Integer-Magic Spectra of Sun Graphs

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Abstract

Let $\mathbb{N}$ be the set of all positive integers, and Z_n=$\{0, 1, 2, \cdots, n-1\}$. For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be Z_h-magic if there exists a labeling $f : E \to Z_h \setminus \{0\}$ such that the induced vertex labeling $f^+ : V \to Z_h$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, is a constant map. The integer-magic spectrum of G is the set $IM(G) = \{h \in \mathbb{N} | G \text{ is } Z_h\text{-magic }\}$. A *sun graph* is obained from attaching a path to each pair of adjacent vertices in an n-cycle. In this paper we showed that the integer-magic spectra of sun graphs are completely determined.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. Let A be a nontrivial abelian group with identity 0. A mapping $f : E \to A \setminus \{0\}$ is called an edge labeling of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$ for each $v \in V$. If there exists an edge labeling f whose induced map f^+ on V is a constant map, then f is an A-magic labeling and G is an A-magic graph. The corresponding constant is called an A-magic value. If $A = Z_h$, then we call G Z_h-magic or h-magic in short. The integer-magic spectrum of G is the set $IM(G) = \{h \in \mathbb{N} | G \text{ is } h\text{-magic}\}$, where $\mathbb{N}$ is the set of all positive integers. By convention, Z-magic graphs are considered to be Z_1-magic.

The original concept of an A-magic graph is due to J. Sedláček [11,12], who defined it to be a graph with a real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the

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labels of the edges incident to a particular vertex is the same for all vertices. A characterization of magic graphs is given by Jenzy and Trenkler [4]. A graph G is called non-magic if G is not A-magic for any abelian group A. Some classes of non-magic graphs are presented in [1]. Z-magic graphs were considered by Stanley [14,15].Recently, there has been considerable amount of research articles on graph labeling. Interested readers can refer to [2,16]. Research related to $IM(G)$ can be found in [5~10]. The aim of this paper is to find the integer-magic spectra of sun graphs.

2 Definitions and Preliminaries

Let $m \geq 3$ and $t \geq 2$. An m-cycle C_m with vertices $v_1, v_2, \cdots, v_m$ and edges v_1v_2, v_2v_3, $\cdots$, v_mv_1 is denoted by $(v_1, v_2, \cdots, v_m)$. A t-path P_t with vertices $u_1, u_2, \cdots, u_t$ and edges $u_1u_2, u_2u_3, \cdots, u_{t-1}u_t$ is denoted by $\langle u_1, u_2, \cdots, u_t\rangle$. A graph is called a *sun graph* of *index* n if it is formed by attaching a path of length at least 2 to the end vertices of each edge of an n-cycle. Let $C_n = (v_1, v_2, \cdots, v_n)$ and for each $i, 1 \leq i \leq n$, let $H_i = \langle u_{i,1}, u_{i,2}, \cdots, u_{i,t_i+1}\rangle$ be a path of length $t_i \geq 2$. For each $i = 1, 2, \cdots, n$, attach H_i to the cycle C_n by identifying v_i and v_{i+1} with $u_{i,1}$ and u_{i,t_i+1} respectively, we obtain a sun graph with index n and parameters $(t_1, t_2, \cdots, t_n)$, denoted by $C_n(t_1, t_2, \cdots, t_n)$. It is easy to see that a sun graph $C_n(t_1, t_2, \cdots, t_n)$ can be partitioned into n edge-disjoint cycles $H_i \cup \{v_iv_{i+1}\}$ of length $t_i + 1$.

In this paper, if f is an A-magic labelling of $C_n(t_1, t_2, \cdots, t_n)$ with A-magic value m, for some abelian group A, we define $f(v_iv_{i+1}) = x_i$ if $v_iv_{i+1} \in E(C_n)$ and $f(u_{i,j}u_{i,j+1}) = y_{i,j}$ if $u_{i,j}u_{i,j+1} \in E(H_i)$, for each $j = 1, 2, \cdots, t_{i-1}$ and $i = 1, 2, \cdots, n$. Then $x_i, y_{i,j} \in A \setminus \{0\}$ and the following conditions hold:

$$\begin{cases} y_{i,j} + y_{i,j+1} = m \\ x_i + x_{i+1} + y_{i,t_i} + y_{i+1,1} = m \end{cases} \qquad (*)$$

In [13], Shiu and Low obtained the following result.

