On the number of genus embeddings of complete tripartite graph $K_{n,n,l}$*

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Abstract On the basis of lit.[9], by the joint tree model, the lower bound of the number of genus embeddings for complete tripartite graph $K_{n,n,l}$ $(l \geq n \geq 1)$ is got.

Keywords: minimum genus, orientable surface, orientable embedding, joint tree

1 Introduction

The graph genus problem is NP-complete as shown by Thomassen [10]. The solution of genus embedding for complete graph K_n was a long and different way and gave birth to modern topological graph theory. Until now, for certain special classes of graphs only with good symmetry, formulae for their genera have been given. For example, complete graph K_n, complete bipartite graph $K_{m,n}$ etc. And for complete tripartite graphs, only partial results for the genera of them are known. White conjectured the orientable genus of the complete tripartite graph $K_{l,m,n}$ is $\lceil \frac{(l-2)(m+n-2)}{4} \rceil$ and proved it is true for $K_{l,m,n}$, where $m+n \leq 6$, and for $K_{mn,n,n}$, where $m, n \in N$. Ringel and Youngs also proved the orientable conjecture for $K_{n,n,n}$. Stahl and White proved it holds for $K_{n,n,n-2}$ when $n \geq 2$ is even, and for $K_{2n,2n,n}$ when $n \geq 1$. In 1998, Craft proved that the orientable conjecture holds for $K_{2l,m,n}$ when $l \geq m+n-2$ and $m+n$ is even. And there were surprisingly few results about the number of genus embeddings of graphs[6]. In 2003, Liu[4] set up the joint tree model, so that the corresponding relation was established between the joint trees and the embeddings. Based on it, the minimum genera of further types of graphs, especially complete tripartite graph $K_{n,n,l}(l \geq n \geq 1)$, have been got, such as [7-9].

In this article, further, on the basis of the joint tree model, by dividing the associated surfaces into segments layer by layer and doing a sequence of exchangers in the layer division, we study the number of genus embeddings

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for complete tripartite graph $K_{n,n,l}(l \geq n \geq 1)$ and find a lower bound for such numbers.

A *surface* is a compact 2-dimensional manifold without boundary. Since it can be obtained by identifying each of pairs of edges along a given direction on a polygon with even number of edges, throughout this paper, an *orientable surface* is regarded as a cyclic order P of letters such that both a and a^- occur once on P for each $a \in P$.

Use $o(S)$ to denote the genus of surface S. Let $\mathcal{S}$ be the set of orientable surfaces and let $o_{\min}(\mathcal{S}) = \min_{S \in \mathcal{S}} o(S)$. By an *elementary transformation*[3] on S, we shall mean one of the following three operations and their inverses:

El.0 $\forall S \in \mathcal{S}$, $S = Aaa^-B$, $A \neq 0$, or $B \neq 0 \Longrightarrow S = AB$.

El.1 $\forall S \in \mathcal{S}$, $S = AabBb^-a^-C \Longrightarrow S = AaBa^-C$.

El.2 $\forall S \in \mathcal{S}$, $S = AaBCa^-D \Longrightarrow S = BaADa^-C$.

If one of two surfaces S_1 and S_2 can be transformed into another by a series of elementary transformations, then we say that S_1 and S_2 are *equivalent*[3], denoted by $S_1 \sim S_2$. And S_1 and S_2 have same orientability and genus.

Lemma 1.1[3]. *For* a, b, a^-, b^- *not belonging to* $\{A, B, C, D, E\}$, *we have*

$$AaBbCa^-Db^-E \sim ADCBEaba^-b^-.$$

Proof According to El.0-El.2,

$$AaBbCa^-Db^-E \sim BaADb^-Ea^-bC \sim Eb^-BaADCba^-$$

$$\sim ADCaEb^-Ba^-b \sim EaADCba^-b^-B \sim aADCba^-b^-BE$$

$$\sim baADCBEb^-a^- \sim ADCBEb^-a^-ba \sim ADCBEaba^-b^-.$$

According to the three operations and Lemma 1.1, it can be seen that each orientable surface is equivalent to only one of the following canonical forms:

$$O_k = \begin{cases} a_0a_0^-, & \text{if the genus of a surface is 0;} \\ \prod_{i=1}^{k} a_ib_ia_i^-b_i^-, & \text{if the genus of a surface is } k. \end{cases}$$

Lemma 1.2[4]. *Let* S_1 *and* S_2 *be surfaces,* $a, b, a^-, b^- \notin S_2$. *If* $S_1 \sim S_2aba^-b^-$, *then* $o(S_1) = o(S_2) + 1$.

An *embedding* of a graph G into a surface S is a homeomorphism τ: $G \to S$, such that each component of $S - \tau(G)$ is an open disc. The embedding is called *orientable* if S is orientable. Throughout this article, whenever we use the term embedding, we are referring to an orientable embedding. Two embeddings τ_1: $G \to S$ and τ_2: $G \to S$ of a graph G

in the orientable surface S are *equivalent* if there is a homeomorphism h: $S \to S$ such that $h\tau_2 = \tau_1$. The *minimum genus* $\gamma(G)$ of a graph G is minimum genus of the surface into which G has an orientable embedding. Throughout this paper, "number of embedding" really means number of equivalence class of embeddings.

