On the average eccentricity of unicyclic graphs *

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Abstract

Let G be a connected graph on n vertices. The average eccentricity of a graph G is defined as $\xi(G) = \frac{1}{n}\sum_{v\in V(G)} \varepsilon(v)$, where $\varepsilon(v)$ is the eccentricity of the vertex v which is the maximum distance from it to any other vertex. In this paper, we characterize the extremal unicyclic graphs among n–vertex unicyclic graphs having the minimal and the second minimal average eccentricity.

Key words: eccentricity; average eccentricity; unicyclic graph; girth

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As we know, a communication network can be represented to be a graph by regarding the nodes as vertices and regarding the links as edges. Graphs stand for the essential properties of a network. Moreover, graphs can be used to describing the mathematical concepts in networks. Let G be a simple connected graph with the vertex set $V(G)$. For vertices $u, v \in V(G)$, the *distance* $d(u, v)$ is defined as the length of the shortest path between u and v in G. The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. The *radius* $r(G)$ of a graph is the minimum eccentricity of any vertex, while the *diameter* $d(G)$ of a graph is the maximum eccentricity of any vertex in the graph.

In the communication network, there is an interesting question: *Minimizing the average, taken over all nodes in the network, of the maximum time delay of a message sending from a node.* This can be regarded as a pure graph-theoretical problem about the average eccentricity of the corresponding graph. The average eccentricity of graph G, denoted by $\xi(G)$, is defined as the mean eccentricities of vertex in G, i.e., $\xi(G) = \frac{1}{n}\sum_{v\in V(G)} \varepsilon(v)$. Clearly, $r(G) \leq \xi(G) \leq d(G)$.

Recently the study of average eccentricity attract a few attention and some results are derived. In [1], Peter Dankelmann shown that a graph with order n and minimum degree δ has average eccentricity at most $\frac{9}{4}n/(\delta+1)+O(1)$. Moreover he/she gave a Nordhaus-Gaddum bound for average eccentricity. As we know, removing edges does not decrease the eccentricity, Torbett et al. [2] characterized those edges which can be added to a tree without affecting the eccentricity of any vertex. In addition, Dankelmann [1] derived an upper bound for the average eccentricity after removing an edge. Let T be a spanning tree of G, T is called *eccentricity-preserving* if $\varepsilon_T(v) = \varepsilon_G(v)$ for every vertex $v \in V(G)$. T is called average-preserving if $\xi(T) = \xi(G)$. Clearly, a spanning tree is average-preserving if and only if it is *eccentricity-preserving*. In [3], Nandakumar and Parthasarathy characterized graphs which have average-preserving spanning trees. In [5], the present authors investigate the average eccentricity of trees and determine the extremal trees with the minimal average eccentricity. We also characterize the extremal trees with the minimal dence number, respectively. The authors [6] investigate the average eccentricity of graphs with fixed diameter.

Let $\mathcal{U}_n(k)$ be the set of n-vertex unicyclic graphs of order n with girth k and $\mathcal{U}_n$ be the set of all unicyclic graphs of order n. For $G \in \mathcal{U}_n(k)$, if $k = n, n-1$, then G is unique. So in the following we assume $3 \le k \le n-2$. By $L_{n,k}$ we denote the graph obtained from C_k and P_{n-k+1} by identifying a vertex of C_k with one end vertex of P_{n-k+1}. Ilić in [4] proved that $L_{n,k}$ maximizes the average eccentricity among all connected unicyclic graphs with given girth k.

Theorem 1.1 *[4] Let $G \in \mathcal{U}_n(k)$ be a unicyclic graph of order $n > 5$. Then $\xi(G) \le \xi(L_{n,k})$. The equality holds if and only if $G \cong L_{n,k}$.*

In this paper, we shall we characterize the extremal unicyclic graphs among n–vertex unicyclic graphs with given girth having the minimal and the second minimal average eccentricity. In addition, we investigate the minimal and the second minimal average eccentricity in $\mathcal{U}_n$ and characterize the extremal graphs.

2 The minimal average eccentricity of unicyclic graphs

We first give a graph transformation which decrease the average eccentricity. Suppose that $p \ge 1$ is an integer. Let u be a vertex of a connected graph H with at least two vertices. Let G be the graph obtained by identifying u and a pendent vertex of a star S_{p+2}. Let v be the center of S_{p+2}. Assume that $vv_1, vv_2, \dots, vv_p$ are pendant edges incident with v in G. We form a graph $G' = \sigma(G, v)$ by removing edges $vv_1, vv_2, \dots, vv_p$ and adding new edges $uv_1, uv_2, \dots, uv_p$. We say that G' is *a σ-transform of G* (see Figure 1).

