Lattice of all subsets of a finite set and partition lattice of a n-set *

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Abstract

This paper uses research methods in the subspace lattices, making a deep research to the lattices of all subsets of a finite set and partition of a n-set. At first, the inclusion relations between different lattices are studied. Then, a characterization of elements contained in a given lattice is given. Finally, the characteristic polynomials of the given lattices computed.

1 Introduction

In this paper, we use terminology and notation in [1, 2, 4]. The characteristic polynomials of partition lattices and subspaces lattices and lattices of all subsets of the finite set have been given by [2]. Lattices generated by orbits of subspaces have been detailed discussed in [3, 4], and obtained some important results. The purpose of this paper, with the way of [4], is further studied the lattices of all subsets of the finite set and partition lattices. At first, we shall study the inclusion relations between different lattices. Then, we shall give a characterization of elements contained in a given lattice. Finally, we shall compute the characteristic polynomials of the given lattices.

Definitions of partially ordered set(Poset) and lattices are seen to [1] or [2]. Assume that P is a partially ordered set, $a, b \in P$, $a < b$. If there exists no

*Foundation item: Supported by the Science Foundation of Hainan Province China(No.109006, 610230) and Natural Science Research Project in Hebei Provincial Colleges and Universities(No.Z2010185). **corresponding author Email address: zjkzyb@tom.com, Postal address: No.59 MaLoudong, Gaoxin District, Zhangjiakou , Hebei ,075000, P. R. China.

$c \in P$ such that $a < c < b$, then b is called a cover of a and denoted by $a < \cdot b$.

Definition 1.1. *Let P be a partially ordered set containing* 0, *and* $\mathbb{N}_0$ *be a non-negative integer set. The mapping*

$$r : \begin{array}{ccc} P & \to & \mathbb{N}_0 \\ a & \mapsto & r(a) \end{array}$$

is called the rank function in P, if the following conditions (i) and (ii) hold:

(i) $r(0) = 0$;

(ii) $r(b) = r(a) + 1$, *for* $a, b \in P$ *and* $a < \cdot b$.

Let L be a poset containing 0, *elements covering* 0 *is saled to be atoms of L. The lattice containing* 0 *is saled to be the atom lattice, if for every $a \in L \backslash \{0\}$, all a's are the supremum of some atoms in L, i.e.,*

$$a = \vee \{p \in L | 0 < \cdot p \leq a\}.$$

Definition 1.2. *Let L be the finite lattice containing* 0. *L is called the geometric lattice , if the following conditions G_1 and G_2 hold:*

G_1 L is a atom lattice;

G_2 there exists the rank function r in L, and $r(x \wedge y) + r(x \vee y) \leq r(x) + r(y)$, for all $x, y \in L$.

Definition 1.3. *Let P be a finite poset, K be a field of characteristic* 0, *and $\mu(x, y)$ be a binary function defined in P and taking values from K. Assume that $\mu(x, y)$ satisfies the following three conditions (i), (ii), and (iii):*

(i) for any $x \in P$, there is always $\mu(x, x) = 1$;

(ii) for $x, y \in P$, if $x \nleq y$, then $\mu(x, y) = 0$;

(iii) for $x, y \in P$, if $x < y$, then $\sum_{x \leq z \leq y} \mu(x, z) = 0$.

Then $\mu(x, y)$ is called Möbius function defined in P and taking values from K.

Definition 1.4. *Let P be a poset containing* 1 *and* 0, *and there exist the rank function r and the Möbius function μ. Then the polynomial*

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is called characteristic polynomial in P.

2 Lattice of all subsets of a finite set

Let $S(n) = \{x_1, x_2, \cdots, x_n\}$, $\mathcal{P}(S(n))$ be power set of $S(n)$. For $X, Y \in \mathcal{P}(S(n))$, if $X \subseteq Y$, defining $X \leq Y$, then $\mathcal{P}(S(n))$ generates a geometric lattice by determinate partially ordered relation $\leq$[2]. If the partial order of $\mathcal{P}(S(n))$ is determined by reverse inclusion relation, then there are the similar results.