Theorem 2.1. *([13]) Let $G = C_n(t_1, t_2, \cdots, t_n)$. If n is even and $t_i \geq 1$, then $IM(G) = \mathbb{N}$.*

Since $C_n(t_1, t_2, \cdots, t_n)$ is an even graph, there is an eulerian circuit e_1, $e_2, \cdots, e_p$, where $p = |E(C_n(t_1, t_2, \cdots, t_n))|$, passing through all edges of $C_n(t_1, t_2, \cdots, t_n)$. Let g be an Z_h-magic labeling of $C_n(t_1, t_2, \cdots, t_n)$. If p is even, then define $g(e_i) = a$ and $g(e_{i+1}) = -a$ where $a \in Z_h \setminus \{0\}$. Then we have a labeling of $C_n(t_1, t_2, \cdots, t_n)$ which is h-magic with magic value 0. Thus we have the following result.

Lemma 2.2. *Let $G = C_n(t_1, t_2, \cdots, t_n)$. If n is odd and $\Sigma_{i=1}^{n} t_i$ is odd, then $IM(G) = \mathbb{N}$.*

Next we consider the case when n is odd and $\Sigma_{i=1}^{n} t_i$ is even. From [13], we have the following:

Lemma 2.3. *([13]) Let $G = C_n(t_1, t_2, \cdots, t_n)$. If n is odd and $t_i \in 2\mathbb{N}$, then $IM(G) = \mathbb{N} \setminus \{3\}$.*

A vertex of degree k is called a *k-vertex*. Suppose u and v are two adjacent 2-vertices of G. Let w be another vertex adjacent to v. Let $G|_{u,v}$ be the graph obtained from G by deleting two edges uv and vw and identifying u and w. In [13], Shiu and Low used this shrinking technique to prove the following result.

Theorem 2.4. *([13]) Let A be an abelian group and G a graph, where u and v are two adjacent 2-vertices of G. Then G is A-magic with A-magic value m if and only if $G|_{u,v}$ is A-magic with A-magic value m.*

From the above result, any sun graph $G = C_n(t_1, t_2, \cdots, t_n)$ can be shrunk to a sun graph $G' = C'_n(t'_1, t'_2, \cdots, t'_n)$ where $t'_i \in \{2, 3\}$ for each i such that the integer-magic spectrum of G is equal to the integer-magic spectrum of G'. Thus in what follows we will consider the sun graph $C_n(t_1, t_2, \cdots, t_n)$ where $t_i = 2$ or 3, for each $i = 1, 2, \cdots, n$. Let $\psi_2(G) = \{i | t_i = 2\}$ and $\psi_3(G) = \{i | t_i = 3\}$. The final case that we need to consider is that $|\psi_2(G)|$ is odd and $|\psi_3(G)|$ is even with $n \geq 2$.

Lemma 2.5. *([13]) Let $G = C_n(t_1, t_2, \cdots, t_n)$. If n is odd with $n \geq 5$ and $\Sigma_{i=1}^{n} t_i$ is even, then G is not Z_3-magic for $|\psi_2(G)| = 3$, 5, or 7.*

After this result, Shiu and Low[13] gave a conjecture:

Conjecture A. *Let $G = C_n(t_1, t_2, \cdots, t_n)$. If n is odd with $n \geq 5$ and $\Sigma_{i=1}^{n} t_i$ is even, then G is not Z_3-magic for $|\psi_2| \geq 9$.*

3 Main results

In this section, we try to show that Conjecture A is true for the sun graph of index n, $C_n(t_1, t_2, \cdots, t_n)$, with $t_i \in \{2, 3\}$ for each $i = 1, 2, \cdots, n$.