In what follows, the joint tree model is introduced.

Given a graph $G = (V, E)$, suppose that a subgraph T of G is a tree. Then T is called a *spanning tree* of G if $V(T) = V(G)$. $(V, E\backslash E(T))$ is called a *cotree* of G, and the number of cotree edges is called *Betti number*, denoted by β.

For a spanning tree T of G, split the cotree edge (u_i, v_i) into two semi-edges $(u_i, \bar{u}_i)$ and $(v_i, \bar{v}_i)$, which are, respectively, incident with u_i and v_i for $1 \leq i \leq \beta$ to obtain a new tree $\hat{T} = (V + V_1, E(T) + E_1)$, where $E_1 = \{(u_i, \bar{u}_i), (v_i, \bar{v}_i) | 1 \leq i \leq \beta\}$ and $V_1 = \{\bar{u}_i, \bar{v}_i | 1 \leq i \leq \beta\}$. Let $\sigma = \{\sigma_v | \ v \in V(G)\}$ be a rotation of G, where σ_v is a cyclic permutation of edges incident with v. Then $\hat{T}_\sigma$ determines an embedding of $\hat{T}$ on the plane.

Write $\hat{T}^\dagger$ instead of $\hat{T}$, when edges $(u_i, \bar{u}_i)$ and $(v_i, \bar{v}_i)$ are labeled by a same letter with different indices: $+$(always omitted) or $-$ for $1 \leq i \leq \beta$. The tree $\hat{T}^\dagger$ with a rotation σ of G is called a *joint tree*[5] of G, denoted by $\hat{T}^\dagger_\sigma$.

On $\hat{T}^\dagger_\sigma$, according to a given orientation (clockwise or counterclockwise), write down the letters used to denote semi-edges, then a cyclic order of 2β letters is got, and called an associated surface[5] of G. Two associated surfaces are *the same* is meant that they have the same cyclic order. Otherwise, distinct.

As shown K_4 in Fig.1.1 and a joint tree in Fig.1.2, an associated surface of K_4 is $aba^-cc^-b^-$.

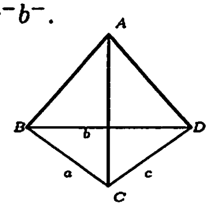

Fig.1.1 K_4

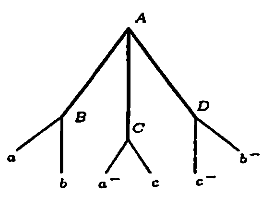

Fig.1.2 A joint tree of K_4

From[5], for a fixed spanning tree T of the graph G, there is a 1-to-1 correspondence between the associated surfaces and the embeddings of G. It is soon seen that an embedding can be represented by a joint tree, further by an associated surface.

Let G_σ be the embedding determined by a rotation σ. We will refer to $P_{\hat{T}^\dagger_\sigma}$ as the associated surface corresponding to $\hat{T}^\dagger_\sigma$. Let $(\Sigma, \hat{T}^\dagger) = \{P_{\hat{T}^\dagger_\sigma} | \sigma \in$

$\Sigma\}$ where Σ is the set of all rotation for G.

Lemma 1.3[4]. *For any $\sigma_1 \neq \sigma_2$, the embeddings G_{σ_1} and G_{σ_2}, as well as $\hat{T}_{\sigma_1}$ and $\hat{T}_{\sigma_2}$ are distinct.*

Lemma 1.4[4]. *Let T_1, T_2 be two spanning trees of G. There is a bijection between $(\Sigma, \hat{T}_1^{\dagger})$ and $(\Sigma, \hat{T}_2^{\dagger})$.*

Based on the lemmas above, the topological problem for enumerating the number of genus embeddings of a graph is transformed into a combinatorial problem for counting the number of distinct associated surfaces of the graph. To find the number of genus embeddings for complete tripartite graphs, it suffices to calculate the number of all distinct associated surfaces of them in the equivalent class of minimum genus.

In order to obtain the associated surface of minimum genus, a layer division of an associated surface is defined for establishing an operation to transform this surface into another associated surface. Given a spanning tree T of G and $v^0 \in T$ with $d_T(v^0) > 1$. Any vertex of T and v^0 are connected by a unique path. Then divide the associated surface S of G into segments layer by layer. The 0th layer contains only one segment, i.e. S, denoted by S_{v^0}. Suppose that $N(v^0) = \{v_1^1, v_2^1, \cdots, v_{l_1}^1\}$. The 1st layer is obtained by dividing S_{v^0} into l_1 segments, i.e., $S_{v^0} = \langle S_{v_1^1} S_{v_2^1} \cdots S_{v_{l_1}^1} \rangle$, where $S_{v_j^1} (1 \leq j \leq l_1)$ is called a layer segment incident with v^0. Suppose that $k-1$st layer is denoted $\langle S_{v_1^{k-1}} S_{v_2^{k-1}} \cdots S_{v_{l_{k-1}}^{k-1}} \rangle$. For $S_{v_j^{k-1}}$, we assume that