Theorem 2.1. *Assume that* $X, Y \in \mathcal{P}(S(n))$, *and* $X \leq Y$ *is uniquely determined by* $X \supseteq Y$. *Then* $\mathcal{P}(S(n))$ *is generated a lattice by the determinate partial order* $\leq$, *denoted by* $\mathcal{L}_R(S(n))$, *and it is a geometric lattice.*

Theorem 2.2. *Let* $n \geq 1$, *then the characteristic polynomial of lattice* $\mathcal{L}_R(S(n))$ *is*

$$\chi(\mathcal{L}_R(S(n)), t) = (t-1)^n.$$

Proposition 2.3. *Let* P *be a finite poset with maximum element* 1, *and* K *be the field of characteristic 0. Let* $\mu(x, y)$ *be the Möbius function defined in* P *and taking values from* K, $f(x)$ *be the function defined in* P *and taking values from* K. *For any* $x \in P$, *let*

$$g(x) = \sum_{x \leq y} f(y). \tag{1}$$

Then

$$f(x) = \sum_{x \leq y} g(y)\mu(x, y). \tag{2}$$

Conversely, assume that $g(x)$ is the function defined in P and taking values from K. For $x \in P$, define $f(x)$ by (2). Then (1) holds. (1) and (2) are said to be Möbius inversion of (2) and (1), respectively.

Theorem 2.4. *Let* $\mathcal{M}(S(m, n)) = \{X \in \mathcal{P}(S(n)) | |X| = m\}$. *Denote the set consisting of the intersections of subsets in* $\mathcal{M}(S(m, n))$ *by* $\mathcal{L}(S(m, n))$. *We regard the intersection of 0 subset in* $\mathcal{M}(S(m, n))$ *to be* $S(n)$. *If define partial order* $\leq$ *of* $\mathcal{L}(S(m, n))$ *by reverse inclusion relation, i.e., for* $X, Y \in \mathcal{L}(S(m, n))$, $X \leq Y \Leftrightarrow X \supseteq Y$, *then* $\mathcal{L}(S(m, n))$ *is a finite lattice, denoted by* $\mathcal{L}_R(S(m, n))$. *It's said to be lattice generated by reverse inclusion relation in* $\mathcal{M}(S(m, n))$, *and* $S(n)$ *is the minimum element of* $\mathcal{L}_R(S(m, n))$.

Proof. Clearly, $|\mathcal{L}_R(S(m, n))| < \infty$, and it is a poset generated by the determinate relation " $\leq$ ". $\cap_{X \in \mathcal{M}(S(m,n))} X$ and $S(n)$ are its maximum element and minimum element, respectively. For $X_1, X_2 \in \mathcal{L}_R(S(m, n))$, define as follows:

$$X_1 \vee X_2 = X_1 \cap X_2, \; X_1 \wedge X_2 = \cap\{Z \in \mathcal{L}_R(S(m, n)) | X_1 \cup X_2 \subseteq Z\}.$$

Since X_1 and X_2 are the intersection of elements in $\mathcal{M}(S(m, n))$, respectively, $X_1 \vee X_2 = X_1 \cap X_2 \in \mathcal{L}_R(S(m, n))$. As $X_1 \cup X_2 \subseteq S(n)$, and $S(n) \in \mathcal{L}_R(S(m, n))$, and in $\mathcal{L}_R(S(m, n))$ the intersection of elements containing $X_1 \cup X_2$ also contains $X_1 \cup X_2$, there exist exactly one element containing $X_1 \cup X_2$ in $\mathcal{L}_R(S(m, n))$, i.e., $X_1 \wedge X_2 \in \mathcal{L}_R(S(m, n))$. Thus $\mathcal{L}_R(S(m, n))$ is a finite lattice. Clearly, $S(n)$ is the minimum element of $\mathcal{L}_R(S(m, n))$. □

Similar to the results of §2.4 in [4], we have the following results

Theorem 2.5. *Let $n > m \geq 0$. $\mathcal{L}_R(S(m, n)) \supseteq \mathcal{L}_R(S(m_1, n))$ if and only if $m \geq m_1 \geq 0$.*