Lemma 3.1. *Let n be odd and $G = C_n(t_1, t_2, \cdots, t_n)$ with $\Sigma_{i=1}^{n} t_i$ even. If $3 \in IM(G)$, then G is Z_3-magic with magic value 0.*

Proof. Suppose G is Z_3-magic with magic value 1 or 2. If G is Z_3-magic with magic value 1, then each $y_{i,j}$ must be 2 for $1 \leq j \leq t_i$ and $x_i + x_{i+1} \equiv 0 \pmod 3$ for $1 \leq i \leq n$. Similarly, if G is Z_3-magic with magic value 2, then each $y_{i,j}$ must be 1 for $1 \leq j \leq t_i$ and $x_i + x_{i+1} \equiv 0 \pmod 3$ for $1 \leq i \leq n$. Without loss of generality, we may assume that $x_1 = 1$, then $x_i = 1$ for i odd and $x_i = 2$ for i even. Since n is odd, it results in $x_n + x_1 \equiv 2 \pmod 3$ contradictng $x_i + x_{i+1} \equiv 0 \pmod 3$. □

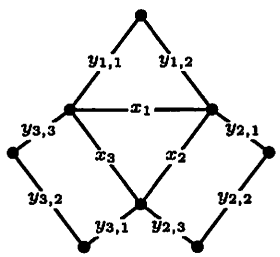

Figure 1. Labeled $C_3(2,3,3)$

Lemma 3.2. $IM(C_3(2,3,3)) = 2\mathbb{N}$.

Proof. Let $C_3(2,3,3)$ be Z_h-magic for some positive integer h with the magic value m. We label $C_3(2,3,3)$ as in Figure 1. Since $y_{2,1} + y_{2,2} = y_{2,2} + y_{2,3} = m$, we have $y_{2,1} = y_{2,3}$. Similarly, $y_{3,1} = y_{3,3}$. Since $x_1 + y_{1,2} + x_2 + y_{2,1} = m = x_2 + y_{2,3} + x_3 + y_{3,1}$, we get $x_1 + y_{1,2} = x_3 + y_{3,3}$. Since $x_3 + y_{3,3} + x_1 + y_{1,1} = m$ and $y_{1,1} + y_{1,2} = m$, it implies $2x_1 \equiv 0 \pmod h$. Thus h should be a multiple of 2, i.e, $IM(G) \subseteq 2\mathbb{N}$. If we label each edge with $h/2$, we will get a Z_h-magic labelling of $C_3(2,3,3)$. Therefore we have $IM(G) = 2\mathbb{N}$. □

Let $G = C_n(t_1, t_2, \cdots, t_n)$ be a sun graph of index n. Then G can be partitioned into n edge disjoint cycles and each cycle contains exactly one edge of C_n. The sun graph $G' = C_{n-(j-i)}(t_1, t_2, \cdots, t_{i-1}, t_j, \cdots, t_n)$ of index $n - (j - i)$ is derived from G by deleting the cycles which contain one of the edges in $\{v_i v_{i+1},\ v_{i+1} v_{i+2},\ \cdots,\ v_{j-1} v_j\}$ and identifying two vertices v_i and v_j. Thus if n-cycle in G is $C_n = (v_1, v_2, \cdots, v_n)$, then the $n-(j-i)$-cycle in G' is $(v_1, v_2, \cdots, v_i, v_{j+1}, v_{j+2}, \cdots, v_n)$. In short, we will just say that $G = C_n(t_1, t_2, \cdots, t_n)$ with $C_n = (v_1, v_2, \cdots, v_n)$ is shrunk to $G' = C_{n-(j-i)}(t_1, t_2, \cdots, t_{i-1}, t_j, \cdots, t_n)$ with the $n - (j - i)$-cycle in G' being $(v_1, v_2, \cdots, v_i, v_{j+1}, v_{j+2}, \cdots, v_n)$.

