cases.

Case 1. d_2 is even.

$$\begin{aligned} s(s+3)/2 &\geq \textstyle(\sum_{k=1}^{d_2^r} 2k + \sum_{k=1}^{d_2^c} 2k) + (\sum_{k=d_2^r+1}^{d_2^r+d_1^r} k + \sum_{k=d_2^c+1}^{d_2^c+d_1^c} k) \\ &\geq \textstyle\sum_{k=1}^{d_2/2} (2k+2k) + \sum_{k=d_2/2+1}^{d_2/2+d_1/2} 2k \\ &= \textstyle\sum_{k=1}^{d_2/2} 2k + \sum_{k=1}^{d_2/2+d_1/2} 2k \\ &= d_2^2/2 + d_1^2/4 + d_1 d_2/2 + (d_1 + 2d_2)/2 \end{aligned}$$

Case 2. d_2 is odd.

$$\begin{aligned} s(s+3)/2 &\geq \textstyle(\sum_{k=1}^{d_2^r} 2k + \sum_{k=1}^{d_2^c} 2k) + (\sum_{k=d_2^r+1}^{d_2^r+d_1^r} k + \sum_{k=d_2^c+1}^{d_2^c+d_1^c} k) \\ &\geq \textstyle\sum_{k=1}^{(d_2-1)/2} 2k + \sum_{k=1}^{(d_2+1)/2} 2k + \sum_{k=(d_2-1)/2+1}^{(d_2-1)/2+d_1/2} k + \sum_{k=(d_2+1)/2+1}^{(d_2+1)/2+d_1/2} k \\ &= \textstyle\sum_{k=1}^{(d_2-1)/2} k + \sum_{k=1}^{(d_2+1)/2} k + \sum_{k=1}^{(d_2-1)/2+d_1/2} k + \sum_{k=1}^{(d_2+1)/2+d_1/2} k \\ &= d_2^2/2 + d_1^2/4 + d_1 d_2/2 + (d_1 + 2d_2)/2 + 1/2 \end{aligned}$$

In either case, $s(s+3)/2 \geq d_2^2/2 + d_1^2/4 + d_1 d_2/2 + (d_1 + 2d_2)/2$. Substituting Equations (1) and (2) into this inequality, it turns out to be

$$\begin{aligned} s(s+3)/2 &\geq [(2rs)/(2r+1)]^2/2 + [(2s)/(2r+1)]^2/4 \\ &\quad + [(2rs)/(2r+1)][(2s)/(2r+1)]/2 + s \\ &= [1 + 1/(2r+1)^2] s^2/2 + s. \end{aligned}$$

This implies that $s \leq (2r+1)^2$. So we have the proof. □

In order to apply Lemma 2.3, we investigate the ratio $r = d_2/d_1$ used above for some specific matrices. Let's call the ratio of the number of lines of degree two to the number of lines of degree one in $\mathbf{E}$ the *two-one ratio* of $\mathbf{E}$. The number of lines of degree i in $\mathbf{E}$ is in fact the same as the number of vertices of degree i in $<\mathbf{E}>$. The two-one ratio of the matrix $\mathbf{E}$ is also called the two-one ratio of the graph $<\mathbf{E}>$. The next result will be used later in the proof of Theorem 2.2. The graph kP_q is the union of k copies of the path P_q of order q.

Lemma 2.4. *Let r_G be the two-one ratio of a subgraph G of kP_q, then $r_G \leq (q-2)/2$ for $k \geq 1$ and $q \geq 2$.*

Proof. Among all subgraphs of P_q, the one of greatest two-one ratio is P_q itself because it has the most vertices of degree two and the least vertices of degree one. Since P_q has $q-2$ vertices of degree two and two vertices of degree one, we have the result for $k = 1$.

When $k \geq 2$, let the j-th copy of the path P_q in kP_q be $P_q^{(j)}$ and d_i^j be the number of vertices of degree i in $G \cap P_q^{(j)}$, $i = 1, 2$ and $j \in [1, k]$. Since the subgraph $G \cap P_q^{(j)}$ of P_q has the two-one ratio $d_2^j/d_1^j \leq (q-2)/2$ for each $j \in [1, k]$, we conclude that the two-one ratio r_G of G is $r_G = (\sum_{j=1}^k d_2^j)/(\sum_{j=1}^k d_1^j) \leq (q-2)/2$. □

Now, we are ready to give the sufficient conditions for the nonexistence of α-labelings for a comet.