Theorem 2.6. *Assume that $n > m \geq 0$, then $\mathcal{L}_R(S(m, n))$ consists of $S(n)$ and all subsets of the number of elements $\leq m$ in $S(n)$.*

Proof. From definition of $\mathcal{L}_R(S(m, n))$ we deduce $S(n) \in \mathcal{L}_R(S(m, n))$, and $\forall P \in \mathcal{L}_R(S(m, n))\backslash\{S(n)\}$, P is the intersection of subsets with m elements in $S(n)$. Then $|P| \leq m$.
Contrarily, assume that P is subsets with k elements in $S(n)$, and $0 \leq k \leq m$, then

$$P \in \mathcal{M}(S(k, n)) \subseteq \mathcal{L}_R(S(k, n)).$$

By theorem 2.5,

$$\mathcal{L}_R(S(k, n)) \subseteq \mathcal{L}_R(S(m, n)).$$

Then $P \in \mathcal{L}_R(S(m, n))$. □

Corollary 2.7. *Let $n > m \geq 0$, then $\varnothing \in \mathcal{L}_R(S(m, n))$. Therefore there exists maximum element $\bigcap_{X \in \mathcal{M}(S(m,n))} X = \varnothing$ in $\mathcal{L}_R(S(m, n))$.*

Corollary 2.8. *Let $m \geq 0$. Then $\mathcal{L}_R(m, m+1) = \mathcal{L}_R(S(m+1))$.*

Theorem 2.9. *Let $1 \leq m < n$. For $X \in \mathcal{L}_R(S(m, n))$, define as follows:*

$$r'(X) = \begin{cases} m+1-|X|, & \text{if } X \neq S(n), \\ 0, & \text{if } X = S(n). \end{cases}$$

Then $r' : \mathcal{L}_R(S(m, n)) \longrightarrow \mathbb{N}_0$ is the rank function of lattice $S(m, n)$.

Theorem 2.10. *Let $0 < m < n$. Then the characteristic polynomial of $\mathcal{L}_R(S(m, n))$ is*

$$\chi(\mathcal{L}_R(S(m, n)), t) = (t-1)^{m+1} + \left(\sum_{j=0}^{m} \binom{m+1}{j} - \sum_{j=0}^{m} \binom{n}{j}\right)(t-1)^j.$$

Theorem 2.11. *Let $1 \leq m \leq n-1$. Then*

(a) $\mathcal{L}_R(S(1, n))$ and $\mathcal{L}_R(S(n-1, n))$ are the finite geometric lattice.

(b) for $2 \leq m \leq n-2$, $\mathcal{L}_R(S(m, n))$ is a finite atom lattice, but it is not a finite geometric lattice.

3 Partition lattice

For the definition of the partition lattice of a n-set, we see [2]. Obviously, the partition consisting of just the single block is unique coarsest, whereas the fineness partition is the one in which all block are singletons, and the coarsest partition and finest partition is 1 and 0, respectively. Further $\pi <\cdot\, \sigma$ if and only if σ consists of the same block as π save one pair of block of π which are joined into a single block in σ. If $0 <\cdot\, \pi$, then π is the atom of $\mathcal{P}(S)$. All ρ satisfying $\pi \le \rho \le \sigma$ is wrote $[\pi, \sigma]$, and is called the interval with π, σ for the endpoint, for simplicity, is said to be interval .

Proposition 3.1. [2] *Let* $S = \{s_1, s_2, \cdots, s_n\}$. *Then* $[\pi, 1] \simeq \mathcal{P}(b(\pi))$.

Proposition 3.2. [2] *Assume that* $S = \{s_1, s_2, \cdots, s_n\}$, $\pi \in \mathcal{P}(n)$. *Let*

$$r : \mathcal{P}(n) \longrightarrow \mathbb{N}_0$$
$$\pi \longmapsto r(\pi) = n - b(\pi).$$

Then r *is the rank function of* $\mathcal{P}(n)$.