Lemma 3.3. *Let n be odd with $n \geq 5$ and $\Sigma_{i=1}^{n} t_i$ is even. If there exists k, $1 \leq k \leq n$ such that $t_k = t_{k+1} = 3$, then $G = C_n(t_1, t_2, \cdots, t_n)$ is Z_h-magic if and only if $G' = C_{n-2}(t_1, t_2, \cdots, t_{k-1}, t_{k+2}, t_{k+3}, \cdots, t_n)$ is Z_h-magic.*

Proof. Let f be the magic labelling of G as in (*). If $G = C_n(t_1, t_2, \cdots, t_n)$ is Z_h-magic with magic value m and there exists k, $1 \le k \le n$ such that $t_k = t_{k+1} = 3$, then we have $y_{k,1} = y_{k,3}$ and $y_{k+1,1} = y_{k+1,3}$. From the following equalities: $x_{k-1} + y_{k-1,t_{k-1}} + x_k + y_{k,1} = m$, $x_k + y_{k,3} + x_{k+1} + y_{k+1,1} = m$, and $x_{k+1} + y_{k+1,3} + x_{k+2} + y_{k+2,1} = m$, we obtain that $x_{k-1} + y_{k-1,t_{k-1}} + x_{k+2} + y_{k+2,1} = m$. Thus we delete the two cycles which contain one of the edges in $\{v_k v_{k+1}, v_{k+1} v_{k+2}\}$ and identify the two vertices v_k and v_{k+2} in G. We obtain $G' = C_{n-2}(t_1, t_2, \cdots, t_{k-1}, t_{k+2}, t_{k+3}, \cdots, t_n)$ with $(n-2)$-cycle $C_{n-2} = (v_1, v_2, \cdots, v_k, v_{k+3}, v_{k+4}, \cdots, v_n)$. Define a labelling f' of G' as $f'(e) = f(e)$ for e an edge of G'. Then f' is an h-magic labelling of G' with magic value m.
Let $G' = C_{n-2}(t_1, t_2, \cdots, t_{n-2})$ be a sun graph with $(n-2)$-cycle $C_{n-2} = (v_1, v_2, \cdots, v_{n-2})$ and the attaching path to the edge $v_i v_{i+1}$ be H_i with length t_i. If G' is Z_h-magic with magic value m, let f' be the magic labelling of G' as in (*), then $x_i + y_{i,t_i'} + x_{i+1} + y_{i+1,1} = m$, for each $i = 1, 2, \cdots, n-2$. Consider the graph $G = C_n(t_1, t_2, \cdots, t_i, 3, 3, t_{i+1}, \cdots, t_{n-2})$ with n-cycle $C_n = (v_1, v_2, \cdots, v_i, v_{i+1}, w, v'_{i+1}, v_{i+2}, \cdots, v_{n-2})$ where the path H_{i+1} is attached to the edge $v'_{i+1} v_{i+2}$ and the paths $\langle v_{i+1}, u'_{i+1,2}, u'_{i+1,3}, w \rangle$ and $\langle w, u'_{i+2,2}, u'_{i+2,3}, v'_{i+1} \rangle$ are attached to the edges $v_{i+1} w$ and $w v'_{i+1}$ respectively. Define a labelling f of G as follows: $f(e) = f'(e)$ for e an edge of G',

$$\begin{cases} f(v_{i+1}w) = f(v'_{i+1}v_{i+2}) = x_{i+1}, f(wv'_{i+1}) = x_i \\ f(v'_{i+1}u_{i+1,2}) = y_{i+1,1}, f(v_{i+1}u'_{i+1,2}) = f(u'_{i+1,3}w) = y_{i+1,1} \\ f(u'_{i+1,2}u'_{i+1,3}) = m - y_{i+1,1}, f(wu'_{i+2,2}) = f(u'_{i+2,3}v'_{i+1}) = y_{i,t_i} \\ f(u'_{i+2,2}u'_{i+2,3}) = m - y_{i,t_i}. \end{cases}$$

Then f is an h-magic labeling of G. Therefore G is Z_h-magic. □

From *Lemma* 3.3, we can reduce $G = C_n(t_1, t_2, \cdots, t_n)$ to a new $G' = C_m(t'_1, t'_2, \cdots, t'_m)$ such that m is odd and $t'_i + t'_{i+1} \in \{4, 5\}$ for each i and $|\psi_2(G')| > |\psi_3(G')|$.

