Chromatic Uniqueness of K_4-Homeomorphic Graphs

S. Catada-Ghimire[1], H. Roslan[2]

[1,2]*School of Mathematical Sciences*
Universiti Sains Malaysia, 11800 Penang, Malaysia

[1]*E-mail: sabinacg@usm.my*
[2]*E-mail: hroslan@cs.usm.my*

ABSTRACT

Graphs which are derived from the same graph are called homeomorphic graphs or simply homeomorphs. A K_4-homeomorph denoted by $K_4(a,b,c,d,e,f)$ is obtained by subdividing the six paths of a complete graph with four vertices into a,b,c,d,e,f number of segments, respectively. In this paper, we shall study the chromaticity of $K_4(a,b,c,d,e,f)$ with exactly two non-adjacent paths of length two. We also give a sufficient and necessary condition for all the graphs in this family to be chromatically unique.

Keywords : Chromatic polynomial, Chromatic uniqueness, K_4-homeomorphic graphs.

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1 Introduction

All graphs considered here are finite, undirected and simple. For such a graph G, a polynomial in λ is the number of vertex colouring in not more than λ colours. This polynomial is called the chromatic polynomial of G denoted by $P(G,\lambda)$ or simply $P(G)$. The chromaticity of graphs is the term used referring to the question of chromatic equivalence and chromatic uniqueness of graphs. Two graphs G and H are chromatically equivalent or simply $\chi-$equivalent, denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$ (or simply $P(G) = P(H)$). A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such that $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

The chromaticity of $K_4(a,b,c,d,e,f)$ (see Figure 1) with at least two paths of length one has been studied by many authors (see [4,8,9,14,16]). The study of the chromatic uniqueness of $K_4(a,b,c,d,e,f)$ where there do not exist three distinct numbers in the set $\{a,b,c,d,e,f\}$ has been done by Whitehead and Zhao in [16]. Ren and Zhang [15] and Li [8] discussed the chromatic uniqueness of $K_4(a,b,c,d,e,f)$ where exactly four numbers among $\{a,b,c,d,e,f\}$ are the same. In [13], Ren fulfilled the study of the chromaticity of $K_4(a,b,c,d,e,f)$ where exactly three of $\{a,b,c,d,e,f\}$ are of the same length and greater than 2. Peng Yanling fulfilled the study of the chromaticity of $K_4(a,b,c,d,e,f)$ which have exactly two paths of

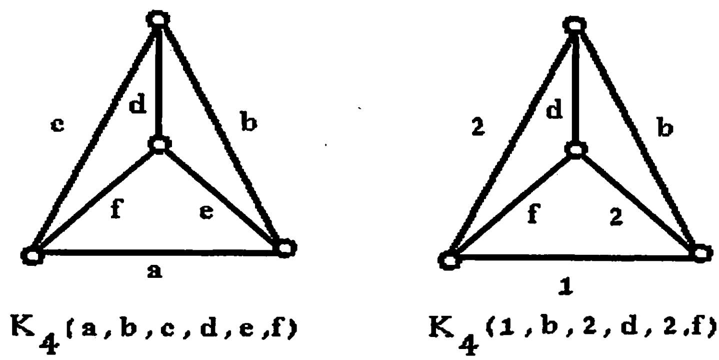

Figure 1: K_4 - Homeomorphs

length 1 (see [12]). In [5], Dong, Koh and Teo summarized some known results on the chromaticity of K_4-homeomorphic graphs and presented the following problems:
(1) Study the chromaticity of K_4-homeomorphs for exactly 2 paths of length $s \geq 2$;
(2) Study the chromaticity of K_4-homeomorphs for exactly one path of length 1.

Motivated by such problems, we shall investigate the chromaticity of K_4-homeomorphic graphs with exactly one path of length one and exactly two non-adjacent paths of length two, that is, $K_4(1, b, 2, d, 2, f)$ (see Figure 1). As a consequence of this study, we shall bring into completion the study of the chromaticity of K_4-homoemorphs with exactly two non-adjacent paths of length two.

2 Preliminary results

In this section, we give some known results used in the sequel.

Lemma 2.1 *Assume that G and H are χ−equivalent. Then the following statements are proven to be true.*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [7]);*

(2) *Let $g(G)$ and $g(H)$ denote the girths of G and H, respectively. Then $g(G) = g(H)$ and G and H have the same number of cycles with length equal to their girth (see [17]);*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [3]);*

(4) *Let* $G = K_4(a,b,c,d,e,f)$ *and* $H = K_4(a',b',c',d',e',f')$, *then*

(i) *min* $\{a,b,c,d,e,f\}$ = *min* $\{a',b',c',d',e',f'\}$ *and the number of times that this minimum occurs in the list* $\{a,b,c,d,e,f\}$ *is equal to the number of times that this minimum occurs in the list* $\{a',b',c',d',e',f'\}$ *(see[16]);*

(ii) *if* $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ *as multisets, then* $H \cong G$ *(see [8]).*

Lemma 2.2 *(Ren and Liu [14]) Let* $G = K_4(a,b,c,d,e,f)$ *(see Figure 1) when exactly three of* a,b,c,d,e,f *are the same. Then* G *is not chromatically unique if and only if* G *is isomorphic to* $K_4(r,r,r-2,1,2,r)$ *or* $K_4(r,r-2,r,2r-2,1,r)$ *or* $K_4(t,t,1,2t,t+2,t)$ *or* $K_4(t,t,1,2t,t-1,t)$ *or* $K_4(t,t+1,t,2t+1,1,t)$ *or* $K_4(1,t,1,t+1,3,1)$ *or* $K_4(1,1,t,2,t+2,1)$, *where* $r \geq 3$, $t \geq 2$.

