A STUDY ON STABLE, POSITIVE AND NEGATIVE EDGES WITH RESPECT TO IRREGULARITY STRENGTH OF A GRAPH

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ABSTRACT. In this paper we discuss how the addition of a new edge changes the irregularity strength in $K(3,n)$, tK_3 and tP_4.

1. INTRODUCTION

In this paper we consider simple undirected and connected graphs with no K_2 components and at most one isolated vertex. Let $G=(V,E)$ be a graph. A network $G(f)$ consists of the graph together with an assignment $f : E(G) \to Z^+$. The sum of the labels of the edges incident with a vertex is called the weight of that vertex. If all the weights are pair wise distinct, $G(f)$ is called an irregular network. The strength of the network $G(f)$ is defined by $s(G(f)) = \max_{e \in E}\{f(e)\}$. The irregularity strength $s(G)$ of G is defined as $s(G) = \min\{s(G(f)) / G(f) \text{ is irregular}\}$.

The problem of finding irregularity strength of graphs was proposed by Chartrand et al., [2] and has proved to be difficult, in general. There are not many graphs for which the irregularity strength is known. The readers are advised to refer the survey of Lehel [9] and the papers [1,6,10,11]. R.J.Faudree, M.S. Jacobson, J. Lehel and R.H.Schelp studied the irregularity strength of tK_3 in [4]. A. Gyárfás [5] determined the irregularity strength of $K_n - mK_2$. Stanislav Jendrol', and Michal Tkáč [11] studied the irregularity strength of the union of t copies of the complete graph K_p.

Definition 1.1 [7]. Let G be any graph which is not complete, e be any edge of $\overline{G}$, then e is called a positive edge if $s(G+e)>s(G)$, e is called a negative edge if $s(G+e)<s(G)$ and e is called a stable edge if $s(G+e)=s(G)$.

Definition 1.2 [7]. If all the edges of $\overline{G}$ are positive, (negative, stable) edges of G, then G is called a positive (negative, stable) graph. Otherwise G is called a mixed graph.

Example 1.3. Star graphs $K_{1,n}$ are negative graphs for $n \geq 3$.

Example 1.4. P_3 is a positive graph.

Example 1.5. G:

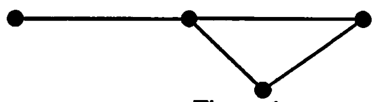

Figure 1

G+e:

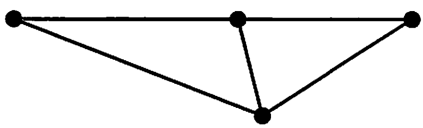

Figure 2

It is easy to verify that $s(G) = 2$ and $s(G+e) = 2$.
Hence G is a stable graph.

Example 1.6. Consider P_4. It is easy to verify that $s(P_4) = 2$.
Let v_1, v_2, v_3 and v_4 be the consecutive vertices of P_4.
For any $e \in \overline{P_4}$, $P_4 + e$ is isomorphic to either G_1 or G_2.

G_1:

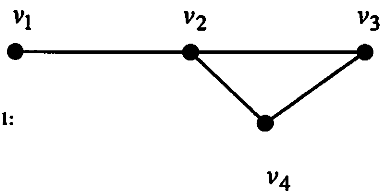

Figure 3

G_2:

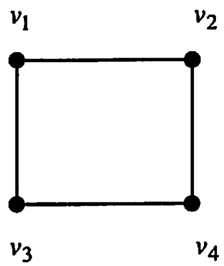

Figure 4

Also $s(G_1) = 2$ and $s(G_2) = 3$.
Thus $v_2 v_4$ is a stable edge and $v_1 v_4$ is a positive edge.
Hence P_4 is a mixed graph.

Authors in [7] proved that $P_n, n \geq 4$ is a mixed graph and the cycle C_n is a negative graph for any $n \geq 4$.

In this paper we discuss positive, negative and stable edges of three families of graphs namely t copies of K_3, friendship graph and t copies of P_4 and we try to characterize positive graphs.

2. Stable and negative graphs

We shall use the following function to simplify the later notations.

Let $\alpha(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$

First we prove that tK_3 is a negative graph.

Theorem 2.1. For $t \geq 2$, tK_3 is a negative graph.