From *Lemma* 3.2 and *Lemma* 3.3 we have the following result.

Theorem 3.4. *If k is odd, then $IM(C_k(2, 3, 3, \cdots, 3)) = 2\mathbb{N}$.*

Next, we consider the case that there are at least three paths of length 2 attached to the edges of the cycle.

Lemma 3.5. *Let $G = C_n(t_1, t_2, \cdots, t_n)$ where $t_i + t_{i+1} \in \{4, 5\}$ for each i and $|\psi_2(G)| > |\psi_3(G)|$. If n is odd and $|\psi_2(G)|$ is odd with $|\psi_2(G)| \ge 3$, then $3 \notin IM(G)$.*

Proof. From *Lemma* 2.5, $3 \notin IM(C_n(t_1, t_2, \cdots, t_n))$ for n odd, $t_i \in \{2, 3\}$ for each i, and $|\psi_2|$ is 3, 5, or 7. Suppose that there is a Z_3-magic labelling f for $G = C_n(t_1, t_2, \cdots, t_n)$ and $|\psi_2(G)| = k$, $k > 7$. Since n is odd and $|\psi_2(G)|$ is odd, $\Sigma_{i=1}^{n} t_i$ is even. By *Lemma* 3.1, the magic value must be zero. Thus $x_i, y_{i,j} \in \{-1, 1\}$, $x_i + y_{i,t_i} + x_{i+1} + y_{i+1,1} = 0$ and $y_{i,j} + y_{i,j+1} = 0$ for $1 \le i \le n$ and $1 \le j \le t_i - 1$. Consider the cardinality of $\psi_3(G)$. By *Lemma* 2.3 we know that $|\psi_3(G)| > 0$, that is $|\psi_3(G)| \ge 2$. From the definition of $G = C_n(t_1, t_2, \cdots, t_n)$, we know that $\psi_3(G)$ contains no consecutive integers. Then there exists some j such that $(t_j, t_{j+1}, t_{j+2}, t_{j+3}) = (2, 2, 3, 2)$ and the graph can be labelled as in Figure 2.

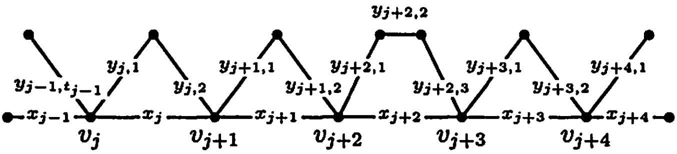

Figure 2

From this labeling of G, we can apply the reduction process to obtain new sun graphs as follows.

(i) if $x_j = a$ and $y_{j,1} = -a$ where $a \in \{-1, 1\}$, then we can get a subgraph of G labelled as in Figure 3, where $b, c, d \in \{-1, 1\}$. If we shrink the sun graph G of index n to a sun graph $G' = C_{n-2}(t_1, t_2, \cdots, t_{j-1}, t_{j+2}, \cdots, t_n)$ of index $n-2$ with $(n-2)$-cycle $C_{n-2} = (v_1, v_2, \cdots, v_j, v_{j+3}, v_{j+4}, \cdots, v_n)$, then $|\psi_2(G')| = |\psi_2(G)| -2$ and G' is Z_3-magic.

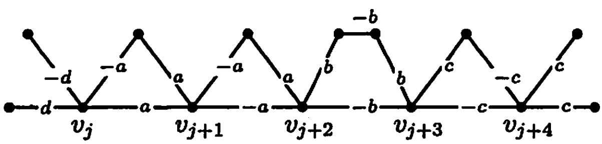

Figure 3

(ii) if $x_j = a$ and $y_{j,1} = a$ where $a \in \{-1, 1\}$, then we obtain a subgraph of G labelled as in Figure 4.