Proposition 3.3. [2] *Assume that* $n \ge 2$. *Then the characteristic polynomial of* $\mathcal{P}(n)$ *is*

$$\chi(\mathcal{P}(n), t) = (t-1)(t-2)\cdots(t-n+1).$$

Let (π) be equivalent a relation associating with the partition π of S, $\forall \pi, \sigma \in \mathcal{P}(S)$, $a, b \in S$,

$$a(\pi \wedge \sigma)b \Longleftrightarrow a(\pi)b \text{ and } a(\sigma)b, a(\pi \vee \sigma)b \Longleftrightarrow \exists\, a = u_0, u_1, \cdots, u_t = b,$$

such that $u_i(\pi)u_{i+1}$ or $u_i(\sigma)u_{i+1}$ for all $i\,(0 \le i \le t-1)$. Hence $\mathcal{P}(n)$ is a finite lattice, and it is a geometric lattice ([2]). Assume that $S = \{s_1, s_2, \cdots, s_n\}$, and the set consisting of all $k(1 \le k \le n)-$part partitions in $\mathcal{P}(n)$ is denoted by $\mathcal{M}(\mathcal{P}(n,k))$, and the set generated by the intersection of subsets in $\mathcal{M}(\mathcal{P}(n, k))$ is denoted by $\mathcal{L}(\mathcal{P}(n, k))$. We agree that the intersection of 0 subset in $\mathcal{M}(\mathcal{P}(n, k))$ to be 1, i.e., $1 \in \mathcal{L}(\mathcal{P}(n,k))$. We use the part order relation $\le$ in $\mathcal{P}(n)$ to define the partial order relation $\le$ in $\mathcal{L}(\mathcal{P}(n, k))$. Clearly, $\mathcal{L}(\mathcal{P}(n, k))$ is a finite poset, and $\cap_{X \in \mathcal{M}(\mathcal{P}(n,k))} X$ and 1 are its the minimum element and the maximum element, respectively . For $\pi, \sigma \in \mathcal{L}(\mathcal{P}(n, k))$, then $\pi, \sigma \in \mathcal{L}(\mathcal{P}(n))$. Write $\wedge$ and $\vee$ for the intersection and union in $\mathcal{L}(\mathcal{P}(n))$, respectively. In $\mathcal{L}(\mathcal{P}(n, k))$, we define as follows:

$$\pi \wedge_1 \sigma := \pi \wedge \sigma,\ \pi \vee_1 \sigma := \cap\{\rho \in \mathcal{L}(\mathcal{P}(n, k)) | \rho \ge \pi \vee \sigma\}.$$

Since $\wedge$ is closed in $\mathcal{L}(\mathcal{P}(n))$, $\wedge_1$ is closed in $\mathcal{L}(\mathcal{P}(n, k))$. Consider that $\pi \vee \sigma \in \mathcal{L}(\mathcal{P}(n))$, $1 \ge \pi \vee \sigma$ and $1 \in \mathcal{L}(\mathcal{P}(n, k))$, $\{\rho \in \mathcal{L}(\mathcal{P}(n, k)) | \rho \ge \pi \vee \sigma\}$ is non-empty, and $\cap\{\rho \in \mathcal{L}(\mathcal{P}(n, k)) | \rho \ge \pi \vee \sigma\}$ containing π, σ is minimum in $\mathcal{L}(\mathcal{P}(n, k))$, i.e., for $\pi, \sigma \in \mathcal{L}(\mathcal{P}(n, k))$, $a, b \in S$, $\exists a = u_0, u_1, \cdots, u_t = b$, such that $u_i(\pi)u_{i+1}$ or $u_i(\sigma)u_{i+1}$ $(i = 0, 1, \cdots, t-1)$ holds. Then $a(\pi \vee_1 \sigma)b$ contains in a block of the partition of $\mathcal{L}(\mathcal{P}(n, k))$. Hence $\vee_1$ is closed in $\mathcal{L}(\mathcal{P}(n, k))$, and satisfy conditions of the lattice. Thus $\mathcal{L}(\mathcal{P}(n, k))$ is a finite lattice. In the following, always denote $\wedge_1$ and $\vee_1$ by $\wedge$ and $\vee$, respectively.