Proof. The irregularity strength of tK_3 was determined in [4]

$$s(tK_3) = \begin{cases} \left\lceil \dfrac{3t+1}{2} \right\rceil + 2 & \text{for } t \equiv 3 \pmod 4 \\ \left\lceil \dfrac{3t+1}{2} \right\rceil + 1 & \text{otherwise} \end{cases}$$

Let us consider t copies of K_3.
Let v_{i1}, v_{i2} and v_{i3} be the vertices of i^{th} copy of K_3 for $1 \leq i \leq t$.

It is easy to see that, for any $e \in \overline{tK_3}$, $tK_3 + e$ is isomorphic to $tK_3 + v_{11}v_{t1}$.

So, it is enough to prove that $s(tK_3 + v_{11}v_{t1}) < s(tK_3)$.

Define the edge label $f : E(tK_3 + v_{11}v_{t1}) \to Z^+$ as follows.

For $2 \le i \le t\text{-}1$,

$$f(v_{i1}\, v_{i2}) = \begin{cases} \dfrac{3i-1}{2}, \textit{if} & i \equiv 1 \,(\text{mod } 4) \\ \dfrac{3i-3}{2}, \textit{if} & i \equiv 3 \,(\text{mod } 4) \\ \dfrac{3i-2}{2}, & \textit{otherwise} \end{cases}$$

$$f(v_{i2}\, v_{i3}) = \begin{cases} \dfrac{3i}{2}, \textit{if} & i \equiv 0 \,(\text{mod } 4) \\ \dfrac{3i+2}{2}, \textit{if} & i \equiv 2 \,(\text{mod } 4) \\ \dfrac{3i+1}{2}, & \textit{otherwise} \end{cases}$$

$$f(v_{i1}\, v_{i3}) = \begin{cases} \dfrac{3i+3}{2}, \textit{if} & i \equiv 1 \,(\text{mod } 4) \\ \dfrac{3i+5}{2}, \textit{if} & i \equiv 3 \,(\text{mod } 4) \\ \dfrac{3i+4}{2}, & \textit{otherwise} \end{cases}$$

Case (1). Suppose $t \equiv 0 \; (mod\, 4)$ then

$$f(v_{11}\, v_{12}) = f(v_{11}\;\; v_{t1}) = 2$$

$$f(v_{12}\, v_{13}) = f(v_{13}\, v_{11}) = 1$$

$$f(v_{t1}\, v_{t2}) = f(v_{t2}\, v_{t3}) = \frac{3t+2}{2}$$

$$f(v_{t1}\, v_{t3}) = \frac{3t}{2}$$

By case (1), we have an irregular network $tK_3 + v_{11}\, v_{t1}$ with maximum label $\frac{3t+2}{2} < \left\lceil \frac{3t+1}{2} \right\rceil + 1$ if $t \equiv 0 \,(\text{mod } 4)$.

Hence the edge $v_{11}v_{t1}$ is negative if $t \equiv 0 \,(\text{mod } 4)$.

Case (2). Suppose $t \equiv 1 \; (mod\, 4)$ then

$$f(v_{12}\, v_{13}) = f(v_{13}\, v_{11}) = 1$$

$$f(v_{12}\, v_{11}) = f(v_{11}\, v_{t1}) = 2$$

$$f(v_{t1}\, v_{t2}) = f(v_{t2}\, v_{t3}) = \frac{3t+1}{2}$$

$$f(v_{t1}\, v_{t3}) = \frac{3t-1}{2}$$

By case (2), we have an irregular network $tK_3 + v_{11}\, v_{t1}$ with maximum label

$\frac{3t+1}{2} < \left\lceil \frac{3t+1}{2} \right\rceil + 1$ if $t \equiv 1 \pmod 4$.

Hence the edge $v_{11}\, v_{t1}$ is negative if $t \equiv 1 \pmod 4$.

Case (3). Suppose $t \equiv 2 \pmod 4$ then

$$f(v_{12}\, v_{13}) = f(v_{13}\, v_{11}) = f(v_{11}\, v_{t1}) = 1$$

$$f(v_{12}\, v_{11}) = 2$$

$$f(v_{t1}\, v_{t2}) = f(v_{t2}\, v_{t3}) = \frac{3t}{2}$$

$$f(v_{t1}\, v_{t3}) = \frac{3t+2}{2}$$

By case (3), we have an irregular network $tK_3 + v_{11}\, v_{t1}$ with maximum label

$\frac{3t+2}{2} < \left\lceil \frac{3t+1}{2} \right\rceil + 1$ if $t \equiv 2 \pmod 4$.