Theorem 2.2. *A comet $S_{k,q}$ fails to admit α-labelings for $k > 4(q-1)$ and $q \geq 2$.*

Proof. Suppose that a comet $S_{k,q}$ has an α-labeling. Then by Theorem 2.1, there exists an $m \times n$ α-labeling matrix $\mathbf{E} = (e_{ij})$ such that $<\mathbf{E}> = (U, V)$ is isomorphic to the comet $S_{k,q}$. Without loss of generality, we may assume that the center is in V. By Remark 1, $<\mathbf{E}>$ is itself the matrix graph of $\mathbf{E}$ and then

$$m = |U| \qquad \text{and} \qquad n = |V| = k\lfloor q/2 \rfloor + 1. \tag{5}$$

Let the l-th column c_l be the center of $<\mathbf{E}>$, and $\mathbf{E}' = \mathbf{E} - c_l$. Then $<\mathbf{E}'>$ is isomorphic to kP_q because $q \geq 2$. If $l > 1$, we let r be the two-one ratio of $\mathbf{E}[1, l-1] = \mathbf{E}'[1, l-1]$. Since $<\mathbf{E}'[1, l-1]>$ is isomorphic to a subgraph of kP_q, $r \leq (q-2)/2$ by Lemma 2.4. On the other hand, each component of $<\mathbf{E}'>$ is a path of order q, the degree of each line in $\mathbf{E}'$ does not exceed two. Now we have that $\mathbf{E}$ is an α-labeling matrix and each line in $\mathbf{E}'[1, l-1] = \mathbf{E}[1, l-1]$ has degree less than three. Applying Lemma 2.3, we obtain

$$l - 1 \leq (2r+1)^2 \leq (2(q-2)/2 + 1)^2 = (q-1)^2. \tag{6}$$

If $l < n$, let $\mathbf{D} = \mathbf{E}[m+l, m+n-1] = \mathbf{E}'[m+l, m+n-1]$. Then $\mathbf{D}^t = \mathbf{E}^t[1, n-l] = (\mathbf{E}')^t[1, n-l]$. Since $\mathbf{E}^t$ is also an α-labeling matrix, by a similar argument as above, we have

$$n - l \leq (2(q-2)/2 + 1)^2 = (q-1)^2. \tag{7}$$

Note that Inequality (6) and Inequality (7) also holds for $l = 1$ and $l = n$ respectively. We now combine (5), (6) and (7) and derive

$$k\lfloor q/2 \rfloor = |V| - 1 = n - 1 = (l-1) + (n-l) \leq 2(q-1)^2.$$

This implies $k \leq 2(q-1)^2/\lfloor q/2 \rfloor \leq 2(q-1)^2/((q-1)/2) = 4(q-1)$ and we have the proof. □

Lemma 2.5. *Let* $\mathbf{E} = (e_{ij})$ *be an* $m \times n$ *α-labeling matrix, and let* $\mathbf{D} = \mathbf{E}[1, s] = (d_{ij})$*, where* $s \in [1, m+n-1]$*. Then either all* 1*-entries of* $\mathbf{D}$ *are in the same line or there are three distinct* 1*-entries* $d_{x_0y_0}$*,* $d_{x_1y_1}$ *and* $d_{x_2y_2}$ *of* $\mathbf{D}$ *where* $x_0 = x_1$ *and* $y_1 = y_2$*.*