Theorem 3.4. *Assume that* $n \ge m \ge 2$

$$\mathcal{L}(\mathcal{P}(n, m_1)) \subseteq \mathcal{L}(\mathcal{P}(n, m)) \tag{3}$$

if only and if

$$n \ge m_1 \ge m. \tag{4}$$

Proof. We first prove sufficiency. When $m_1 = 2$, it is clear that (3) holds. In the following let $m_1 \geq 3$. At first, we prove

$$\mathcal{L}(\mathcal{P}(n,\, m_1)) \subseteq \mathcal{L}(\mathcal{P}(n,\, m_1 - 1)). \tag{5}$$

Let $\sigma \in \mathcal{M}(\mathcal{P}(n,\, m_1))$, i.e., σ is a m_1-part partition of S. Without loss of generality, assume that

$$\sigma = S_1|S_2|\cdots|S_{m_1}.$$

Let

$$\sigma_1 = S_1 \cup S_2|\cdots|S_{m_1},\ \sigma_2 = S_1 \cup S_3|S_2|S_4|\cdots|S_{m_1}.$$

Since $m_1 \geq 3$, there exist the partitions σ_1 and σ_2 of S and $\sigma_1, \sigma_2 \in \mathcal{M}(\mathcal{P}(n,\, m_1 - 1))$. Therefore $\sigma = \sigma_1 \wedge \sigma_2 \in \mathcal{L}(\mathcal{P}(n, m_1 - 1))$. Then (5) holds.

Now let us prove (3). When $m = m_1$, clearly, (3) holds. Let $m_1 > m$. From (5), we obtain

$$\mathcal{L}(\mathcal{P}(n,\, m_1)) \subseteq \mathcal{L}(\mathcal{P}(n,\, m_1 - 1)) \subseteq \cdots \subseteq \mathcal{L}(\mathcal{P}(n,\, m)).$$

Then (3) holds.

Next we prove necessity. For any $\sigma \in \mathcal{M}(\mathcal{P}(n,\, m_1))$, then σ is a m_1-part partition of S, and

$$\sigma \in \mathcal{L}(\mathcal{P}(n,\, m_1)) \subseteq \mathcal{L}(\mathcal{P}(n,\, m)).$$

Hence σ is the intersection of m-partial partitions in $\mathcal{L}(\mathcal{P}(n,\, m))$. Notice that the partition of the intersection is finer than those partitions of the partitions in the intersection. Therefore the partition σ is finer than m-part partition. Hence $n \geq m_1 \geq m$, i.e., (4) holds. □

Theorem 3.5. *Assume that $n \geq m \geq 2$. Then $\mathcal{L}(\mathcal{P}(n,\, m))$ consists of 1 and k-part partitions of S, where $n \geq k \geq m$.*

Proof. According to the definition of $\mathcal{L}(\mathcal{P}(n,\, m))$, we have $1 \in \mathcal{L}(\mathcal{P}(n,\, m))$. For $\sigma \in \mathcal{L}(\mathcal{P}(n,\, m))\backslash\{1\}$, σ is the intersection of m-partial partitions of S. Hence the partitions in $\mathcal{L}(\mathcal{P}(n,\, m))\backslash\{1\}$ is not more coarse than the m-partial partitions. If σ is a k-partial partition, then $n \geq k \geq m$.

Conversely, assume that ρ is a k-part partition of S, $n \geq k \geq m$. Then $\rho \in \mathcal{M}(\mathcal{P}(n,\, k)) \subseteq \mathcal{L}(\mathcal{P}(n,\, k))$. By Theorem 3.4,

$$\mathcal{L}(\mathcal{P}(n,\, k)) \subseteq \mathcal{L}(\mathcal{P}(n,\, m)),$$

Hence $\rho \in \mathcal{L}(\mathcal{P}(n,\, m))$.

Clearly, when $n \geq m \geq 2$, 1 is the maximum element of $\mathcal{L}(\mathcal{P}(n,\, m))$. □

But for the minimum element of $\mathcal{L}(\mathcal{P}(n,\, m))$, we have

Corollary 3.6. *Assume that $n \geq m \geq 2$. Then*

$$0 \in \mathcal{L}(\mathcal{P}(n,\, m)),$$

and the minimum element of $\mathcal{L}(\mathcal{P}(n,\, m))$ is $\cap_{X \in \mathcal{M}(\mathcal{P}(m,\, n))} X = 0$.