Hence the edge $v_{11}\, v_{t1}$ is negative if $t \equiv 2 \pmod 4$.

Case (4). Suppose $t \equiv 3 \pmod 4$ then

$$f(v_{11}\, v_{t1}) = f(v_{12}\, v_{13}) = f(v_{11}\, v_{13}) = 1$$

$$f(v_{11}\, v_{12}) = 2$$

$$f(v_{t1}\, v_{t2}) = f(v_{t2}\, v_{t3}) = \frac{3t+1}{2}$$

$$f(v_{t1}\, v_{t3}) = \frac{3t+3}{2}$$

By case (4), we have an irregular network $tK_3 + v_{11}\, v_{t1}$ with maximum label

$\frac{3t+3}{2} < \left\lceil \frac{3t+1}{2} \right\rceil + 2 = s(tK_3)$ if $t \equiv 3 \pmod 4$.

Hence the edge $v_{11} v_{t1}$ is negative if $t \equiv 3 \pmod 4$.

Thus tK_3 is a negative graph. ∎

Definition 2.2 [8]. For any $n \geq 2$, identify n copies of K_3 with a common vertex, then the resulting graph is called friendship graph *K (3,n)*.

Examples 2.3. *K(3,5):*

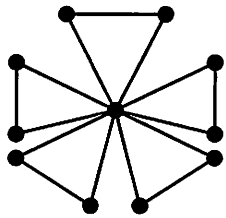

Figure 5

In the next theorem we determine the irregularity strength of *K(3,n)*.

Theorem 2.4. The irregularity strength of *K(3,n)* is *n+1* for all $n \geq 2$.

Proof. Let $v_1, v_2, ...v_{2n+1}$ be the 2n+1 vertices of *K(3,n)* where we assume that v_{2n+1} is the vertex of degree 2n and all other vertices are of degree 2.
Define the edge label $f : E \to Z^+$ by

$$f(v_i\ v_{i+1}) = i, \qquad i = 1,3,5,.........n - \alpha(n+1)$$
$$f(v_{n+1-\alpha(n+1)+i}\ v_{n+2-\alpha(n+1)+i}) = n, \qquad i = 1,3,5 ....n - 2 + \alpha(n+1)$$
$$f(v_i\ v_{2n+1}) = \begin{cases} 1, & i = 1,3,5,.........n - \alpha(n+1) \\ 2, & i = 2,4,6........n + 1 - \alpha(n+1) \end{cases}$$
$$f(v_{n+i}\ v_{2n+1}) = i + 1, \qquad 1 + \alpha(n) \leq i \leq n$$

By the edge labeling f, we have an irregular network *K (3,n)* whose weights of the vertices are 2,3,4.............2n+1, $\frac{n^2+6n-1}{2}$ if n is odd and 2,3,4,.....2n+1, $\frac{n^2+4n}{2}$ if n is even. Thus $s(K(3,n)) \leq n+1$.
Since there are 2n vertices having degree 2 in *K(3,n)*, it is not possible to find an irregular assignment with fewer than label n+1. Therefore $s(K(3,n)) \geq n+1$.
Hence $s(K(3,n)) = n+1$. ■

Theorem 2.5. Friendship graph *K(3,n)* is a negative graph for $n \geq 2$.

Proof. For any edge e of $\overline{K(3,n)}$, $K(3,n)+e$ is isomorphic to $K(3,n)+v_{2n}v_1$
Therefore it is enough to prove that $v_{2n}v_1$ is a negative edge. Assign the same edge labeling as given in the previous theorem except for the edges $v_{2n}v_{2n+1}$, $v_{2n}v_1$ and $v_2\, v_{2n+1}$ and assign the label n to the edge $v_{2n}\, v_{2n+1}$ and 1 to the remaining edges.
Thus $s(K(3,n)+v_{2n}\, v_1) \leq n < s(K(3,n))$.
Thus $v_{2n}v_1$ is a negative edge.
Hence the friendship graph *K(3,n)* is a negative graph. ∎

Let us determine the irregularity strength of tP_4.

Theorem 2.6. Irregularity strength of tP_4 is 2t.