Lemma 2.3 *(Ren and Zhang [15] and Li [8]) The graph* $K_4(a,b,c,d,e,f)$ *is* $\chi-$*unique if exactly four numbers among a,b,c,d,e,f are the same.*

Lemma 2.4 *(Whitehead, Jr. and Zhao [16])The graph* $K_4(a,b,c,d,e,f)$ *is* $\chi-$*unique if the positive integers* a,b,c,d,e,f *assume no more than two distinct values.*

Lemma 2.5 *(Li [8] and Whitehead and Zhao [16]) The chromatic polynomial of* $G = K_4(a,b,c,d,e,f)$ *is* $P(G,\lambda) = \frac{1}{\lambda^2}(-1)^m w[w^{m-1} + Q(G,w) - (w+1)(w+2)]$, *where* $w = m-1$, $m = \mid E(G) \mid$, $Q(G,w) = -(w^{a+f+c} + w^{a+b+e} + w^{b+c+d} + w^{d+e+f} + w^{a+d} + w^{b+f} + w^{c+e}) + (1+w)(w^a + w^b + w^c + w^d + w^e + w^f)$. $Q(G,w)$ *or simply* $Q(G)$ *is called the essential polynomial of* G.

Lemma 2.6 *(Li [8]) Two* K_4*-homeomorphs with the same order are* χ*-equivalent if they have the same essential polynomial.*

Lemma 2.7 *(Guo and Whitehead Jr. [6])* K_4-*homeomorph* $K_4(1,b,c,1,e,f)$ *is not* $\chi-$*unique if and only if it is* $K_4(1,t+2,t,1,2,2)$ *or* $K_4(1,s+1,s+3,1,2,s)$ *or* $K_4(1,s+2,t,1,2,s)$, *where* $s \geq 2$, $t \geq 1$, *and*

$$K_4(1,t+2,t,1,2,2) \sim K_4(3,1,1,2,t,t+1),$$
$$K_4(1,s+1,s+3,1,2,s) \sim K_4(s+1,1,1,s,3,s+2),$$
$$K_4(1,s+2,t,1,2,s) \sim K_4(s+1,1,1,t,3,s).$$

Lemma 2.8 *(Peng and Liu [12])* K_4-*homeomorph* $K_4(a,1,1,d,e,f)$ *where min*$\{a,d,e,f\} \geq 2$, *is not* $\chi-$*unique if and only if it is* $K_4(s,1,1,s+t+1,t,t+1)$, $K_4(s,1,1,t,t+2,s+t)$, $K_4(s+1,1,1,s+3,2,s)$, $K_4(s+2,1,1,s,2,s+2)$, $K_4(3,1,1,2,t,t+1)$, $K_4(s+1,1,1,s,3,s+2)$ *or* $K_4(s+1,1,1,t,3,s)$, *where* $s \geq 2$, $t \geq 2$, *and*

$$K_4(s,1,1,s+t+1,t,t+1) \sim K_4(s,1,1,t,t+2,s+t),$$
$$K_4(s+1,1,1,s+3,2,s) \sim K_4(s+2,1,1,s,2,s+2),$$
$$K_4(3,1,1,2,t,t+1) \sim K_4(1,t+2,t,1,2,2),$$
$$K_4(s+1,1,1,s,3,s+2) \sim K_4(1,s+1,s+3,1,2,s),$$
$$K_4(s+1,1,1,t,3,s) \sim K_4(1,s+2,t,1,2,s).$$

Lemma 2.9 *(Catada-Ghimire, Roslan and Peng [1]) K_4-homeomorph $K_4(a,b,2,d,2,f)$, where $min\{a,b,d,f\} \geq 3$, is χ-unique if and only if it is not isomorphic to $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$. Moreover, each of the following sets is $\chi-$equivalence class:*

$$\{K_4(3,4,2,4,2,6), K_4(3,5,4,2,2,5)\},$$
$$\{K_4(3,4,2,4,2,8), K_4(3,4,2,7,5,2)\},$$
$$\{K_4(3,4,2,8,2,4), K_4(3,4,2,5,7,2)\}.$$

3 Main result

The following are notations that will be used in the sequel.

L_t^+ = set of the exponents of the positive terms in the left hand side of equation t,
L_t^- = set of the exponents of the negative terms in the left hand side of equation t,
R_t^+ = set of the exponents of the positive terms in the right hand side of equation t,
R_t^- = set of the exponents of the negative terms in the right hand side of equation t,
$-L_t$ = left side of Equation t with negative terms,
L_t = left side of Equation t with positive terms.

Elements can be repeated in a set.