Corollary 3.7. *Assume that $n \geq 2$. Then $\mathcal{L}(\mathcal{P}(n,\, 2)) = \mathcal{L}(n)$.*

Theorem 3.8. *Let $n \geq m \geq 2$. For $\pi \in \mathcal{L}(\mathcal{P}(n,\, m))$, we define as follows*

$$r(\pi) = \begin{cases} n - b(\pi), & \text{if } \pi \neq 1, \\ n - m + 1 & \text{if } \pi = 1. \end{cases}$$

Then $r : \mathcal{L}(\mathcal{P}(n,\, m)) \longrightarrow \mathbb{N}_0$ is the rank function of $\mathcal{L}(\mathcal{P}(n,\, m))$.

Proof. Clearly, $r(0) = 0$. For $\pi, \sigma \in \mathcal{L}(S(n,\, m))$, and $\pi \lessdot \sigma$. When $\sigma \neq 1$, we have $b(\pi) = b(\sigma) + 1$. Hence

$$r(\sigma) = n - b(\sigma) = n - b(\pi) + 1 = r(\pi) + 1.$$

When $\sigma = 1$, we have $b(\sigma) = 1$, $b(\pi) = m$. Therefore

$$r(\sigma) = n - m + 1 = r(\pi) + 1.$$

By Definition 1, $r : \mathcal{L}(\mathcal{P}(n, m)) \to \mathbb{N}_0$ is the rank function of $\mathcal{L}(\mathcal{P}(n, m))$.□
Now we discuss the characteristic polynomial of $\mathcal{L}(\mathcal{P}(n,\, m))$. We have the following the proposition.

Proposition 3.9. [2] *Assume that $S = \{s_1,\, s_2,\, \cdots,\, s_n\}$ and $\pi \in \mathcal{P}(n)$, $\pi = S_1|S_2|\cdots|S_{b(\pi)}$, where $|S_i| = n_i$, $\sum_{i=1}^{b(\pi)} n_i = n$. Then*

$$[0,\, \pi] \simeq \prod_{i=1}^{b(\pi)} \mathcal{P}(n_i).$$

Proposition 3.10. [2] *Assume that L_1 and L_2 are two finite lattices, and characteristic polynomials are $\chi(L_1,\, t)$ and $\chi(L_2,\, t)$, respectively. Then the characteristic polynomial of $L_1 \times L_2$ is*

$$\chi(L_1 \times L_2,\, t) = \chi(L_1,\, t) \times \chi(L_2,\, t).$$

Theorem 3.11. *Assume that $S = \{s_1,\, s_2,\, \cdots,\, s_n\}$, π is a $(m-1)$-part partition of S, i.e., $\pi = S_1|S_2|\cdots|S_{m-1}$, and $|S_i| = n_i$. Let $\sigma \in [0,\, \pi]$ be a k-part partition, and σ_i is the k_i-part partition corresponding to σ in S_i. Then the number of k-part partitions of S in $[0,\, \pi]$ is $\prod_{i=1}^{m-1} S(n_i,\, k_i)$, where $S(n_i,\, k_i)$ is Stirling number of the second kind (see [2] or [6]).*

Proof. Since $\pi \in \mathcal{L}(\mathcal{P}(n))$, $\pi = S_1|S_2|\cdots|S_{m-1}$, and $|S_i| = n_i$, $i = 1, 2, \cdots, m-1$, for $\sigma \in [0,\, \pi]$, the partition σ_i in S_i corresponding to σ is a k_i-part partition, and the number of k_i-partial partitions in S_i is $S(n_i,\, k_i)$. Hence the number of k-part partitions of S in $[0,\, \pi]$ is $\prod_{i=1}^{m-1} S(n_i,\, k_i)$, where the representative of $S(n_i,\, k_i)$ is given by [2] or [6]. □