Proof. Consider t copies of P_4.
Let $v_{i\,1}, v_{i\,2}, v_{i\,3},$ and $v_{i\,4}$ be the consecutive vertices of the i^{th} copy of P_4 where $1 \leq i \leq t$.
Define the label $f : E(tP_4) \to Z^+$ by

$$f(v_{i\,1}\; v_{i\,2}) = 2i-1, \qquad 1 \leq i \leq t$$

$$f(v_{i\,2}\; v_{i\,3}) = 2t$$

$$f(v_{i\,3}\; v_{i\,4}) = 2i, \qquad 1 \leq i \leq t$$

Hence the weights of the vertices are 1, 2,3,...,4 t.
All the weights are different and hence tP_4 is irregular.
Therefore $s(tP_4) \leq 2t$.

In tP_4, there are 2t pendent edges therefore it is not possible to obtain an irregular assignment with fewer than label 2t. Therefore $s(tP_4) \geq 2t$.
Hence $s(tP_4)=2t$. ∎

Theorem 2.7. tP_4 is a stable graph for $t \geq 2$.

Proof. Addition of any edge $e \in \overline{tP_4}$ to tP_4 will be isomorphic to any one of the following graphs.

1) $P_8 \cup (t-2)P_4$.
2) Path on 7 vertices with one pendent edge attached to the third vertex from one end union t – 2 copies of P_4.
3) P_6 with 2 pendent edges attached to 3rd and 4th vertices of the path union t – 2 copies of P_4.

4) $C_4 \cup (t-1)P_4$.

5) C_3 with a pendent edge attached to any vertex of C_3 union t – 1 copies of P_4.

Case (1). Suppose $e = v_{t\,4}\ v_{(t-1)\,4}$

Define the labeling $f : E(tP_4 + v_{t4}\ v_{(t-1)4}) \to Z^+ by$

$$f(v_{11}\ v_{12}) = 1, \qquad f(v_{12}\ v_{13}) = f(v_{13}\ v_{14}) = 2$$

$$f(v_{i1}\ v_{i2}) = 2i+1, \quad 2 \le i \le t-2.$$

$$f(v_{i2}\ v_{i3}) = 2t-4, \quad 2 \le i \le t-2$$

$$f(v_{i3}\ v_{i4}) = 2i+2, \quad 2 \le i \le t-2$$

$$f(v_{t1}\ v_{t2}) = f(v_{t2}\ v_{t3}) = f(v_{(t-1)2}\ v_{(t-1)3}) = 2t,$$

$$f(v_{(t-1)1}\ v_{(t-1)2}) = 2t-1,$$

$$f(v_{(t-1)3}\ v_{(t-1)4}) = f(v_{t4}\ v_{(t-1)4}) = 2t-2,$$

$$f(v_{t3}\ v_{t4}) = 2t-3.$$

By the above edge labeling, we have an irregular network $tP_4 + v_{t\,4}v_{(t-1)\,4}$ whose weights are 1,2,3 ....,4 t. Thus $s(tP_4 + v_{t\,4}v_{(t-1)\,4}) \le 2t$.

The minimum possible weights of $tP_4 + v_{t\,4}v_{(t-1)\,4}$ are 1,2,3........4 t. Since the maximum degree of $tP_4 + v_{t\,4}v_{(t-1)\,4}$ is 2, it is not possible to obtain the weight 4t with fewer than label 2t. Thus $s(tP_4 + v_{t\,4}v_{(t-1)\,4}) \ge 2t$.

Hence $s(tP_4 + v_{t\,4}v_{(t-1)\,4}) = 2t$.

Hence the edge $v_{t\,4}v_{(t-1)\,4}$ is stable.

Case (2). Suppose $e = v_{(t-1)\,3}v_{t\,4}$

Define the labeling $f : E(tP_4 + v_{(t-1)\,3}\ v_{t\,4}) \to Z^+ by$

$$f(v_{t1}\ v_{t2}) = 2t-2, \qquad f(v_{t3}\ v_{t4}) = f(v_{(t-1)3}\ v_{(t-1)4}) = 2t-1$$

$$f(v_{(t-1)3}\ v_{t4}) = 1$$

For other edges assign the labels as in theorem 2.6. By the above labeling, we have an irregular network $tP_4 + v_{(t-1)\,3}\ v_{t\,4}$ with maximum label 2t. Thus $s(tP_4 + v_{(t-1)\,3}\ v_{t\,4}) \le 2t$.

The minimum possible weights of $tP_4 + v_{(t-1)3}\ v_{t4}$ are 1, 2, 3 ...,4t.