We now give the main result of the paper.

Theorem 3.1 *Let G be a K_4-homeomorph with exactly two non-adjacent paths of length two. Then G is χ-unique if and only if it is not isomorphic to $K_4(4,1,2,1,2,4)$ or $K_4(1,s+2,2,1,2,s)$ or $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$ or $K_4(1,t,2,4,2,4)$, where $s \geq 3$, $t \geq 3$. Moreover, each of the following sets is $\chi-$equivalence class:*

$$\{K_4(4,1,2,1,2,4), K_4(2,1,1,5,2,3)\},$$
$$\{K_4(1,s+2,2,1,2,s), K_4(s+1,1,1,2,3,s)\},$$
$$\{K_4(3,4,2,4,2,6), K_4(3,5,4,2,2,5)\},$$
$$\{K_4(3,4,2,4,2,8), K_4(3,4,2,7,5,2)\},$$
$$\{K_4(3,4,2,8,2,4), K_4(3,4,2,5,7,2)\},$$
$$\{K_4(1,t,2,4,2,4), K_4(1,2,t,2,3,5)\}.$$

Proof.
Let $G = K_4(a,b,2,d,2,f)$, where $a \neq 2$, $b \neq 2$, $d \neq 2$, $f \neq 2$. We shall consider the following types of G according to the conditions on the paths a, b, d, f.

(i) Type 1: All four paths are of length one, e.i., $G = K_4(1,1,2,1,2,1)$,
(ii) Type 2: Three paths are of length one and one path is of length greater than two, e.i., $G = K_4(a,1,2,1,2,1)$, where $a > 2$,
(iii) Type 3: Two paths are of length one and two paths are of length greater than two, e.i., $G : K_4(a,1,2,1,2,f)$, where $a \geq 3$, $f \geq 3$, or $G : K_4(1,b,2,1,2,f)$, where $b \geq 3$, $f \geq 3$,
(iv) Type 4: Four paths are of length greater than two, e.i., $G_5 : K_4(a,b,2,d,2,f)$, where $a \geq 3$, $b \geq 3$, $b \geq 3$, $d \geq 3$, $f \geq 3$,
(v) Type 5: One path is of length one and three paths are of length greater than two, e.i., $G = K_4(1,b,2,d,2,f)$, where $b \geq 3$, $d \geq 3$, $f \geq 3$.

We then obtain the following
(i) If G is the graph of Type 1, then by Lemma 2.3, G is χ–unique;
(ii) If G is the graph of Type 2, then by Lemma 2.2, G is χ–unique;
(iii) If G is the graph of Type 3, then by Lemmas 2.7 and 2.8, G is not χ–unique if and only if it is isomorphic to $K_4(4,1,2,1,2,4)$ or $K_4(1,s+2,2,1,2,s)$, where $s \geq 3$ and each of the following sets is χ–equivalence class:

$$\{K_4(4,1,2,1,2,4), K_4(2,1,1,5,2,3)\},$$
$$\{K_4(1,s+2,2,1,2,s), K_4(s+1,1,1,2,3,s)\};$$

(iv) If G is the graph of Type 4, then by Lemma 2.9, G is χ-unique if and only if it is not isomorphic to $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$, and each of the following sets is χ–equivalence class:

$$\{K_4(3,4,2,4,2,6), K_4(3,5,4,2,2,5)\},$$
$$\{K_4(3,4,2,4,2,8), K_4(3,4,2,7,5,2)\},$$
$$\{K_4(3,4,2,8,2,4), K_4(3,4,2,5,7,2)\}.$$

To complete the study on the chromaticity of $G = K_4$-homeomorphs with exactly two non-adjacent paths of length two, we need to consider G of Type 5.

If there exists a graph H such that $H \sim G$, by Lemmas 2.1(1),(3) and 2.6, we know that H is a K_4-homeomorph and

$$|E(G)| = |E(H)|, Q(G) = Q(H) \tag{1}$$

Let $H = K_4(i_1, j_1, k_1, l_1, m_1, n_1)$, without loss of generality, we assume that $i_1 = \min\ \{i_1, j_1, k_1, l_1, m_1, n_1\} = 1$. We define $R' = \{j_1, k_1, l_1, m_1, n_1\}$ and $R'' = \{j_1+1, k_1+1, l_1+1, m_1+1, n_1+1\}$.

Let $G = K_4(a,b,2,d,2,f) = K_4(i,j,2,k,2,l)$. Without loss of generality, let i =min $\{i,j,2,k,2,l\}$ and by Lemma 2.1(4), $i = i_1 = 1$. Then after simplification, Equation (1) yields

$$4+j+k+l=j_1+k_1+l_1+m_1+n_1 \tag{2}$$

$$-w^{l+3}-w^{j+3}-w^{j+k+2}-w^{l+2+k}-w^{k+1}-w^{l+j}-w^4+w^j+w^2+ w^k+w^2+w^2+w^l+w^{j+1}+w^3+w^{k+1}+w^3+w^{l+1}=-w^{1+n_1+k_1}- w^{1+m_1+j_1}-w^{k_1+j_1+l_1}-w^{n_1+m_1+l_1}-w^{1+l_1}-w^{n_1+j_1}-w^{k_1+m_1}+ w_1^j+w_1^k+w_1^l+w_1^m+w_1^n+w^{j_1+1}+w^{k_1+1}+w^{l_1+1}+w^{m_1+1}+ w^{n_1+1} \tag{3}$$

where $R_3^+=R'\cup R''$.