Assume that $S = \{s_1,\, s_2,\, \cdots,\, s_n\}$, π_i is a $(m-1)$-part partition of S, $i = 1, 2, \cdots$, $\mathcal{A}_i$ be the set generated by all m-partial partition of S contained in π_i. Let $\mathcal{L}(\mathcal{A}_i)$ be the set generated by the intersections of partition in $\mathcal{A}_i$, and we agree that $\pi_i \in \mathcal{L}(\mathcal{A}_i)$. For $\pi, \sigma \in \mathcal{L}(\mathcal{A}_i)$, we define

$\pi \leq \sigma \Longleftrightarrow \pi \subseteq \sigma$. It is easy to verify that $\leq$ is the partially ordered relation of $\mathcal{L}(\mathcal{A}_i)$, and it forms a lattice, denoted by $\mathcal{L}_O(\mathcal{A}_i)$. For simplicity we write $\mathcal{L}_0 = \mathcal{L}(\mathcal{P}(n, m))$, $\mathcal{L}_i = \mathcal{L}_O(\mathcal{A}_i)$. For $\pi \in \mathcal{L}_0$ and $\pi \in \mathcal{L}_i$, we define that

$$\mathcal{L}_0^{\pi} = \{\sigma \in \mathcal{L}_0 | \sigma \leq \pi\} \text{ and } \mathcal{L}_i^{\pi} = \{\sigma \in \mathcal{L}_i | \sigma \leq \pi\},$$

respectively. Clearly, $\mathcal{L}_0^1 = \mathcal{L}_0$, $\mathcal{L}_i^{\pi_i} = \mathcal{L}_i$. For $\pi \in \mathcal{L}_0 \backslash \{1\}$, there is $\pi \in \mathcal{L}_j$, $j = 1, 2 \cdots$. Without loss of generality let $j = 1$, then $\mathcal{L}_0^{\pi} = \mathcal{L}_1^{\pi}$. Assume that π_l is a $l-$part partition of S, i.e., $\pi_l = S_{l1}|S_{l2}| \cdots |S_{lb(\pi_l)}$, where $|S_{li}| = n_{li}$, $i = 1, 2, \cdots, b(\pi_l)$, $\sum_{i=1}^{b(\pi_l)} n_{li} = n$. From Proposition 3.9 we deduce $[0, \pi_l] \simeq \prod_{i=1}^{b(\pi_l)} \mathcal{P}(n_i)$. Since $\mathcal{L}_0^{\pi_l} = [0, \pi_l]$, $\mathcal{L}_0^{\pi_l} \simeq \prod_{i=1}^{b(\pi_l)} \mathcal{P}(n_{li})$. From Proposition 3.3, we have $\chi(\mathcal{P}(n_i), t) = (t-1)(t-2)\cdots(t-n_i+1)$. Again from Proposition 3.10, we have

$$\chi(\mathcal{L}_0^{\pi_l}, t) = \prod_{i=1}^{b(\pi_l)} (t-1)(t-2)\cdots(t-n_i+1). \tag{6}$$

By definition in Theorem 3.8, $r : \mathcal{L}_0 \to \mathbb{N}_0$ is the rank function of lattice $\mathcal{L}_0 = \mathcal{L}(\mathcal{P}(n, m))$, and 1 and 0 are its maximum element and minimum element, respectively. Then

$$\chi(\mathcal{L}_0^1, t) = \sum_{\sigma \in \mathcal{L}_0^1} \mu(0, \sigma) t^{r(1)-r(\sigma)}.$$

By Möbius inversion

$$t^{n-m+1} = \sum_{\sigma \in \mathcal{L}_0^1} \chi(\mathcal{L}^{\sigma}, t) = \sum_{\sigma \in \mathcal{L}_0} \chi(\mathcal{L}^{\sigma}, t). \tag{7}$$

For $\pi \in \mathcal{L}_1$, define as follows

$$r(\pi) = \begin{cases} n - b(\pi), & \text{if } \pi \neq \pi_i, \\ n - m + 1, & \text{if } \pi = \pi_i. \end{cases}$$

