Edge choosablity and total choosability of toroidal graphs without intersecting triangles*

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Abstract

Two cycles are said to be intersecting if they share at least one common vertex. Let $\chi_l'(G)$ and $\chi_l''(G)$ denote the list edge chromatic number and list total chromatic number of a graph G, respectively. In this paper, we proved that for any toroidal graph G without intersecting triangles, $\chi_l'(G) \le \Delta(G) + 1$ and $\chi_l''(G) \le \Delta(G) + 2$ if $\Delta(G) \ge 6$, and $\chi_l'(G) = \Delta(G)$ if $\Delta(G) \ge 8$.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow [3] for terminology and notation not defined here. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A k-, k^+- or k^--vertex is a vertex of degree k, at least k or at most k, respectively. A k-cycle is a cycle of length k, and a 3-cycle is usually called a triangle.

A *total k-coloring* of a graph $G = (V, E)$ is a coloring of $V \cup E$ using at most k colors such that all adjacent or incident elements receive distinct

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colors. The total chromatic number of G, denoted by $\chi''(G)$, is the smallest integer k such that G has a total k-coloring. It is clear that $\chi''(G) \geq \Delta(G)+1$. Behzad [1] and Vizing [17] independently conjectured that $\chi''(G) \leq \Delta+2$ for every graph G. This conjecture was verified for planar graphs with maximum degree $\Delta \neq 6$ ([4], [12], [13], [16], [20], [22]).

A *total assignment* of G is a mapping L that assigns to each element $x \in V \cup E$ a list $L(x)$ of colors. Given a total assignment L of G, an *L-total-coloring* of G is a proper total-coloring such that each element receives a color from its own list. A graph G is *k-total-choosable* if G has a proper L-total-coloring for every total assignment L with $|L(x)| \geq k$ for every $x \in V \cup E$. The *list total chromatic number* or *total choosability* of G, denoted $\chi''_l(G)$, is the smallest integer k such that G is k-total-choosable. The *list edge chromatic number* or *edge choosablity* $\chi'_l(G)$ is defined similarly in terms of coloring only edges, the ordinary *edge chromatic number* is denoted $\chi'(G)$. Clearly, $\chi''_l(G) \geq \chi''(G) \geq \Delta+1$ and $\chi'_l(G) \geq \chi'(G) \geq \Delta$.

Suppose that G is a graph. Vizing, Gupta, Albertson and Collins, Bollobás and Harris (see [2] or [10]) independently conjectured that $\chi'_l(G) = \chi'(G)$. It is well known as the *List Coloring Conjecture*. Borodin et al. [5], and Juvan et al. [11] posed independently the conjecture which we refer to as the *Total List Coloring Conjecture* (or simply $TLCC$) that $\chi''_l(G) = \chi''(G)$. The conjecture has been proved only for a few special cases. Borodin et al. [5] proved $\chi'_l(G) = \chi'(G) = \Delta$ and $\chi''_l(G) = \chi''(G) = \Delta+1$ for graphs with $\Delta \geq 12$ which can be embedded in a surface of nonnegative characteristic. Wang and Lih [21] confirmed $TLCC$ for outerplanar graphs. List edge and list total colorings are extensively studied and quite a few interesting results about these two colorings of planar graphs have been obtained in recent years (see [7], [8], [9], [14], [15], [18], [19]). In [21], Wang and Lih proved that a planar graph without intersecting triangles is edge-$(\Delta+1)$-choosable where $\Delta \neq 5$. In [7], Cranston proved that, for G being a planar graph without triangles sharing a common edge, $\chi''_l(G) \leq \Delta(G)+2$ if $\Delta(G) \geq 7$, and $\chi''_l(G) = \Delta(G)+1$ if $\Delta(G) \geq 9$. Cranston also noted that his result can be extended to graphs embedded into surfaces of nonnegative characters. Liu et al. [15] prove that $\chi'_l(G) = \Delta(G)$ and $\chi''_l(G) = \Delta(G)+1$ if $\Delta(G) \geq 8$ for a planar graph G without intersecting 4-cycles.