Proof. Since $\mathbf{E}$ is an α-labeling matrix, the entry $d_{m1} = e_{m1}$ must be an 1-entry. Now we let R be the largest integer such that $d_{mj} = 1$ for all $j \in [1, R]$, then $1 \leq R \leq n$ and $R \leq s$. If $m = 1$ or $n = 1$ or $s = 1$ or $R = s$, then all 1-entries are in the same line of $\mathbf{D}$. Now, assuming that $m > 1$, $n > 1$, $s > 1$ and $R < s$, we consider the cases as follows.
Case 1. $2 \leq R < s$. If $R = n < s$, then all entries in the bottom row are 1-entries. If $R < n$, then $d_{m(R+1)} = 0$, that is, the bottom entry in the $(R+1)$-th diagonal $\mathbf{D}[R+1]$ is zero. In either case, there must exist an unique 1-entry $d_{i_0j_0}$ in the diagonal $\mathbf{D}[R+1]$ with $i_0 < m$. Since $j_0 - i_0 + m = R+1$, $j_0 = R+1-(m-i_0) \leq R$. Therefore, we have at least three distinct 1-entries $d_{i_0j_0}$, d_{mj_0} and d_{mj}, $j \in [1, R] - \{j_0\}$.
Case 2. $R = 1 < s$. In this case, $d_{m2} = 0$. So we have $d_{m1} = d_{(m-1)1} = 1$. Let t be the largest integer such that $d_{(m-i+1)1} = 1$ for all $i \in [1, t]$, then $2 \leq t \leq m$ and $t \leq s$. If $t = s$, then all 1-entries are in the first column of $\mathbf{D}$. Now, let's assume that $t < s$. In the case when $t = m < s$, all entries in the first column are 1-entries. In the case when $t < m$, we have $d_{(m-t)1} = 0$, that is, the leftmost entry in the $(t+1)$-th diagonal $\mathbf{D}[t+1]$ is zero. In either case, there must exist an unique 1-entry $d_{i_0j_0}$ in the diagonal $\mathbf{D}[t+1]$ with $j_0 > 1$ and $j_0 - i_0 + m = t+1$, then $i_0 = m - t + (j_0 - 1) \geq m - t + 1$. Therefore, we have at least three distinct 1-entries $d_{i_0j_0}$, d_{i_01} and d_{i1}, $i \in [m-t+1, m] - \{i_0\}$. □

A disconnected graph G ia called a *star forest* if every component of G is a star.

Lemma 2.6. *Let* $\mathbf{E}$ *be an* $m \times n$ $(0,1)$*-matrix. If there exists an* $s \in [1, m+n-1]$ *such that* $< \mathbf{E}[1, s] >$ *is a star forest, then* $\mathbf{E}$ *is not an α-labeling matrix.*

Proof. Suppose that $\mathbf{E}$ is an α-labeling matrix and $s \in [1, m+n-1]$. Then, by Lemma 2.5, one of the following two conditions is satisfied.

(i) All 1-entries of $< \mathbf{E}[1, s] >$ are in the same line.

(ii) There are three distinct 1-entries e_1, e_2 and e_3 of $< \mathbf{E}[1, s] >$ such that e_1 and e_2 are in the same row and e_2 and e_3 are in the same column.

If condition (i) is satisfied, then $< \mathbf{E}[1,s] >$ is a star. If condition (ii) is true, then $< \mathbf{E}[1,s] >$ contains a path of three edges as a subgraph. In either case, $< \mathbf{E}[1,s] >$ is not a star forest. □

Before we give the sufficient conditions for the nonexistence of α-labelings for trees of diameter at most six. Let's review the definition of the center of a graph G which will be used later. The *eccentricity* of a vertex v in G is the maximum distance from v over all vertices in G. The *center* of G is the subgraph induced by the vertices of minimum eccentricity. It is known that the center of a tree is either a vertex or an edge.

Theorem 2.3. *Suppose that $T = (U, V)$ is a tree of size $q \geq 2$ with the center $c \in V$. Let $T_1, T_2, \cdots, T_r$ be all the components of $T - \{c\}$. If each T_i is either a star or an isolated vertex and $|E(T_i)| + |E(T_j)| < |V| - 1$ for all $i, j \in [1, r]$ and $i \neq j$, then T admits no α-labelings.*

Proof. Suppose that T has an α-labeling f. Then there exists an $m \times n$ α-labeling matrix $\mathbf{E} = (e_{ij})$ such that $< \mathbf{E} >$ is isomorphic to T and $< \mathbf{E} >$ is itself the matrix graph of $\mathbf{E}$. So $m = |U|$ and $n = |V|$. Let the l-th column c_l be the center of $< \mathbf{E} >$. Then $< \mathbf{E} > -\{c_l\}$ is isomorphic to $T - \{c\}$.

Let $\mathbf{E}' = \mathbf{E} - c_l$. Then $< \mathbf{E}' >$ is a subgraph of $< \mathbf{E} > -\{c_l\}$. Note that $< \mathbf{E}' >$ has no isolated vertices, whereas $< \mathbf{E} > -\{c_l\}$ may have some. Let $T'_1, T'_2, \cdots, T'_s$ be the components of $< \mathbf{E}' >$. If $s = 1$, then all but one component of $T - \{c\}$ are isolated vertices. The only one star component must have $n - 1$ edges because $|V| = n$. However, this is impossible by the hypotheses. So, $2 \leq s \leq r$ and we have the following two facts.