In Equation (3), the positive terms (resp. negative terms) in the left hand side can only be canceled by the negative terms (resp. positive terms) in the left hand side or by the positive terms (resp. negative terms) in the right hand side. Since 2,2,3,3 are not equal to the exponents in L_3^- or simply 2,2,3,3 $\notin L_3^-$, we have 2,2,3,3 $\in R_3^+$, where $R_3^+=R'\cup R''$. Note that the two lowest terms in the left hand side of Equation (3) are equal to w^2. So, 2,2 $\in R'$ and 3,3 $\in R''$.Observe that the term $-w^4\in -L_t$ can be canceled by the terms w^j, w^k, w^l, w^{j+1}, w^{k+1}, w^{l+1} in L_3. If one of the exponents $j+1$, $k+1$, $l+1$ is equal to 4 then correspondingly one of j, k, l is equal to 3 which implies that $3,3,3\cup R_3^+$. This is a contradiction. Let us now consider one of j, k, l being equal to 4.

There are six cases to be considered, namely, Case 1: $2<j\le k\le l$, Case 2: $2<j\le l\le k$, Case 3: $2<k\le j\le l$, Case 4: $2<k\le l\le j$, Case 5: $2<l\le j\le k$, Case 6: $2<l\le k\le j$.

Table 1 shows the possible values of exponents belonging to L_3^+ in relation to the values of some exponents in L_3^- following the assumptions in Case 1.

Case 1. Claim: $2<j\le k\le l$.

Subcase 1.1 If $l=4$ then by our claim $2<j\le k\le l$, $j=3$ or $j=4$.
Subcase 1.1.1 Suppose $j=3$. Then $k=3$ or $k=4$.
Subcase 1.1.1(a) Let $j=3$, $k=3$ and $l=4$. Then we have $G=K_4(1,3,2,3,2,4)$, where $g(G)=6$. Since $\{l,k+1\}\subset L_3^-$, we have $\{2,2,3,3,l+1,j,j+1,k\}\subset R_3^+$. So $R'=\{2,2,j,l+1,k-1\}=\{2,2,3,5,2\}$. By Lemma 3.2, the only case of H which is not χ-unique is $K_4(1,2,2,2,3,5)$, where $K_4(1,2,2,2,3,5)\sim K_4(1,2,2,2,4,2)$. Obviously, $G=K_4(1,3,2,3,2,4)$ is not isomorphic to $K_4(1,2,2,2,4,2)$. This is a contradiction. For all other cases of H being χ-unique, since $H\sim G$, $H\cong G$. But H is not isomorphic to G. This is again a contradiction.
Subcase 1.1.1(b) Let $j=3$, $k=4$ and $l=4$. Then $G=K_4(1,3,2,4,2,4)$, where $g(G)=6$ and G has only one cycle of length six. We have $R'=\{2,2,j,l+1,k-1\}=\{2,2,3,5,3\}$. There are four non-isomorphic cases of H, namely: $K_4(1,2,3,2,5,3)$, $K_4(1,2,3,2,3,5)$,

Subcase	$j+3$	$j+k+2$	4
1.1			l
1.2.1		$l+1$	k
1.2.2	$l+1$		k
1.2.3		l	k
1.2.4	l		k
1.3.1		$l+1$	j
1.3.2	k	$l+1$	j
1.3.3	k		j
1.3.4	$l+1$		j
1.3.5		l	j
1.3.6	k	l	j
1.3.7	l		j

Table 1: $2 < j \leq k \leq l$.

$K_4(1,2,3,3,5,2)$ and $K_4(1,2,3,3,2,5)$. Thus, we obtain a solution for Equation (3) where $G \cong K_4(1,3,2,4,2,4)$ and $H \cong K_4(1,2,3,2,3,5)$. For brevity, throughout the proof, the algebraic solution to show whether Equation (3) is satisfied or not by the cases of H and G shall be left to the reader.

Subcase 1.1.2 Suppose $j = 4$. Then by our claim $2 < j \leq k \leq l$, we have $k = l = 4$, $G = K_4(1,4,2,4,2,4)$, $g(G) = 7$ and G has two cycles of length seven. We consider $R' = \{2,2,j,l+1,k-1\} = \{2,2,4,5,3\}$. By Lemma 2.1(2), $g(H) = 7$ and H must have two cycles of length seven as well. The following are the non-isomorphic cases of H: $K_4(1,2,4,3,2,5)$, $K_4(1,2,4,2,3,5)$. Therefore, we obtain a solution where $G \cong K_4(1,4,2,4,2,4)$ and $H \cong K_4(1,2,4,2,3,5)$. Note that this solution can be verified by Lemma 2.2.