In this paper, we consider toroidal graphs with no triangles sharing a common vertex. In 2008, Wu and Wang [19] proved that $\chi''_l(G) \leq \Delta(G)+1$ for $\Delta \geq 8$. We consider $\Delta \geq 6$ and obtain Theorem 1.1, and follow the idea of Wang and Wu, get Theorem 1.2.

Theorem 1.1. *Suppose that G is a toroidal graph without intersecting triangles. If $\Delta(G) \geq 6$, then $\chi'_l(G) \leq \Delta(G)+1$ and $\chi''_l(G) \leq \Delta(G)+2$.*

Theorem 1.2. *Suppose that G is a toroidal graph without intersecting triangles. If $\Delta(G) \geq 8$, then $\chi'_l(G) = \Delta(G)$.*

2 Proofs

We use the following notation for convenience. Let $G = (V, E, F)$ be a toroidal graph, where F is the face set of G. For a face $f \in F$, the degree of f, denoted by $d(f)$, is the length of a boundary walk around f (where each cut-edge is counted twice). A k-, k^+- or k^--face is a face of degree k, at least k or at most k, respectively. Let $v_1, v_2, \ldots, v_k$ be the vertices incident with a k-face f such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_k)$. We define $D(f) = (d(v_1), d(v_2), \cdots, d(v_k))$, $\delta(f) = d(v_k)$ (the minimum degree of vertices incident with f), and call f a $(d(v_1), d(v_2), \cdots, d(v_k))$-face. Let $f_k(v)$, $f_{k^+}(v)$ or $f_{k^-}(v)$ denote the number of k-faces, k^+-faces or k^--faces incident with a vertex v, and let $n_k(v)$ and $n_{k^+}(v)$ denote the number of k-vertices and k^+-vertices adjacent to v, respectively.

In the proofs we use the discharging method. We assign an initial charge for each element. Then we design appropriate rules and redistribute weights accordingly. Once the discharging is finished, we get the new charge for each element which is nonnegative, and the sum of all the new charges is positive. But by Euler's formula, the total charge is still negative, which is a contradiction.

We first introduce the following lemmas. A graph G is minimally non-k-totally choosable if it is not k-totally choosable but all its proper subgraphs are k-totally choosable. It is similar to define minimally non-k-edge choosable graphs.

Lemma 2.1. *(Chang et al. [6]) The following hold for any minimally non-k-totally choosable graph G with maximum degree $\Delta \leq k-1$.*

(a) G is connected.

(b) If $e = uv$ is an edge in G with $d(u) \leq \frac{k-1}{2}$, then $d(u) + d(v) \geq k+1$. In particular, $\delta(G) \geq k + 1 - \Delta$ and so G has no 1-vertex.

(c) G has no even cycle $v_1v_2 \cdots v_{2t}v_1$ with $d(v_i) \leq \min\{\frac{k-1}{2}, k+1-\Delta\}$ for each odd i.

In the same spirit of previous results in [5] and [6], we have the following lemma similar as Lemma 2.1.

Lemma 2.2. *The following hold for any minimally non-k-edge choosable graph G with maximum degree $\Delta \leq k$.*

(a) G is connected.

(b) If uv is an edge of G, then $d(u) + d(v) \geq k+2$.

(c) G has no even cycle $v_1v_2 \cdots v_{2t}v_1$ with $d(v_i) \leq k+2-\Delta$ for each odd i.

Proof. Clearly, (a) is true for the choice of G. We will show (b) and (c) as follows.

(b) Suppose to the contrary that there is an edge uv such that $d(u) + d(v) \leq k+1$. Suppose that L is an edge-list-assignment of G such that $|L(e)| \geq k$ for each $e \in E$. Let $G' = G - uv$. By the minimality of G, G' admits an L-edge coloring ϕ. Since the neighbors of uv use at most $k-1$ colors with respect to ϕ, ϕ can be extended to G, a contradiction.

