On a graph partition result of Kühn and Osthus *

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Abstract

Kühn and Osthus [2] proved that for every positive integer ℓ, there exists integer $k(\ell) \leq 2^{11} \cdot 3\ell^2$, such that the vertex set of every graph G with $\delta(G) \geq k(\ell)$ can be partitioned into subsets S and T with the properties that $\delta(G[S]) \geq \ell \leq \delta(G[T])$ and every vertex of S has at least ℓ neighbors in T. In this note, we improve the upper bound to $k(\ell) \leq 2^4 \cdot 17\ell^2$.

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Given a graph G, a partition of G is a family of pairwise disjoint subsets $V_1, \ldots, V_k$ of $V(G)$ such that $V(G) = \bigcup_{i=1}^{k} V_i$. Graph partition problems ask for a partition of a graph with various requirements.

Let S and T be a partition of a graph G. We use $(S,T)_G$ to denote the bipartite subgraph of G consisting of the edges between S and T. It is well known that the minimum degree of the bipartite subgraph $(S,T)_G$ of G can be greater than half the minimum degree of G. So it is an interesting problem that whether $G[S], G[T]$ and $(S,T)_G$ have large minimum degree simultaneously. In [2], Kühn and Osthus gave a negative answer to this question (see Proposition 6 in [2]). In light of this negative result, one may relax the constraints and ask that only one side of $(S,T)_G$ has large minimum degree while both the minimum degree of $G[S]$ and $G[T]$ are large. Kühn and Osthus answered it affirmatively.

Theorem 1. *(Kühn and Osthus [2]) For every $\ell \in \mathbb{N}$, there exists an integer $k(\ell) \leq 2^{11} \cdot 3\ell^2$ such that $\delta(G) \geq k(\ell)$ guarantees the existence of a*

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partition of $V(G)$ into S and T such that $\delta(G[S]) \geq \ell \leq \delta(G[T])$ and every vertex of S has at least ℓ neighbors in T.

In this note, we improve Kühn and Osthus's bound, and prove the following theorem.

Theorem 2. *For each pair of positive integers s and t with $s \geq t$, there exists an integer $k(s,t) \leq 17 \cdot 2^4 st$ such that $\delta(G) \geq k(s,t)$ guarantees the existence of a partition of $V(G)$ into S and T such that $\delta(G[S])$ and $\delta(G[T])$ are at least $s+1$ and t, respectively, and every vertex in S has at least t neighbors in T.*

The proof technique of Theorem 2 is the same as that used by Kühn and Osthus but with a much sharper analysis. All graphs considered in this paper are finite, undirected and simple. For a graph, we denote by $V(G)$, $E(G)$, $d(G)$ and $\delta(G)$ the vertex set, edge set, average degree and the minimum degree of G, respectively. We use $e(G)$ to denote the number of edges in a graph G. For a vertex $x \in V(G)$, we use $N(x)$ to denote the neighbor set of x in G. For a subset $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and use $e(S)$ to denote the number of edges in $G[S]$. For two disjoint subsets S and T of $V(G)$, we use $e(S,T)$ to denote the number of edges in $(S,T)_G$.

In [3], Stiebitz proved that for integers $s,t \geq 1$, each graph G with $\delta(G) \geq s+t+1$ admits a partition of $V(G)$ into S and T such that $\delta(G[S]) \geq s$ and $\delta(G[T]) \geq t$. As per Proposition 1.2.2 of [1], each graph G with $d(G) \geq 2k$ has a subgraph of minimum degree at least $k+1$. Combining this result, we obtain the following conclusion in the spirit of Lemma 10 of [2] by replacing 3ℓ with $s+t+1$.

Lemma 3. *Let G be a graph with $\delta(G) \geq s+t+1$, where $s \geq t$. Then $V(G)$ can be partitioned into two non-empty sets S and T such that $\delta(G) \geq s$, $\delta(G) \geq t$ and for every subgraph $H \subseteq G[S]$ has average degree $\delta(H) \leq 2(s+t)$.*

A technical lemma (Lemma 11) in [2] plays a very important role. With the same argument as that of [2], we reduce the requirement on the minimum degree of G from $2^6 c\ell r$ to $2^3 c\ell r$ by a sharper analysis. This improvement enables us to reduce the bound $2^{11} \cdot 3\ell^2$ of Theorem 1 to $17 \cdot 2^4 \ell^2$.

Lemma 4. *Let $c \geq 2$ be a real, k, ℓ, r be positive integers such that $\ell \geq r$ and $k \geq 2^3 c\ell r$. Let G be a graph of minimum degree at least k and let S,T be a partition of $V(G)$ such that $d(G[S]) \geq \ell$, $\delta(G[T]) \geq r$ and every subgraph of $G[S]$ has average degree less than $c\ell$. Then there exists $S' \subseteq S$ such that, writing $T' = V(G) \backslash S'$, every vertex in S' has at least r neighbors in T', $d(G[S']) \geq \ell/8$ and $\delta(G(T')) \geq r$. Moreover, T' can be obtained from*

T by successively adding vertices having at least r neighbors in the superset of T already constructed.