Subcase 1.2 If $k = 4$ then $j = 3$ or $j = 4$, where $l \geq 4$. Since $k, k+1 \in L_3^-$, we have $j, j+1, l, l+1, 2, 2, 3, 3 \in R_3^+$. Thus, $R' = \{2,2,j,k,l\}$. Hence, $\{i,2,2,j,k,l\}$ and $\{i_1,j_1,k_1,l_1,m_1,n_1\}$ are the same as multisets. By Lemma 2.1(4(ii)), $G \cong H$. Let us now consider conditions where $R' \neq \{2,2,j,k,l\}$.

Subcase 1.2.1 Suppose $k = 4$ and $l+1 = j+2+k$, i.e., $l = j+5$. Then $G = K_4(1,j,2,4,2,j+6)$, $g(G) = j+3$ and G has only one cycle of length $j+3$, where $j = 3,4$. Since $k, l+1, k+1 \in L_3^-$, we have $l, j, j+1, 2, 2, 3, 3 \in R_3^+$. So, $R' = \{2,2,j,l-1,k+1\} = \{2,2,j,j+4,5\}$ and the non-isomorphic cases of H are: $K_4(1,2,j,2,j+4,5)$, $K_4(1,2,j,2,5,j+4)$, $K_4(1,2,j,5,j+4,2)$, $K_4(1,2,j,5,2,j+4)$, $K_4(1,2,j,j+4,2,5)$ and $K_4(1,2,j,j+4,5,2)$. Equation (3) is not satisfied by any of these cases of H for the values of $j = 3,4$.

Subcase 1.2.2 Suppose $k = 4$ and $l + 1 = j + 3$. Then $l = j + 2$, $G = K_4(1, j, 2, 4, 2, j + 2)$, $g(G) = j + 3$ and G has only one cycle of length $j + 3$, where $j = 3, 4$. We have $R' = \{2, 2, j, l - 1, k + 1\} = \{2, 2, j, j + 1, 5\}$. The following are the non-isomorphic cases of H: $K_4(1, 2, j, 2, j + 1, 5)$, $K_4(1, 2, j, 2, 5, j + 1)$, $K_4(1, 2, j, 5, j + 1, 2)$, $K_4(1, 2, j, 5, 2, j+1)$, $K_4(1, 2, j, j+1, 2, 5)$ and $K_4(1, 2, j, j+1, 5, 2)$. Equation (3) is not satisfied by any of these cases of H for the values of $j = 3, 4$.

Subcase 1.2.3 Suppose $k = 4$ and $l = j + 2 + k = j + 6$. Then $G = K_4(1, j, 2, 4, 2, j+6)$, $g(G) = j+3$ and G has only one cycle of length $j+3$, where $j = 3, 4$. We know that $k, l, k+1 \in L_3^-$, thus, $j, j+1, l+1, 2, 2, 3, 3 \in R_3^+$. So, $R' = \{2, 2, j, l + 1, k - 1\} = \{2, 2, j, j + 7, 3\}$. By Lemma 2.1(2), $g(H) = j+3$ and H must have only one cycle of length $j+3$. The following are the non-isomorphic cases of H: $K_4(1, j, 2, 2, 3, j+7)$, $K_4(1, j, 2, 3, 2, j+7)$, where $j = 3, 4$; $K_4(1, j, 2, 2, j+7, 3)$, $K_4(1, j, 2, 3, j+7, 2)$, where $j = 3$. None of these cases of H satisfies Equation (3) for the values of $j = 3, 4$.

Subcase 1.2.4 Suppose $k = 4$ and $l = j+3$. Then $G = K_4(1, j, 2, 4, 2, j+3)$, $g(G) = j + 3$ and G has only one cycle of length $j + 3$, where $j = 3, 4$. We have $R' = \{2, 2, j, l + 1, k - 1\} = \{2, 2, j, j + 4, 3\}$. The following are the non-isomorphic cases of H: $K_4(1, j, 2, 2, 3, j+4)$, $K_4(1, j, 2, 3, 2, j+4)$, where $j = 3, 4$; $K_4(1, j, 2, 2, j + 4, 3)$, $K_4(1, j, 2, 3, j + 4, 2)$, where $j = 3$. Equation (3) is not satisfied by any of these cases of H for the values of $j = 3, 4$.

Subcase 1.3 If $j = 4$ then we can consider the following subcases.

Subcase 1.3.1 Suppose $l + 1 = j + k + 2$. Then $l = k + 5$, $G = K_4(1, 4, 2, k, 2, k + 5)$, where $k \geq 4$, $g(G) = 7$ and G has only one cycle of length seven. We know that $j, k + 1, l + 1 \in L_3^-$, so, $j + 1, k, l \in R_3^+$. Thus, $R' = \{j+1, k-1, l, 2, 2\} = \{5, k-1, k+5, 2, 2\}$ or $R' = \{j+1, k, l-1, 2, 2\} = \{5, k, k + 4, 2, 2\}$.