(c) Suppose to the contrary that G contains an even cycle $C =: v_1v_2 \cdots v_{2t}v$ with $d(v_i) \leq k+2-\Delta$ for each odd i. Suppose that L is a edge-list-assignment of G such that $|L(e)| \geq k$ for each $e \in E$. By the minimality of G, $G - E(C)$ has an L-edge coloring. Then each edge in C has at least $k-(\Delta-2)-(k+2-\Delta-2) = 2$ permissible colors. Since any even cycle is 2-choosable, we can color the edges of C, hence G has an L-edge coloring, a contradiction. ∎

Lemma 2.3. *(Wang and Wu [19]) Let G be a minimally non-Δ-edge choosable graph or a minimally non-$(\Delta+1)$-totally choosable graph. For any integer $2 \leq k \leq \lfloor \frac{\Delta}{2} \rfloor$, let $X_k = \{x \in V(G) \mid d(x) \leq k\}$ and $Y_k = \cup_{x \in X_k} N(x)$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for each $x \in X_k$ and $d_{M_k}(y) \leq k-1$ for each $y \in Y_k$.*

2.1 Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into two parts. First, we will show the former part of the Theorem. Suppose the theorem is false. Let $G = (V, E)$ be a minimal counter-example. Let L be an edge assignment of G with $|L(e)| = \Delta + 1$ for any edge $e \in E$ such that G is not edge-L-colorable. We still use G to denote a toroidal embedding of G and let F be the face set of G. Let $f \in F$ be a face of G. By Lemma 2.2(b), $\delta(G) \geq 3$ and each 3-vertex must be adjacent to three Δ-vertices. It yields that f is incident with at most $\lfloor \frac{2k}{3} \rfloor$ Δ-vertices who is adjacent to at least one 3-vertex incident f.

We use a discharging procedure. The initial charge of an element $x \in V \cup F$ of G is $\omega(x) = d(x) - 4$. Since G is a toroidal graph, by the Euler's formula $|V| + |F| - |E| \geq 0$, the total charge of the vertices and faces of G is equal to

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) \leq 0.$$

Then, we transfer the charges between vertices and faces, leading to a new charge $\omega'(x)$, by applying the following rules.

(R1) Each 3-vertex receives charge $\frac{1}{3}$ from each of its adjacent Δ-vertices.

(R2) Each 3-face receives charge $\frac{1}{2}$ from each of its incident 5^+-vertices.

(R3) Each 5^+-faces f sends charge $\frac{1}{3}$ to its incident Δ-vertex if the Δ-vertex is adjacent to at least one 3-vertex incident with f.

Let $f \in F$ be a k-face of G. It is easy to see that $\omega'(f) \geq 0$ for $k = 4$. First, we suppose $k \geq 5$. Then we have $\omega'(f) \geq \omega(f) - \lfloor \frac{2k}{3} \rfloor \times \frac{1}{3} \geq \frac{7k}{9} - 4 \geq \frac{2}{3} > 0$ if $k \geq 6$, and $\omega'(f) \geq \omega(f) - \lfloor \frac{2k}{3} \rfloor \times \frac{1}{3} = k - 5 \geq 0$ if $k = 5$ by (R3).

Now we may suppose that $k = 3$. Then $\omega(f) = -1$. Note that the neighbors of a 4^--vertex are $(\Delta - 1)^+$-vertices by Lemma 2.2(b). So if there is a 4^--vertex incident with f, then it is incident with at least two 5^+-vertices. Thus by (R2), $\omega'(f) \geq -1 + 2 \times \frac{1}{2} = 0$. Otherwise, there are at least three 5^+-vertices incident with f and so $\omega'(f) \geq -1 + 3 \times \frac{1}{2} = \frac{1}{2} > 0$ by (R2).

Suppose that v is a k-vertex of G. Since G has no intersecting triangles, each k-vertex is incident with at most one 3-face. If $k = 3$, then by (R1), v receives $\frac{1}{3}$ from each of its adjacent Δ-vertex, so $\omega'(v) \geq \omega(v) + 3 \times \frac{1}{3} = 0$. If $k = 4$, then $\omega'(v) \geq \omega(v) = 0$. If $5 \leq k \leq \Delta - 1$, since v is incident with at most one 3-face, $\omega'(v) \geq \omega(v) - \frac{1}{2} \geq k - 4 - \frac{1}{2} \geq \frac{1}{2} > 0$ by (R2).