Lemma 2.1 *Let $G' = \sigma(G, v)$ be a σ-transform of a graph G. Then $\xi(G) > \xi(G')$.*

Proof. Note that $\varepsilon_{G'}(w) \le \varepsilon_G(w)$ for any $w \in V(H)$ and $\varepsilon_G(v) = \varepsilon_{G'}(v)$. So we have $\sum_{w \in V(H)} (\varepsilon_{G'}(w) - \varepsilon_G(w)) \le \sum_{w \in V(H)} (\varepsilon_G(w) - \varepsilon_G(w)) = 0$; $\sum_{w \notin V(H)} (\varepsilon_{G'}(w) - \varepsilon_G(w)) = \varepsilon_{G'}(v) - \varepsilon_G(v) + \sum_{v_i} (\varepsilon_{G'}(v_i) - \varepsilon_G(v_i)) = \sum_{v_i} ((\varepsilon_G(v_i) - 1) - \varepsilon_G(v_i)) = -p$. It follows that $\xi(G') -$

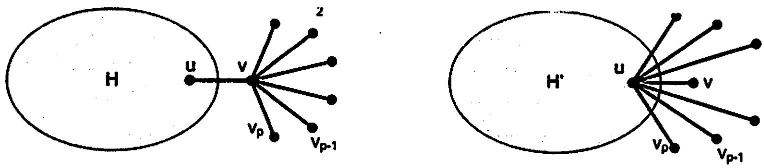

Figure 1: *σ-transform applied to G at vertex v.*

$\xi(G) = \frac{1}{n}\left[\sum_{w\in V(H)}(\varepsilon_{G'}(w)-\varepsilon_G(w))+\sum_{w\notin V(H)}(\varepsilon_{G'}(w)-\varepsilon_G(w))\right] \le -\frac{p}{n} < 0$. This implies the result. ■

We denote by $H_{n,k}$ the unicyclic graph obtained from C_k by adding $n-k$ pendant vertices to a vertex of C_k. Let $H(n,k;n_1,n_2,\ldots,n_k)$ be a unicyclic graph on n vertices obtained from cycle $C_k = v_1v_2\cdots v_k$ with n_i pendant vertices attached at $v_i (i = 1,2,\ldots,k)$. Clearly, $n-k = \sum_{i=1}^{k} n_i$.

Theorem 2.2 *Let $G \in \mathcal{U}_n(k)$ be a unicyclic graph of order $n > 5$. Then* $\xi(G) \ge \begin{cases} (\frac{1}{2}-\frac{1}{n})k+1+\frac{1}{n} & \text{if } k \text{ is even,} \\ (\frac{1}{2}-\frac{1}{n})k+\frac{1}{2}+\frac{2}{n} & \text{if } k \text{ is odd.} \end{cases}$ *with equality if and only if $G \cong H_{n,k}$.*

Proof. Let $G \in \mathcal{U}_n(k)$ be a unicyclic graph with the minimal average eccentricity. By Lemma 2.1, G must be of the form $H(n,k;n_1,n_2,\ldots,n_k)$. So we have $\xi(G) = \frac{1}{n}\left[\sum_{v_i\in V(C_k)}\varepsilon(v_i) + \sum_{v\in V(G)\setminus V(C_k)}\varepsilon(v)\right]$

$= \frac{1}{n}\left[\sum_{v_i\in V(C_k)}\varepsilon(v_i) + \sum_{v_i\in V(C_k)} n_i(\varepsilon(v_i)+1)\right]$

$= \frac{1}{n}\left[\sum_{v_i\in V(C_k)}(n_i+1)\varepsilon(v_i) + n_i\right] = \frac{1}{n}\left[\sum_{i=1}^{k}(n_i+1)\varepsilon(v_i) + n - k\right]$.