Let $R' = \{5, k - 1, k + 5, 2, 2\}$. $g(H) = 7$ when $k = 4, 6$. If $k = 4$ then $R' = \{5, 3, 9, 2, 2\}$ and the non-isomorphic cases of H are: $K_4(1, 2, 9, 3, 2, 5)$, $K_4(1, 2, 9, 2, 3, 5)$, $K_4(1, 3, 9, 2, 2, 5)$. Therefore, we obtain a solution where $G \cong K_4(1, 4, 2, 4, 2, 9)$ and $H \cong K_4(1, 2, 9, 2, 3, 5)$. If $k = 6$ then $R' = \{5, 5, 11, 2, 2\}$ and it is not possible to have cases of H where $g(H) = 7$.

Let $R' = \{5, k, k + 4, 2, 2\}$. $g(H) = 7$ when $k = 4$, i.e., $R' = \{5, 4, 8, 2, 2\}$. We have the following non-isomorphic cases of H: $K_4(1, 2, 4, 2, 8, 5)$, $K_4(1, 2, 4, 2, 5, 8)$, $K_4(1, 2, 4, 5, 2, 8)$, $K_4(1, 2, 4, 5, 8, 2)$, $K_4(1, 2, 4, 8, 5, 2)$, $K_4(1, 2, 4, 8, 2, 5)$. Equation (3) is not satisfied by any of these cases of H.

Subcase 1.3.2 Suppose $l + 1 = j + k + 2$ and $k = j + 3$. Then $k = 7$, $l = 12$, $K_4(1, 4, 2, 7, 2, 12)$, where $g(G) = 7$ and G has only one cycle of length seven. We know that $j, l+1, k, k+1 \in L_3^-$, so, $j+1, l, 2, 2, 3, 3 \in R_3^+$. Thus, $R' = \{j+1, k, l-1, 2, 2\} = \{5, 7, 11, 2, 2\}$ or $R' = \{2, 2, j+1, l, k-1\} = \{2, 2, 5, 12, 6\}$ or $R' = \{2, 2, j, l - 1, k + 1\} = \{2, 2, 4, 11, 8\}$.

In the first and second cases of R', namely, $R' = \{5, 7, 11, 2, 2\}$ and $R' = \{2, 2, 5, 12, 6\}$, respectively, $g(H) \neq 7$, a contradiction to Lemma 2.1(2). If $R' = \{2, 2, j, l - 1, k + 1\} = \{2, 2, 4, 11, 8\}$, then the non-isomorphic cases of H are: $K_4(1, 2, 4, 2, 11, 8)$, $K_4(1, 2, 4, 2, 8, 11)$, $K_4(1, 2, 4, 11, 2, 8)$, $K_4(1, 2, 4, 11, 8, 2)$,
$K_4(1, 2, 4, 8, 2, 11)$ and $K_4(1, 2, 4, 8, 11, 2)$. Equation (3) is not satisfied by

any of these cases of H.

Subcase 1.3.3 Suppose $k = j+3$, i.e., $k = 7$, then $G = K_4(1,4,2,7,2,l)$ where $l \geq 7$ and $g(G) = 7$. We have $k, j, k+1 \in L_3^-$, thus $2,2,3,3,j+1,l,l+1 \in R_3^+$. By Equation (2), $R' = \{2,2,j+1,k-1,l\}$, i.e., $R' = \{2,2,5,6,l\}$. By Lemma 2.1(2), $g(H) = 7$. But with $R' = \{j_1,k_1,l_1,m_1,n_1\} = \{2,2,5,6,l\}$ and $i_1 = 1$, the girth of H cannot be equal to seven. This is a contradiction.

Subcase 1.3.4 Suppose $l+1 = j+3$, i.e., $l = 6$, then $G = K_4(1,4,2,k,2,6)$, where $k = 4,5,6$ and $g(G) = 7$. Since $l+1, j, k+1 \in L_3^-$, we have $l, j+1, k, 2, 2, 3, 3 \in R_3^-$. So, $R' = \{2,2,j+1,l-1,k\}$ or $R' = \{2,2,j+1,l,k-1\}$.
If $R' = \{2,2,j+1,l-1,k\} = \{2,2,5,5,k\}$, then the non-isomorphic cases of H occur only if $k = 4$ since $g(G) = g(H) = 7$, by Lemma 2.1(2). These cases of H are as follows: $K_4(1,4,2,5,2,5)$, $K_4(1,4,2,5,5,2)$ and $K_4(1,4,2,2,5,5)$. Note that $G = K_4(1,4,2,4,2,6)$. None of these cases satisfies Equation (3).
If $R' = \{2,2,j+1,l,k-1\} = \{2,2,5,6,k-1\}$, then the non-isomorphic cases of H, where $g(H) = 7$, occur only when $k = 4,5$.
Let $k = 4$. The non-isomorphic cases of H are: $K_4(1,2,5,2,3,6)$, $K_4(1,2,6,2,3,5)$ and $K_4(1,2,5,3,2,6)$. Note that $G = K_4(1,4,2,4,2,6)$. Equation (3) is satisfied when $G \cong K_4(1,4,2,4,2,6)$ and $H \cong K_4(1,2,6,2,3,5)$. Thus, we get a solution for Equation (3) where $K_4(1,4,2,4,2,6) \sim K_4(1,2,6,4,2,4)$, i.e., $G \cong K_4(1,6,2,4,2,4)$ and $H \cong K_4(1,2,6,2,3,5)$.
Let $k = 5$, i.e., $R' = \{2,2,5,6,4\}$ and $G = K_4(1,4,2,5,2,6)$. Then the non-isomorphic cases of H are: $K_4(1,2,4,2,5,6)$, $K_4(1,2,4,2,6,5)$, $K_4(1,2,4,6,5,2)$, $K_4(1,2,4,6,2,5)$, $K_4(1,2,4,5,2,6)$ and $K_4(1,2,4,5,6,2)$. It can easily be checked that none of these cases satisfies Equation (3).