Suppose that $k = \Delta$. Note that $f_3(v) \leq 1$. By Lemma 2.2(c), G has no $(\Delta, \Delta, 3, 3)$-faces. If $n_3(v) > \lfloor \frac{\Delta}{2} \rfloor$, then there are at least $n_3(v) - \lfloor \frac{\Delta}{2} \rfloor$ 5^+-faces. Then by (R1) and (R3), we have $\omega'(v) \geq \omega(v) + (n_3(v) - \lfloor \frac{\Delta}{2} \rfloor) \times \frac{1}{3} - n_3(v) \times \frac{1}{3} - \frac{1}{2} \geq \Delta - 4\frac{1}{2} - \frac{\Delta}{6} \geq \frac{1}{2} > 0$ for $\Delta \geq 6$. If $n_3(v) \leq \lfloor \frac{\Delta}{2} \rfloor$, we have $\omega'(v) \geq \omega(v) - n_3(v) \times \frac{1}{3} - \frac{1}{2} \geq \Delta - 4 - \lfloor \frac{\Delta}{2} \rfloor \times \frac{1}{3} - \frac{1}{2} \geq \frac{5\Delta}{6} - 4\frac{1}{2} \geq \frac{1}{2} > 0$.

In all cases, we have $\omega'(x) \geq 0$ for each $x \in V \cup F$, and furthermore $\omega'(v) > 0$ if $d(v) = \Delta \geq 6$. Hence $0 < \sum_{x \in V \cup F} \omega'(x) = \sum_{x \in V \cup F} \omega(x) \leq 0$, a contradiction. This contradiction completes the proof of the former part of Theorem 1.1.

Now, we proceed to prove the latter part. Suppose the theorem is false. Let $G = (V, E)$ be a minimal counterexample. By [7], the assertion holds for $\Delta(G) \geq 7$ since no intersecting triangles yields no adjacent triangles. So it follows that $\Delta(G) = 6$. Thus by Lemma 2.1(b), $\delta(G) \geq 3$, each 3-vertex is adjacent to three 6-vertices, and hence each 4-vertex is adjacent to four 4^+-vertices. Let L be a total assignment of G such that $|L(x)| \geq 8$ for each $x \in V \cup E$ and G is not L-total colorable. We still use G to denote a toroidal embedding of G and let F be the face set of G.

Claim 1. *G has no $(4, 4, 4)$-face.*

Proof. Assume to the contrary that G has a $(4,4,4)$-face f. Let $C = uvw$ be the facial cycle of f. Consider the graph $G' = G \setminus \{uv, uw, vw\}$. By the minimality of G, G' has an L-total coloring. Uncolor u, v and w, then each element incident with C touches at most four colors. This implies that each element incident with C has at least four colors available. It follows from $\chi''(C) = \chi''_l(C) = 3$ that C can be totally colored properly. This implies that G is L-totally colorable, which is a contradiction. ∎

The initial charge of a vertex v of G is $\omega(v) = 2d(v) - 6$, and the initial charge of a face f of G is $\omega(f) = d(f) - 6$. By Euler's formula $|V| + |F| - |E| \geq 0$, the total charge of the vertices and faces of G is equal to

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) \leq 0.$$

Our discharging rules are as follows.

(R4) Each 4-vertex sends charge $\frac{1}{2}$ to each of its incident k-faces f for $3 \leq k \leq 5$.

(R5) From each 5-vertex to each of its incident k-faces f for $3 \leq k \leq 5$, transfers

(R5.1) 2, if $k = 3$;

(R5.2) $\frac{1}{2}$, if $k = 4$ or 5.

(R6) From each 6-vertex to each of its incident k-faces f for $3 \leq k \leq 5$, transfers

(R6.1) 2, if $k = 3$;

(R6.2) $\frac{3}{4}$, if $k = 4$ or 5.