In $tP_4 + v_{(t-1)3}\ v_{t4}$, there are 2t vertices of degree 2, 2t – 1 vertices of degree 1 and one vertex of degree 3. So, it is not possible to obtain the weights 4t and 4t – 1 with fewer than label 2t. Thus $s(tP_4 + v_{(t-1)3}\ v_{t4}) \geq 2t.$

Hence $s(tP_4 + v_{(v-1)3}\ v_{t4}) = 2t.$

Hence the edge $v_{(t-1)3}v_{t4}$ is stable.

Case(3). Suppose $e = v_{t3}\ v_{(t-1)3}$.

Define the labeling $f : E(tP_4 + v_{t3}v_{(t-1)3}) \rightarrow Z^+ by$

$f(v_{t1}\ v_{t2}) = f(v_{t2}\ v_{t3}) = 2t - 1$ and $f(v_{t3}\ v_{(t-1)3}) = 1$

For other edges assign the labels as in theorem 2.6 .

By the above labeling, we have an irregular network $tP_4 + v_{t3}v_{(t-1)3}$ with maximum label 2t. Thus $s(tP_4 + v_{t3}\ v_{(t-1)3}) \leq 2t.$

In $tP_4 + v_{t3}\ v_{(t-1)3}$, there are 2t pendent edges, so it is not possible to obtain an irregular assignment with fewer than label 2t. Thus $s(tP_4 + v_{t3}\ v_{(t-1)3}) \geq 2t.$

Hence $s(tP_4 + v_{t3}\ v_{(t-1)3}) = 2t$.

Hence the edge $v_{t3}\ v_{(t-1)3}$ is stable.

Case (4). Suppose $e = v_{t1}v_{t4}$.

Define the labeling $f : E(tP_4 + v_{t1}v_{t4}) \rightarrow Z^+ by$

$$f(v_{i2}\ v_{i3}) = 2t - 2, \quad 1 \leq i \leq t - 1$$

$$f(v_{t1}v_{t4}) = 2t - 2$$

For other edges assign the labels as in theorem 2.6. By the above labeling, we have an irregular network $tP_4 + v_{t1}\ v_{t4}$ with maximum label 2t. Thus $s(tP_4 + v_{t1}v_{t4}) \leq 2t.$

The minimum possible weights of $tP_4 + v_{t1}v_{t4}$ are 1,2,3........4t and the maximum degree is 2. So it is not possible to obtain the weight 4t with fewer than label 2t. Thus $s(tP_4 + v_{t1}\ v_{t4}) \geq 2t.$

Hence $s(tP_4 + v_{t1}\ v_{t4}) = 2t.$

Hence the edge $v_{t1}\ v_{t4}$ is stable.

Case (5). Suppose $e = v_{t2} v_{t4}$.

Define the labeling $f : E(tP_4 + v_{t2}v_{t4}) \to Z^+ by$

$f(v_{t3} v_{t4}) = 2t-1$ and $f(v_{t2} v_{t4}) = 1$

For other edges assign the labels as in theorem 2.6. By the above labeling, we have an irregular network $tP_4 + v_{t2} v_{t4}$ with maximum label 2t. Thus $s(tP_4 + v_{t2} v_{t4}) \le 2t$.

The minimum possible weights of $tP_4 + v_{t2}v_{t4}$ are 1,2,3,........4 t and the maximum degree is 2. So it is not possible to obtain the weight 4t, with fewer than label 2t. Thus $s(tP_4 + v_{t2} v_{t4}) \ge 2t$.

Hence $s(tP_4 + v_{t2} v_{t4}) = 2t$.

Hence the edge $v_{t2} v_{t4}$ is stable.

By all the five cases, $s(tP_4 + e) = s(tP_4)$ where $e \in \overline{tP_4}$.

Hence tP_4 is a stable graph. ∎

Theorem 2.8. $K_n - e$ is a positive graph for n = 3,4,5.

Proof. In [5] Gyarfas determined that $s(K_n - mK_2) = 3$ unless n = 4m, n = 4m+1, n = 4m – 1. In the exceptional cases, the irregularity strength is 2. Hence $s(K_n - e) = 2$, for $n = 3,4,5$.

In [2] it was shown that $s(K_p) = 3$, for $p \ge 3$

Hence $K_n - e$ is a positive graph for n = 3,4,5. ∎

Based on the above theorem and our experience we conclude with the following conjecture.

Conjecture: A graph G is a positive graph iff G is any one of the following graphs $K_3 - e$, $K_4 - e$ and $K_5 - e$.

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