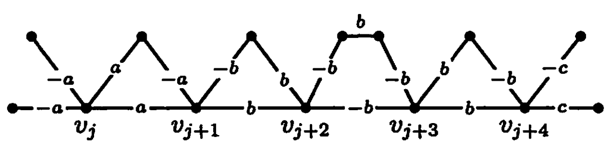

Figure 4

If $t_{j+4} = 3$, then we can get a subgraph of G labelled as in Figure 5, where $b, c, d \in \{-1, 1\}$. If we shrink the sun graph G of index n

to a sun graph $G' = C_{n-4}(t_1, t_2, \cdots, t_j, t_{j+5}, \cdots, t_n)$ of index $n-4$ with $(n-4)$-cycle $C_{n-4} = (v_1, v_2, \cdots, v_{j+1}, v_{j+6}, v_{j+7}, \cdots, v_n)$, then $|\psi_2(G')| = |\psi_2(G)| -2$ and G' is Z_3-magic.

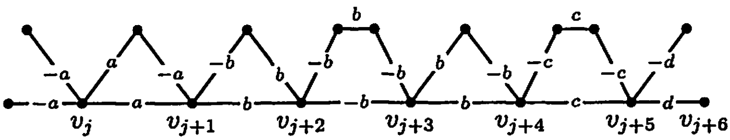

Figure 5

If $t_{j+4} = 2$ and $t_{j+5} = 2$, then we can get a subgraph of G labelled as in Figure 6, where $b, c, d \in \{-1, 1\}$. If we shrink the sun graph G of index n to a sun graph $G' = C_{n-2}(t_1, t_2, \cdots, t_{j+3}, t_{j+6}, t_{j+7}, \cdots, t_n)$ of index $n-2$ with $(n-2)$-cycle $C_{n-2} = (v_1, v_2, \cdots, v_{j+4}, v_{j+7}, v_{j+8}, \cdots, v_n)$, then $|\psi_2(G')| = |\psi_2(G)| - 2$ and G' is Z_3-magic.

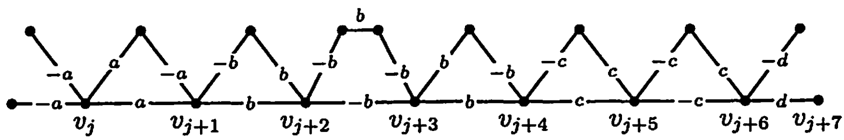

Figure 6

If $t_{j+4} = 2$ and $t_{j+5} = 3$, then $t_{j+6} = 2$. we can get a subgraph of G labelled as in Figure 7, where $b, c, d \in \{-1, 1\}$. If we shrink the sun graph G of index n to a sun graph $G' = C_{n-6}(t_1, t_2, \cdots, t_j, t_{j+7}, \cdots, t_n)$ of index $n-6$ with $(n-6)$-cycle $C_{n-6} = (v_1, v_2, \cdots, v_{j+1}, v_{j+8}, v_{j+9}, \cdots, v_n)$, then $|\psi_2(G')| = |\psi_2(G)| - 4$ and G' is Z_3-magic.

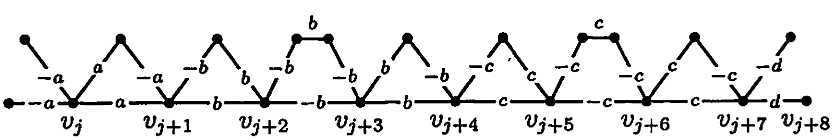

Figure 7

If $|\psi_2(G')| > 7$, we can keep doing the above reduction until $|\psi_2(G)| \leq 7$, then we will get a contradiction to the result in *Lemma* 2.5. Therefore, The proof is complete. □

The above Lemma showed that Conjecture A is true. Next, we will show that $C_n(t_1, t_2, \cdots, t_n)$ can be Z_h-magic labelled for $h > 3$.