Let $f \in F$ be a face of G. Clearly, $\omega'(f) = \omega(f) \geq 0$ if $d(f) \geq 6$. Suppose that $d(f) = 3$. Then $\omega(f) = -3$. If $\delta(f) = 3$, then $D(f) = (6,6,3)$ by Lemma 2.1(b), and thus $\omega'(f) \geq \omega(f) + 2 \times 2 = 1 > 0$ by (R6.1). If $\delta(f) = 4$, since 4-vertex is adjacent to only 4^+-vertices and by Claim 1, we have $D(f) = (5^+, 4^+, 4)$. By (R4), (R5.1) and (R6.1), we have $\omega'(f) \geq -3 + 2 + 2 \times \frac{1}{2} = 0$. If $\delta(f) \geq 5$, note that each vertex incident with f sends 2 to f by (R5.1) and (R6.1), $\omega'(f) \geq -3 + 3 \times 2 = 3 > 0$.

Suppose that $d(f) = 4$. Then $\omega(f) = -2$. First we consider that $\delta(f) = 3$. Note that f is incident with at least two 6-vertices. By Lemma 2.1(c), it is impossible that $D(f) = (6,6,3,3)$. Hence, $D(f) = (6,6,4^+,3)$. Since each 4^+-vertex gives at least charge $\frac{1}{2}$ to f and each 6-vertex give $\frac{3}{4}$ to f, $\omega'(f) \geq -2 + 2 \times \frac{3}{4} + \frac{1}{2} = 0$. If $\delta(f) \geq 4$, by (R4), (R5.2) and (R6.2), then $\omega'(f) \geq -2 + 4 \times \frac{1}{2} \geq 0$.

Suppose that $d(f) = 5$. Then $\omega(f) = -1$. If $\delta(f) = 3$, then f is incident with at least two 6-vertices of which each sends $\frac{3}{4}$ to f, and hence $\omega'(f) \geq -1 + 2 \times \frac{3}{4} = \frac{1}{2} > 0$. If $\delta(f) \geq 4$, then each vertex incident with f sends $\frac{1}{2}$ to it, and $\omega'(f) \geq -1 + 5 \times \frac{1}{2} = \frac{3}{2} > 0$.

Now let v be a k-vertex of G. Since G has no intersecting triangles, each k-vertex is incident with at most one 3-face, i.e., $f_3(v) \leq 1$. If $k = 3$, then $\omega'(v) = \omega(v) \geq 0$. If $k = 4$, then $\omega(v) = 2 \times 4 - 6 = 2$. Note that v sends charge at most $\frac{1}{2}$ to each incident face by (R4). Thus $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 0$. If $k = 5$, then $\omega(v) = 4$ and v is incident with at most one 3-face, so $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} - 2 \geq 0$ by (R5).

At last, If $k = 6$, then $\omega(v) = 6$, and v is incident with at most one 3-faces, then $\omega'(v) \geq \omega(v) - 5 \times \frac{3}{4} - 2 = \frac{1}{4} > 0$ by (R6).

In all cases, we have $\omega'(x) \geq 0$ for each $x \in V \cup F$, and furthermore $\omega'(v) > 0$ if $d(v) = \Delta = 6$. Hence $0 < \sum_{x \in V \cup F} \omega'(x) = \sum_{x \in V \cup F} \omega(x) \leq 0$, a contradiction. This completes the proof of Theorem 1.1. ∎

2.2 Proof of Theorem 1.2

Suppose the theorem is false. Let $G = (V, E)$ be a minimal counterexample. If G is not edge-Δ-choosable, we suppose that L is an edge assignment of G with $|L(e)| = \Delta$ for any edge $e \in E$ such that G is not edge-L-colorable. Suppose that G is not edge-Δ-choosable. It follows from Lemma 2.2(b) that $\delta(G) \geq \Delta(G) + 2 - \Delta = 2$, and hence every 2-vertex is adjacent to two Δ-vertices.

The following notations are used in both [6] and [15]. Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Since $\Delta \geq 8$, G_2 does not contain any odd cycle. Thus, by Lemma 2.1(c), G_2 is a forest. In each component T of G_2, the leaves of T are all Δ-vertices. It follows that G_2 contains a matching M that saturates all 2-vertices. If $uv \in M$ and $d(u) = 2$, then v is called the *2-master* of u and u is called the *dependent* of v. It is easy to see that each 2-vertex has a 2-master and each vertex of maximum degree can be the 2-master of at most one 2-vertex.