Subcase 1.3.5 Suppose $l = j+2+k = k+6$. Then $G = K_4(1,4,2,k,2,k+6)$, where $k \geq 4$ and $g(G) = 7$. Since $j, l, k+1 \in L_3^-$, we have $j+1, l+1, k, 2, 2, 3, 3 \in R_3^+$. So, $R' = \{2,2,j+1,l,k-1\}$ or $R' = \{2,2,j,l+1,k-1\}$.

Subcase 1.3.5(a) Let $R' = \{2,2,j+1,l,k-1\} = \{2,2,5,k+6,k-1\}$, where $k \geq 4$. Then the only possible value of k is 5 for H to have girth seven. If $k = 5$, then $G = K_4(1,4,2,5,2,11)$, $g(G) = 7$, G has only one cycle of length seven and $R' = \{2,2,5,11,4\}$. There are six non-isomorphic cases of H, namely: $K_4(1,2,4,5,11,2)$, $K_4(1,2,4,5,2,11)$, $K_4(1,2,4,2,11,5)$, $K_4(1,2,4,2,5,11)$,
$K_4(1,2,4,11,5,2)$ and $K_4(1,2,4,11,2,5)$. It is easy to check that none of these cases of H satisfies Equation (3).

Subcase 1.3.5(b) Let $R' = \{2,2,j,l+1,k-1\} = \{2,2,4,k+7,k-1\}$, where $k \geq 4$, $g(G) = 7$ and G has only one cycle of length seven. Then the following are the non-isomorphic cases of H: (i) $K_4(1,2,4,2,k-1,k+7)$, $K_4(1,2,4,k-1,2,k+7)$ only when $k > 4$. Since if $k = 4$ then H will have two cycles of length seven which contradicts Lemma 2.1(2). (ii) $K_4(1,2,4,k+7,2,k-1)$, $K_4(1,2,4,k+7,k-1,2)$ only when $k > 5$. Since if $k = 4$ then $g(H) = 6$ and if $k = 5$ then H will have two cycles of length seven which contradicts Lemma 2.1(2). (iii) $K_4(1,2,4,2,k+7,k-1)$,

$K_4(1,2,4k-1,k+7,2)$, where $k \geq 4$. Take note that if $k=4$, then there are two cycles of H that can be considered as its shortest cycles, namely, $(1,2,4)$ and $(2,2,3)$, so, that $g(G)=7$. The non-isomorphic cases of H when $(1,2,4)$ is its shortest cycle are $K_4(1,2,4,2,11,5)$, $K_4(1,2,15,11,2)$. If we consider $(2,2,3)$ as the shortest cycle of H then the other two non-isomorphic cases of H are $K_4(1,2,11,2,3,4)$ and $K_4(1,3,4,2,2,11)$, where $G=K_4(1,4,2,4,2,10)$. None of these cases of H satisfies Equation (3).

Subcase 1.3.6 Suppose $k=j+3$ and $l=j+2+k$, i.e., $k=7$, $j=4$ and $l=13$. Then $G=K_4(1,4,2,7,2,13)$, where $g(G)=7$. Since $k,j,l,k+1 \in L_3^-$, we have $j+1,l+1,2,2,3,3 \in R_3^+$. So, we have $R'=\{2,2,j,l+1,k-2\}$ or $R'=\{2,2,j+1,l,k-1\}$ or $R'=\{2,2,j+1,l+1,k-2\}$.
Let $R'=\{2,2,j,l+1,k-1\}=\{2,2,4,14,6\}$. Then the non-isomorphic cases of H are: $K_4(1,2,4,2,14,6)$, $K_4(1,2,4,2,6,14)$, $K_4(1,2,4,6,2,14)$, $K_4(1,2,4,6,14,2)$, $K_4(1,2,4,14,6,2)$ and $K_4(1,2,4,14,2,6)$. Equation (3) is not satisfied by any of these cases of H. If $R'=\{2,2,j+1,l,k-1\}=\{2,2,5,13,6\}$ or $R'=\{2,2,j+1,l+1,k-2\}=\{2,2,5,14,5\}$, then $g(H)\neq 7$. This contradicts Lemma 2.1(2).

Subcase 1.3.7 Suppose $l=j+3$ and $j=4$. Then $l=7$, $G=K_4(1,4,2,k,2,7)$, where $4 \leq k \leq 7$ and $g(G)=7$.