Note that the eccentricity of the vertex $v_i \in V(C_k)$ is exactly $\lfloor\frac{k}{2}\rfloor$ or $\lfloor\frac{k}{2}\rfloor+1$. If k is even, there exists at least one vertex $v \in V(C_k)$ with eccentricity $\frac{k}{2}+1$; if k is odd, there are at least two vertices with eccentricity $\frac{k+1}{2}$. Hence we have $\sum_{i=1}^{k}(n_i+1)\varepsilon(v_i) \ge$

$$\begin{cases} \frac{k}{2}+1+(k-2)\frac{k}{2}+(n-k+1)\frac{k}{2} & \text{if } k \text{ is even,} \\ 2\frac{k+1}{2}+(k-3)\frac{k-1}{2}+(n-k+1)\frac{k-1}{2} & \text{if } k \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{nk}{2}+1 & \text{if } k \text{ is even,} \\ \frac{n(k-1)}{2}+2 & \text{if } k \text{ is odd,} \end{cases} \quad \text{with equality if and only if } G \cong H_{n,k}.$$

■

Let $f(k) = (\frac{1}{2} - \frac{1}{n})k + 1 - \frac{1}{n}$, $(4 \le k \le n-2)$ and $g(k) = (\frac{1}{2} - \frac{1}{n})k + \frac{1}{2} + \frac{2}{n}$, $(3 \le k \le n-2)$ be two functions with respect to k. It can be checked that $f(k)$ and $g(k)$ are increasing for k. Therefore, $f(k) > f(4)$, $g(k) > g(3)$ and $f(4) > g(3)$, this leads to the following result.

Theorem 2.3 *Let G be a unicyclic graph of order $n > 5$. Then $\xi(G) \ge 2 - \frac{1}{n}$, with equality if and only if $G \cong H_{n,3}$.*

3 On the second minimal eccentricity of unicyclic graphs

Let $H'_{n,k}$ $(3 \le k \le n-2)$ be a graph obtained from cycle $C_k = v_1v_2\cdots v_k$ by attaching $n-k-1$ pendent vertices and a pendent vertex at v_1 and v_2, respectively. In the following, we shall characterize the unicyclic graphs with the second minimal average eccentricity.

Lemma 3.1 *Let $t > 1$ be an integer. Let $H_k(t)$ be a unicyclic graph obtained from $H_{n-t,k}$ and a star S_{t+1} by identifying a pendent vertex u of $H_{n-t,k}$ with the center of S_{t+1}. Then $\xi(H_k(t)) > \xi(H_k(1))$.*

Proof. Let w be the pendent vertex adjacent to u in $H_k(1)$. Let $w, u_1, u_2, \cdots, u_{t-1}$ be the pendent vertices adjacent to u in $H_k(t)$. Clearly, $\varepsilon_{H_k(t)}(v_i) = \varepsilon_{H_k(1)}(v_i)$ for any $v_i \in V(C_k)$ and $\varepsilon_{H_k(t)}(u_i) = \varepsilon_{H_k(1)}(u_i) + 1 (i = 1, 2, \cdots, t-1)$. So we have $\xi(H_k(t)) - \xi(H_k(1)) = \frac{1}{n}\left[\sum_{i=1}^{t-1}\left(\varepsilon_{H_k(t)}(u_i) - \varepsilon_{H_k(1)}(u_i)\right)\right] = \frac{t-1}{n} > 0$. ■

Theorem 3.2 *Among all n-vertices $(n > 5)$ unicyclic graphs with girth k, $H'_{n,k}$ has the second minimal average eccentricity and*

$$\xi(H'_{n,k}) = \begin{cases} (\frac{1}{2} - \frac{1}{n})k + 1 + \frac{2}{n} & \text{if } k \text{ is even,} \\ (\frac{1}{2} - \frac{1}{n})k + \frac{1}{2} + \frac{3}{n} & \text{if } k \text{ is odd,} \\ 3 - \frac{3}{n} & \text{if } k = 3. \end{cases}$$

Proof. Assume that G has the second minimal average eccentricity in $\mathcal{U}_n(k)$. By Lemma 3.1 and Lemma 2.1, G must be of the form $H(n, k; n_1, n_2, \ldots, n_k)$ or $H_k(1)$. In the following we divide into two steps to get the second average eccentricity in $\mathcal{U}_n(k)$.

Step 1: $H'_{n,k}$ has the second minimal average eccentricity among unicyclic graphs of the form $H(n,k;n_1, n_2, \ldots, n_k)$.

Let G be a unicyclic graph of the form $H(n,k;n_1,n_2,\ldots,n_k)$. From the proof of Theorem 2.2, we have $\xi(G) = \frac{1}{n}\sum_{v_i\in V(C_k)}\varepsilon(v_i) + \frac{1}{n}\sum_{v\in V(G)\setminus V(C_k)}\varepsilon(v) = \frac{1}{n}\Big[\sum_{i=1}^{k}(n_i+1)\varepsilon(v_i) + n - k\Big]$.