Lemma 3.6. *Let n be odd and $G = C_n(t_1, t_2, \cdots, t_n)$ and $|\psi_2(G)|$ is odd with $|\psi_2(G)| \geq 3$, then $IM(G) \supseteq \mathbb{N} \setminus \{3\}$.*

Proof. Let $|\psi_2(G)| = \kappa$. If $i_1, i_2, \cdots, i_\kappa \in \psi_2(G)$ and $i_1 < i_2 < \cdots < i_\kappa$. Since $t_i \in \{2, 3\}$ and $t_i + t_{i+1} = 4$ or 5 for each i, the difference $i_{r+1} - i_r = 1$ or 2. Let $H'_k = H_k \cup \{v_k v_{k+1}\}$, for each k. Then H'_k is a 3-cycle or 4-cycle. Since κ is odd, we can partition the set $\psi_2(G) \setminus \{i_1, i_2, i_3\}$ into two element subsets $\{i_t, i_{t+1}\}$, $t = 4, 6, 8, \cdots, \kappa - 1$. Corresponding to this partition, we define graphs $I_1 = H'_{i_1} \cup H'_{i_1+1} \cup \cdots \cup H'_{i_3}$, $I_2 = H'_{i_4} \cup H'_{i_4+1} \cup \cdots \cup H'_{i_5}$, $I_3 = H'_{i_6} \cup H'_{i_6+1} \cup \cdots \cup H'_{i_7}, \cdots, I_{(\kappa-1)/2} = H'_{i_{\kappa-1}} \cup H'_{i_{\kappa-1}+1} \cup \cdots \cup H'_{i_\kappa}$. Then G is partitioned into edge disjoint graphs $I_1, I_2, \cdots, I_{(\kappa-1)/2}$ and some 4-cycles. The graph I_1 is one of the graphs in Figure 8 and I_k, $k > 1$, is one of the graphs in Figure 9. We can give a magic labelling with magic value 0 for the graphs in Figure 8 and Figure 9, and the remaining 4-cycles labelled by $x_i = 1$, $y_{i,1} = -1$, $y_{i,2} = 1$, $y_{i,3} = -1$. Hence G has a Z_h-magic labelling with magic value 0 for $h > 3$. Since G is an even graph, G is Z_2-magic. Therefore, $IM(G) \supseteq \mathbb{N} \setminus \{3\}$. □

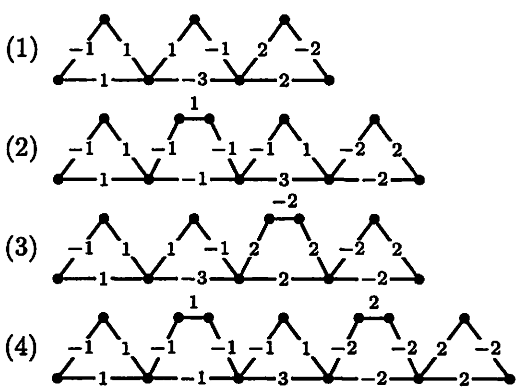

Figure 8

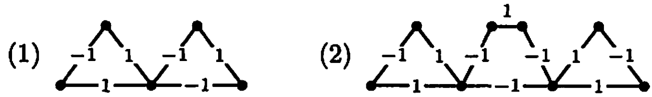

Figure 9

Combining *Lemma* 3.5 and *Lemma* 3.6, we have the following result.

Proposition 3.7. Let n be odd and $G = C_n(t_1, t_2, \cdots, t_n)$. Then $IM(G) = \mathbb{N} \setminus \{3\}$ provided $|\psi_2(G)| \geq 3$ is odd.

Combining the above results, we have shown the following:

Theorem 3.8. *Let $G = C_n(t_1, t_2, \cdots, t_n)$ be a sun graph of index n and $\psi = \{i | t_i \text{ is even}\}$. Then*

(a) $IM(G) = \mathbb{N}$ if n is even;

(b) $IM(G) = \mathbb{N}$ if n is odd and $\Sigma_{i=1}^{n} t_i$ is odd;

(c) $IM(G) = 2\mathbb{N}$ if n is odd, $|\psi| = 1$, and $\Sigma_{i=1}^{n} t_i$ is even;

(d) $IM(G) = \mathbb{N} \setminus \{3\}$ if n is odd, $|\psi| > 1$, and $\Sigma_{i=1}^{n} t_i$ is even.

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