Proof. We start by defining sets as in the proof of Lemma 11 of [2]. These sets, shown in Fig. 1, are as follows: $A = \{a \in S \mid |N(a) \cap S| \geq 4c\ell\}$, $C = S \setminus A$, $B = \{b \in S \mid |N(b) \cap T| < r\}$, $B' = \{b \in B \mid |N(b) \cap C| \geq r\}$, $A_1 = B \setminus B'$ and $A_2 = A \setminus A_1$, $\bar{C} = \cup_{x \in B'} N_x$ where $N_x \subseteq N(x)$ and $|N_x| = r$, $C' = C \setminus \bar{C}$.

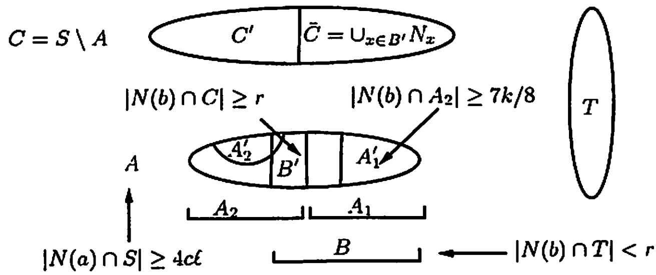

Figure 1: The set-up of the proof of Lemma 4.

We may suppose that $A \neq \emptyset$ and $B \neq \emptyset$ (one can see more details in [2]). By the definitions, we have $e(A)+e(S) \geq 2e(A)+e(A,C) \geq 4c\ell|A| \geq 8e(A)$, i.e.

$$e(A) \leq \frac{e(S)}{7}. \tag{1}$$

Since $\delta(G) \geq k \geq 2^3 c\ell r$, $k - r > 4c\ell$, and thus $B \subseteq A$. By (1),

$$(k-r)|B| \leq 2e(B)+e(S\setminus B, B) \leq e(B)+e(S) \leq e(A)+e(S) \leq \frac{8e(S)}{7}. \tag{2}$$

Let $e_S(\bar{C})$ be the number of edges in $G[S]$ which have at least one end in $\bar{C}$. Since each vertex of C has fewer than $4c\ell$ neighbors in S, and $k-r \geq \frac{15}{2}c\ell r$ as $c \geq 2$,

$$e_S(\bar{C}) < |B'|r \cdot 4c\ell \leq |B|r \cdot 4c\ell \leq \frac{8e(S)}{7(k-r)} \cdot 4c\ell r \leq \frac{64e(S)}{105}. \tag{3}$$

Where the third inequality follows equation 2.

Different from [2], replacing $\frac{1}{2}$ with $\frac{7}{8}$, we define that $A_1' = \{a \in A_1 \mid |N(a) \cap A_2| \geq \frac{7k}{8}\}$. By an argument analgous to that in [2], there exists a subset A_2' of A_2 such that

(*i*) every vertex of A_1' has at least r neighbors in A_2', and (*ii*) $e(A_2', C') \leq 2e(A_2, C')/13$.

Put $T' = T \cup A_1 \cup A_2' \cup \bar{C}$, $S' = V(G) \setminus T'$. Now, we show that $G[S']$ has large average degree. If $e(A_2', C') \geq \frac{e(S)}{44}$, then

$$e(S') \geq e(A_2 \setminus A_2', C') = e(A_2, C') - e(A_2', C') \geq \frac{11}{2} e(A_2', C') > \frac{e(S)}{8}.$$

Otherwise, $e(A_2', C') < \frac{e(S)}{44}$. Since $c \geq 2$, $k \geq 2^3 c\ell r$, and by (2),

$$e(A_1, C') \leq |B| r \leq \frac{8r}{7(k-r)} e(S) \leq \frac{8}{7(16\ell - 1)} \leq \frac{8e(S)}{105}. \tag{4}$$

And thus

$$\begin{aligned} e(S') &\geq e(S) - e(A) - e_S(\bar{C}) - e(A_1, C') - e(A_2', C') \\ &\geq e(S) - \frac{e(S)}{7} - \frac{64e(S)}{105} - \frac{8e(S)}{105} - \frac{e(S)}{44} \quad \text{(by (1), (3) and (4))} \\ &> \frac{e(S)}{8}. \end{aligned}$$

Thus we obtain that $d(G[S']) \geq \ell/8$ since $d(G[S]) \geq \ell$. Furthermore, every vertex of $S' \subseteq S \setminus (B \setminus B')$ has at least r neighbors in $\bar{C} \cup T \subseteq T'$, and it is easy to show that $\delta(G[T']) \geq t$. As for the final statement in the lemma, we can verify the last claim of the Lemma similar as that in [2]. ∎

Proof of Theorem 2. Since $16s + t + 1 < 17 \cdot 16st$, by Lemma 3, we may partition $V(G)$ into S'', T'' such that $\delta(G[S'']) \geq 16s$, $\delta(G[T'']) \geq t$ and every subgraph of $G[S'']$ has average degree less than $32s + t \leq 34s$. Applying Lemma 4 to the partition S'', T'' with parameters $\ell = 16s$, $r = t$ and $c = \frac{17}{8}$, we can find a partition S', T' of $V(G)$ such that $d(G[S']) \geq 2s$, $\delta(G[T']) \geq t$ and each vertex of S' has at least t neighbors in T'. Since $d(G[S']) \geq 2s$, $G[S']$ contains a subgraph H of minimum degree at least $s + 1$ (see Proposition 1.2.2 [1]). Then $S = V(H), T = V(G) \setminus S$ is a partition as required. ∎

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