Note that the eccentricity of the vertex $v_i \in C_k$ is exactly $\frac{k}{2}$ or $\frac{k}{2}+1$. If k is even, there exist at least two vertices $v \in C_k$ with eccentricity $\frac{k}{2}+1$. Therefore, $\sum_{i=1}^{k}(n_i+1)\varepsilon(v_i) \geq (\frac{k}{2}+1)+(\frac{k}{2}+1)+(n-k+2)\frac{k}{2}+(k-4)\frac{k}{2} = \frac{nk}{2}+2$, with equalities if and only if there exist some n_s and n_t such that $n_s \neq 0, n_t \neq 0$ and $n_s + n_t = n-k$ while $n_j = 0$ for $j \neq s,t$. If k is odd, there are at least three vertices in C_k with eccentricity $\frac{k+1}{2}$. Therefore, $\sum_{i=1}^{k}(n_i+1)\varepsilon(v_i) \geq \frac{3(k+1)}{2} + (n-k+2)\frac{k-1}{2}+(k-5)\frac{k-1}{2} = \frac{n(k-1)}{2}+3$, with equalities if and only if there exist some n_s and n_t such that $n_s \neq 0, n_t \neq 0, n_s+n_t = n-k$, v_s is adjacent to v_t, and $n_j = 0$ for $j \neq s,t$.

Therefore, among unicyclic graphs of the form $H(n,k;n_1,n_2,\cdots,n_k)$, $H'_{n,k}$ has the second average eccentricity.

Step 2: Verifying $\xi(H'_{n,k}) < \xi(H_k(1))$.

By direct calculation, we have

$$\xi(H_k(1)) = \begin{cases} (\frac{1}{2}-\frac{1}{n})k+1+\frac{5}{n} & \text{if } k \text{ is even,}\\ (\frac{1}{2}-\frac{1}{n})k+\frac{1}{2}+\frac{7}{n} & \text{if } k \text{ is odd and } k \geq 5,\\ 3-\frac{2}{n} & \text{if } k=3.\end{cases}$$

$$\xi(H'_{n,k}) = \begin{cases} (\frac{1}{2}-\frac{1}{n})k+1+\frac{2}{n} & \text{if } k \text{ is even,}\\ (\frac{1}{2}-\frac{1}{n})k+\frac{1}{2}+\frac{3}{n} & \text{if } k \text{ is odd and } k \geq 5,\\ 3-\frac{3}{n} & \text{if } k=3.\end{cases}$$

Therefore, we have $\xi(H_k(1)) > \xi(H'_{n,k})$. ■

Theorem 3.3 *Let $G \in \mathcal{U}_n$ ($\neq H_{n,3}$) be a unicyclic graph on $n > 5$ vertices. Then $\xi(G) \geq 3-\frac{3}{n}$, with equality if and only if $G \cong H_{n,4}$, or $H_{n,5}$, or $H'_{n,3}$.*

Proof. From the proof of Theorem 2.2 and Theorem 3.2, we only need to compare average eccentricities of $H_{n,4}$, $H_{n,5}$, $H'_{n,3}$, $H'_{n,4}$ and $H_3(1)$.

By Theorem 2.2 and Theorem 3.2, we have $\xi(H_3(1)) = 3-\frac{2}{n}, \xi(H_{n,4}) = 3-\frac{3}{n}, \xi(H_{n,5}) = 3-\frac{3}{n}$, $\xi(H'_{n,3}) = 3-\frac{3}{n}, \xi(H'_{n,4}) = \xi(H'_{n,5}) = 3-\frac{2}{n}$. This implies the result. ■

4 Conclusion

In this paper we present $H_{n,k}$ has the minimal average eccentricity among all $n-$vertex unicyclic graphs with girth k and $H_{n,3}$ has the minimal average eccentricity among all unicyclic graphs of order n. Moreover, we derive that among all n-vertices ($n > 5$) unicyclic graphs with girth k, $H'_{n,k}$ has the second minimal average eccentricity.

There are still many interesting open questions for the further study. It would be interesting to determining extremal regular (cubic) graphs with respect to the average eccentricity. It would be interesting to give the ordering of unicyclic graphs with respect to the average eccentricity. In addition, it would be interesting to investigate the average eccentricity of bicyclic graphs and characterize the extremal graphs with the minimal and the maximal average eccentricity.

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