Then $r : \mathcal{L}_1 \to \mathbb{N}_0$ is the rank function of lattice$\mathcal{L}_1$, and its functional value is the same as corresponding to the functional value in $\mathcal{L}_0$. Since π_i and 0 are its maximum element and minimum element, respectively,

$$\chi(\mathcal{L}_1^{\pi_i}, t) = \sum_{\sigma \in \mathcal{L}_1^{\pi_i}} \mu(0, \sigma) t^{r(1)-r(\sigma)}.$$

By Möbius inversion

$$t^{n-m+1} = \sum_{\sigma \in \mathcal{L}_1^{\pi_i}} \chi(\mathcal{L}_1^{\pi}, t) = \sum_{\sigma \in \mathcal{L}_1} \chi(\mathcal{L}_1^{\sigma}, t). \tag{8}$$

Then by (7) and (8), there exists

$$\chi(\mathcal{L}_0, t) = \sum_{\sigma \in \mathcal{L}_1} \chi(\mathcal{L}_1^{\sigma}, t) - \sum_{\sigma \in \mathcal{L}_0 \backslash \{1\}} \chi(\mathcal{L}^{\sigma}, t).$$

Notice that the number of $k-$part partitions of S is $S(n, k), k = 1, 2, \cdots, n$. Then the number of $l-$part partition in $\mathcal{L}_0$ is $S(n, l), l = n, n-1, \cdots, m$, where $\pi_l = S_{l1}|S_{l2}| \cdots ||S_{lb(\pi_l}, |S_{li}| = n_{li}$ $(i = 1, 2, \cdots, b(\pi_l))$, and $\chi(\mathcal{L}_0^{\pi_l}, t)$

is given by (6). Let the k-part partition in $\mathcal{L}_1$ $\sigma = S_1|S_2|\cdots|S_k, |S_i| = n_{ki}\,(i = 1, 2, \cdots, k))$, σ_i is a k_i-part partition corresponding to σ in S_i and the number of σ_i in S_i is $S(n_{ki}, k_i)$, and

$$\chi(\mathcal{L}_1^{\sigma_i}, t) = \prod_{i=1}^{b(\sigma_i)} (t-1)(t-2)\cdots(t-n_{ki}+1).$$

Since the number of k-part partitions in $\mathcal{L}_1$ is the number of k-part partitions of S in in $[0, \pi]$, by Theorem 3.11, the number of k-partial partitions in $\mathcal{L}_1$ is $\prod_{i=1}^{m-1} S(n_i, k_i)$.
Hence we can deduce

Theorem 3.12. *Let $n > m$. Then*

$$\begin{aligned}
&\chi(\mathcal{L}_0(\mathcal{P}(n, m), t)) = \chi(\mathcal{L}_0, t)\\
&= \sum_{k=m-1}^{n} \prod_{i=1}^{k} S(n_{li}, k_i) \prod_{j=1}^{b(\sigma_k)} (t-1)(t-2)\cdots(t-n_{kj}+1)\\
&- \sum_{l=m}^{n} S(n, l) \prod_{i=1}^{b(\pi_l)} (t-1)(t-2)\cdots(t-n_{li}+1).
\end{aligned}$$

References

[1] G. Birkhoff , Lattice Theory(3rdedition). Prouidence, Amer.Math.soc. coll.Publ. Vol.25, 1967.

[2] M. Aigner, Combinatorial Theory. Berlin: Springer-Verlag, 1979.

[3] Y. Huo, Z. wan, On the geometricity of lattices generated by orbits of Subspaces under finite Classical groups, J.of Algebra 243(2001), 339–359.

[4] Z. Wan, Y. Huo, Lattices generated by orbits of subspaces of classical groups in finite fields(second edition)(Chinese). Beijing: Science Press in Chinese, 2004.

[5] Z. Wan, Geometry of classical groups over finite field, Studentlitteratur, Lund, Sweden, 1993.

[6] J.H.van Lint, R.M.Wilson, A Course in Combinatorics (second edition). China Machine Press, Beijing, 2004.

[7] P.Orlik, L.Solomon, Arrangements in unitary and orthogonal geometry over finite fields. J.Combin. Theory Ser.A 38(1985), 217–229.