(a) $|E(T'_i)| + |E(T'_j)| < n - 1$ for all $i, j \in [1, s]$ and $i \neq j$.

(b) Each component of $< E' >$ is a star.

Now let's split the discussion into the following three cases.

Case 1. $l = n$. Let $\mathbf{D} = \mathbf{E}[1, n-1]$. Then $< \mathbf{D} >$ has $n - 1$ edges and therefore it has at least two nontrivial components by fact(a). Note that $< \mathbf{D} >$ has no isolated vertices and it is a subgraph of $< \mathbf{E}' >$ because $\mathbf{E}[1, n-1] = \mathbf{E}'[1, n-1]$. Fact(b) guarantees that each component of $< \mathbf{D} >$ is a star. In other words, $< \mathbf{D} >$ is a star forest. This gives a contradiction to the assumption by Lemma 2.6.

Case 2. $l = 1$. Let $\mathbf{D}$ be the transpose of $\mathbf{E}[m+1, m+n-1]$. Since $\mathbf{E}[m+1, m+n-1] = \mathbf{E}'[m+1, m+n-1]$, we have $\mathbf{D} = \mathbf{E}^t[1, n-1] = (\mathbf{E}')^t[1, n-1]$. Note that $\mathbf{E}^t$ is also an α-labeling matrix and $< \mathbf{D} >$ is again a subgraph of $< \mathbf{E}' >$. An argument similar to Case 1 also leads to a contradiction.

Case 3. $1 < l < n$. Let $\mathbf{D}_1 = \mathbf{E}[1, l-1]$ and $\mathbf{D}_2 = \mathbf{E}[m+l, m+n-1]$, then both $< \mathbf{D}_1 >$ and $< \mathbf{D}_2 >$ are subgraphs of $< \mathbf{E}' >$. If $< \mathbf{D}_i >$ has more than one components, then, by fact(b), it is a star forest. This contradicts to the fact that $\mathbf{E}$ and $\mathbf{E}^t$ are α-labeling matrices. So $< \mathbf{D}_i >$ is connected. Let T'_{j_i} be the component of < E′ > which contains < $\mathbf{D}_i$ > as a subgraph, $i = 1, 2$. Then $|E(T'_{j_1})| + |E(T'_{j_2})| \geq |E(<\mathbf{D}_1>)| + |E(<\mathbf{D}_2>)| = |\mathbf{D}_1| + |\mathbf{D}_2| = (l-1) + (n-l) = n-1$. This is again a contradiction to fact(a) and we have the proof. □

Now, the result obtained by Rosa [6] follows.

Corollary 2.1. *A tree of diameter four that contains a comet $S_{3,2}$ as a subtree does not admit an α-labeling.*

Proof. Let the tree $T = (U, V)$ where V contains the center c of T. Then each component of $T - \{c\}$ is a star or an isolated vertex. Since T contains $S_{3,2}$, $T - \{c\}$ has at least three star components. Let $T_1, T_2, \cdots, T_r$ be all the star components of $T - \{c\}$, where $r \geq 3$. Then $\sum_{i=1}^{r} |E(T_i)| = |V| - 1$ and the inequality $|E(T_i)| + |E(T_j)| < |V| - 1$ follows for all $i, j \in [1, r]$ and $i \neq j$. Therefore, we have the proof by Theorem 2.3. □

The resuls we obtained so far suggest that a comet $S_{k,q}$ with relatively larger k and smaller q is more likely to fail to admit α-labelings. We next investigate the existence of α-labelings for two families of comets with small k.

Lemma 2.7. *The comets $S_{3,q}$ and $S_{4,q}$ have α-labelings if q is odd.*

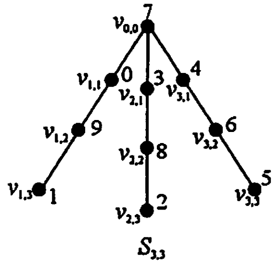
Proof. Let $v_{0,0}$ denote the center of the comet $S_{k,q}$, $k = 3$ or 4, and let $\{v_{0,0}, v_{i,1}, v_{i,2}, ..., v_{i,q}\}$ be the vertex set of the i-th path $P_{q+1}^{(i)}$ of order $q+1$ in $S_{k,q}$, $i \in [1, k]$. Then $S_{k,q} = (U, V)$ where $V = \{v_{0,0}\} \cup \{v_{i,j} : i \in [1, k], j = 2, 4, ..., q-1\}$ and $U = \{v_{i,j} : i \in [1, k], j = 1, 3, ..., q\}$. Define the labeling f as follows:

$$f(v_{i,j}) = \begin{cases} qk - (q-1), & \text{if } i = j = 0; \\ (i-1)(q+1)/2 + (j-1)/2, & \text{if } i = 1, 3 \text{ and } j = 1, 3, ..., q; \\ kq - (i-1)q/2 + 1 - j/2, & \text{if } i = 1, 3 \text{ and } j = 2, 4, ..., q-1; \\ i(q+1)/2 - (j+1)/2, & \text{if } i = 2, 2\lfloor k/2 \rfloor \text{ and } j = 1, 3, ..., q; \\ kq - iq/2 + 1 + j/2, & \text{if } i = 2, 2\lfloor k/2 \rfloor \text{ and } j = 2, 4, ..., q-1. \end{cases}$$

It is routine to verify that $f(U) = [0, k(q+1)/2 - 1]$ and $f(V) = [k(q+1)/2, kq]$. One can easily check that the induced label for the edge $v_{i,j}v_{i,j+1}$ is $q(k-i+1) - j + (3-i)/2$ when $i \in [1, 2]$ and is $q(k-i) + j + (4-i)/2$ when $i \in [3, k]$, $j \in [1, q-1]$. Therefore, the set of induced edge labels on $E(P_{q+1}^{(i)} - v_{0,0}v_{i,1})$ is $[q(k-i)+2, q(k-i+1)]$ for $i \in [1, 2]$ and is $[q(k-i)+1, q(k-i+1)-1]$ for $i \in [3, k]$. In addition, the edges $v_{0,0}v_{i,1}$'s, $i \in [1, k]$, receive the labels $q(k-1)+1, q(k-2)+1$ and $q(k-1-l)$ where $l \in [1, \lfloor k/2 \rfloor]$. Hence, f is an α-labeling of $S_{k,q}$. □

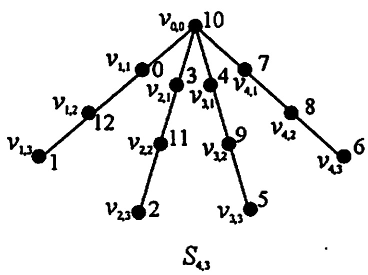
For clarity, we present the α-labelings from Lemma 2.7 and the corresponding α-labeling matrices for $S_{3,3}$ and $S_{4,3}$ in Figure 2 and 3, respectively.

Finally, we give a sufficient and necessary condition for the existence of an α-labeling for a comet $S_{k,3}$.



	$v_{3,2}$ 6	$v_{0,0}$ 7	$v_{2,2}$ 8	$v_{1,2}$ 9
$v_{1,1}$ 0		1		1
$v_{1,3}$ 1				1
$v_{2,3}$ 2			1	
$v_{2,1}$ 3		1	1	
$v_{3,1}$ 4	1	1		
$v_{3,3}$ 5	1			

Figure 2: $S_{3,3}$



	$v_{4,2}$ 8	$v_{3,2}$ 9	$v_{0,0}$ 10	$v_{2,2}$ 11	$v_{1,2}$ 12
$v_{1,1}$ 0			1		1
$v_{1,3}$ 1					1
$v_{2,3}$ 2				1	
$v_{2,1}$ 3			1	1	
$v_{3,1}$ 4		1	1		
$v_{3,3}$ 5		1			
$v_{4,3}$ 6	1				
$v_{4,1}$ 7	1		1		

Figure 3: $S_{4,3}$

Corollary 2.2. *A comet $S_{k,3}$ has an α-labeling if and only if $k \leq 4$.*

Proof. Let the comet $S_{k,3} = (U, V)$ with the center $c \in V$. Then $|V| = k + 1$ and each component of $T - \{c\}$ is a star $K_{1,2}$. By Theorem 2.3, the comet admits no α-labeling when $k \geq 5$. In the case when $k = 1$ or 2, the comet is actually a path and therefore admits an α-labeling. The case when $k = 3$ or 4 follows Lemma 2.7. □

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