Let X be the set of vertices of degree at most 3 and $Y = \bigcup_{x \in X} N(x)$. By Lemma 2.3, we have that G contains a bipartite subgraph $M = (X, Y)$ such that $d_M(x) = 1$ and $d_M(y) = 2$ for all $x \in X$ and $y \in Y$. We call y the 3-master of x if $xy \in M$ and $x \in X$. Therefore, each vertex of degree at most 3 has a 3-master, and each vertex of degree at least $\Delta - 1$ can be the 3-master of at most two vertices.

We use a discharging procedure. The initial charge of an element $x \in V \cup F$ of G is $\omega(x) = d(x) - 4$. Since G is a toroidal graph, by the Euler's formula $|V| + |F| - |E| \geq 0$, the total charge of the vertices and faces of G

is equal to

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(d(f)-4)\le 0.$$

Then, we use the discharging procedure, leading to a final charge $\omega'(x)$, defined by applying the following rules.

(R1) Each 2-vertex receives charge 1 from its 2-master.

(R2) Each k-vertex ($2\le k\le 3$) receives charge 1 from each of its 3-masters.

(R3) Each 3-face receives charge $\frac{1}{2}$ from each of its incident 5^+-vertices.

Let $f\in F$ be a k-face of G. It is easy to see that $\omega'(f)\ge 0$ for $k\ge 4$. Now we may suppose that $k=3$. Then $\omega(f)=-1$. Note that the neighbors of a 4^--vertex are $(\Delta-2)^+$-vertices. Hence every 3-face is incident with at least two 5^+-vertices. So $\omega'(f)\ge -1+2\times\frac{1}{2}=0$.

Suppose that v is a k-vertex of G. First, since G has no intersecting triangles, each k-vertex is incident with at most one 3-face. Lemma 2.1(b) and Lemma 2.2(b) yield that $k\ge 8$ if v is the 2-master of some vertex, and $k\ge 7$ if v is the 3-master of some vertex. If $k=2$, then by (R1) and (R2), v receives 1 from its 2-master and receives 1 from its 3-master. So $\omega'(v)\ge\omega(v)+1+1=0$. If $k=3$, then by (R2), v receives 1 from its 3-master, and thus $\omega'(v)\ge\omega(v)+1=0$. If $k=4$, then $\omega'(v)=\omega(v)=0$. If $5\le k\le\Delta-2$, since v is incident with at most one 3-face, $\omega'(v)\ge\omega(v)-\frac{1}{2}=\frac{1}{2}>0$. If $k=\Delta-1$, then v can be a 3-master of at most two 3-vertices, and so $\omega'(v)\ge\omega(v)-2\times 1-\frac{1}{2}=\frac{1}{2}>0$. If $k=\Delta$, then v can be the 2-master of a 2-vertex and the 3-master of at most two 3^--vertices. Thus $\omega'(v)\ge\omega(v)-3-\frac{1}{2}=\frac{1}{2}>0$.

In all the cases, we obtain that $\omega'(x)\ge 0$ for any element $x\in V\cup F$. Since $\omega'(v)>0$ when $d(v)=\Delta\ge 8$, we see that $\sum_{x\in V\cup F}\omega(x)=\sum_{x\in V\cup F}\omega'(x)>0$, a contradiction. This completes the proof of Theorem 1.2. ∎

In the proofs of Theorems 1.1 and 1.2, since we show that $\sum_{x\in V\cup F}\omega'(x)>0$, the following two corollaries hold naturally.

Corollary 2.4. *Suppose that G is a graph embedded in a surface of nonnegative characteristic and without intersecting triangles. Then $\chi_l'(G)\le\Delta(G)+1$ and $\chi_l''(G)\le\Delta(G)+2$ if $\Delta(G)\ge 6$.*

Corollary 2.5. *Suppose that G is a graph embedded in a surface of nonnegative characteristic and without intersecting triangles. Then $\chi_l'(G)=\Delta(G)$ if $\Delta(G)\ge 8$.*

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