Subcase 1.3.7(a) Let $k=4$, i.e., $j=k=4$. Then by Equation (2), $j \in L_3^-$ and $k \in R_3^+$. Thus, $l,j,k+1 \in L_3^-$ and $2,2,3,3,l+1,j+1,k \in R_3^+$. So we have $R'=\{2,2,j+1,l,k-1\}$ or $R'=\{2,2,j,l+1,k-1\}$.
Assume that $R'=\{2,2,j+1,l,k-1\}=\{2,2,5,7,3\}$. Then we have $G=K_4(1,4,2,4,2,7)$, $g(G)=7$ and the non-isomorphic cases of H (where $g(H)=7$, by Lemma 2.1(2)) are as follows: $K_4(1,2,5,3,2,7)$, $K_4(1,2,7,3,2,5)$,
$K_4(1,2,7,2,3,5)$ and $K_4(1,2,5,2,3,7)$. The solution for Equation (3) is $G \cong K_4(1,2,5,2,3,7)$ and $H \cong K_4(1,2,7,2,3,5)$. Note that $K_4(1,4,2,4,2,7) \cong K_4(1,7,2,4,2,4)$. So, the solution can be expressed as $G \cong K_4(1,7,2,4,2,4)$ and $H \cong K_4(1,2,7,2,3,5)$.
Assume that $R'=\{2,2,j,l+1,k-1\}$, i.e., $R'=\{2,2,4,8,3\}$. Then by Lemma 2.1(2), $g(H)=7$ and H has only one cycle of length seven. The non-isomorphic cases of H are as follows: $K_4(1,2,4,2,8,3)$ and $K_4(1,2,8,2,3,4)$. It is easy to verify that none of these two cases of H satisfies Equation (3).

Subcase 1.3.7(b) Let $j=4$, $l=7$ and $4<k\leq 7$. Then $G=K_4(1,4,2,k,2,7)$ and $g(G)=7$. Since $l,j,k+1 \in L_3^-$, $l+1,j+1,k,2,2,3,3 \in R_3^+$, we have $R'=\{2,2,j+1,l,k-1\}$ or $R'=\{2,2,j,l+1,k-1\}$.
Assume that $R'=\{2,2,j+1,l,k-1\}=\{2,2,5,7,k-1\}$, where $k=5,6,7$. By Lemma 2.1(2), $g(H)=7$. The only possible way to have cases of H with $g(H)=7$ is when $k=5$. If $k=5$ then $G=K_4(1,4,2,5,2,7)$ and $R'=\{2,2,5,7,4\}$. Thus, we have $\{1,4,2,5,2,7\}=\{i_1,j_1,k_1,l_1,m_1,n_1\}$ as mulltisets (since $i_1=1$). By Lemma 2.1(4(ii)), $G\cong H$.
Assume that $R'=\{2,2,j,l+1,k-1\}=\{2,2,4,8,k-1\}$ and $G=K_4(1,4,2,k,2,7\}$, where $k=5,6,7$. Then the following are the non-isomorphic cases of H: $K_4(1,2,4,8,2,k-1\}$ and $K_4(1,2,4,8,k-1,2\}$, where $k=6,7$ (since for $k=5$, there will be two cycles of length 7); $K_4(1,2,4,2,8,k-1\}$, $K_4(1,2,4,2,k-1,8\}$, $K_4(1,2,4,k-1,8,2\}$ and

$K_4(1,2,4,k-1,2,8\}$, where $k=5,6,7$. None of these cases of H satisfies Equation (3) for all values of $k=5,6,7$.

In Cases 2-6, we follow the same procedure and will not be shown in this paper since the process is very long and tedious. For the details of the rest of the proof, the reader may contact the first author or may refer to our technical report [2].

We now summarize our result. Please note that our reference for the solutions derived from Cases 2-6 is our technical report [2].

We have obtained the following solutions:

(i) $K_4(1,3,2,4,2,4) \sim K_4(1,2,3,2,3,5)$ (Cases 1,2,5/Subcases 1.1.1(b),2.2.1(a),5.2.1(a))

(ii) $K_4(1,4,2,4,2,4) \sim K_4(1,2,4,2,3,5)$ (Cases 1,2,5/Subcases 1.1.2,2.2.1(b),5.2.1(b))

(iii) $K_4(1,6,2,4,2,4) \sim K_4(1,2,6,2,3,5)$ (Case 1,3,4/Subcases 1.3.4,3.2.4(a),4.2.4(a))

(iv) $K_4(1,7,2,4,2,4) \sim K_4(1,2,7,2,3,5)$ (Cases 1,3,4/Subcase 1.3.7(a),3.2.2(a),4.2.2(a))

(v) $K_4(1,9,2,4,2,4) \sim K_4(1,2,9,2,3,5)$ (Cases 1,3/Subcases 1.3.1,3.2.5(a))

(vii) $K_4(1,10,2,4,2,4) \sim K_4(1,2,10,2,3,5)$ (Case 3/Subcase 3.2.3(b))

(viii) $K_4(1,b,2,4,2,4) \sim K_4(1,2,b,2,3,5)$,$b>4$ (Cases 3,4/Subcases 3.2.1(d),4.2.1(b))

Clearly, all of these solutions can be expressed in a single result which is $K_4(1,b,2,4,2,4) \sim K_4(1,2,b,2,3,5)$, where $b \geq 3$. Let $b=t$. This completes the proof of Theorem 3.1. □

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