$$N(v_j^{k-1}) \cap \{u \mid d(u, v^0) = k,\ u \in T\} = \{v_{t_{j-1}+1}^k, v_{t_{j-1}+2}^k, \cdots, v_{t_j}^k\},$$

where $t_0 = 0, t_{l_k} = l_k$. Then $S_{v_j^{k-1}} = \langle S_{v_{t_{j-1}+1}^k} S_{v_{t_{j-1}+2}^k} \cdots S_{v_{t_j}^k} \rangle$, where $S_{v_{t_{j-1}+m}^k} (1 \leq m \leq t_j - t_{j-1})$ is called a layer segment incident with vertex v_j^{k-1}. At this time, the kth layer is obtained by dividing each k-1st layer segment as

$$\langle S_{v_1^k} S_{v_2^k} \cdots S_{v_{t_1}^k} S_{v_{t_1+1}^k} S_{v_{t_1+2}^k} \cdots S_{v_{t_2}^k} \cdots S_{v_{t_{(j-1)}+1}^k} \cdots S_{v_{t_j}^k} \cdots S_{v_{t_{(l_k-1)}+1}^k}$$
$$S_{v_{t_{(l_k-1)}+2}^k} \cdots S_{v_{l_k}^k} \rangle.$$

Each layer segment except for the layer segment S_{v_x} which has only one element can be divided into other layer segments in the same way, where v_x is a vertex of T satisfying that $d_T(v_x) = 1$ and S_{v_x} is constituted by those letters used to label the semi-edges incident with v_x. According to a rotation system of G, the positions of any two layer segments incident with a same vertex in a same layer can be interchanged, and every interchange makes an associated surface S be transformed into another associated surface S'. And the operation in a layer division is called an *exchanger*[5], denoted by $S \to S'$. Hence the associated surface of minimum genus can be got by doing a sequence of exchangers on any associated surface.

Note that $\langle\rangle$ used to denote a layer segment does not change the associated surface itself, and is always omitted when the layer segment contains only one letter.

A spanning tree of $K_{2,2,4}$ is represented with thick lines as shown in Fig.1.3 and a joint tree of $K_{2,2,4}$ in Fig.1.4. Denote the cotree edge u_iv_j, w_kv_j and w_ku_2 by b_j^{i-1}, a_{j+1}^k and a_1^k, respectively, for $i,j = 1,2$ and $k = 1,2,3$. Let joint trees of $K_{2,2,4}$ have an anticlockwise rotation at each vertex. Then an associated surface is shown as

$$S = a_1^1a_1^2a_1^3b_1^1b_2^1a_2^1a_2^2a_2^3b_1^{0-}b_1^{1-}a_3^1a_3^2a_3^3b_2^{0-}b_2^{1-}b_1^0b_2^0$$
$$a_1^{1-}a_2^{1-}a_3^{1-}a_1^{2-}a_2^{2-}a_3^{2-}a_1^{3-}a_2^{3-}a_3^{3-}.$$

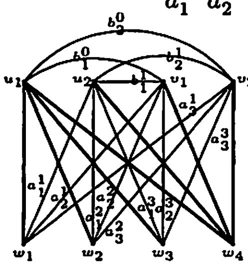

Fig.1.3 $K_{2,2,4}$

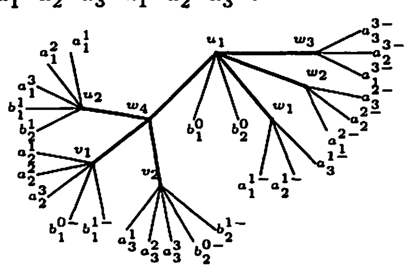

Fig.1.4 A joint tree of $K_{2,2,4}$

Suppose that $S = \langle S_{u_1}\rangle$. Then

$$S = \langle S_{u_1}\rangle = \langle S_{w_4}b_1^0b_2^0S_{w_1}S_{w_2}S_{w_3}\rangle = \langle\langle S_{u_2}S_{v_1}S_{v_2}\rangle b_1^0b_2^0S_{w_1}S_{w_2}S_{w_3}\rangle,$$

where

$$S_{u_2} = \langle a_1^1a_1^2a_1^3b_1^1b_2^1\rangle,\ S_{v_1} = \langle a_2^1a_2^2a_2^3b_1^{0-}b_1^{1-}\rangle,\ S_{v_2} = \langle a_3^1a_3^2a_3^3b_2^{0-}b_2^{1-}\rangle,$$
$$S_{w_i} = \langle a_1^{i-}a_2^{i-}a_3^{i-}\rangle \text{ for } i = 1,2,3.$$

Obviously, the associated surface can also be shown as

$$S = \langle\langle a_1^1a_1^2a_1^3b_1^1b_2^1\rangle\langle a_2^1a_2^2a_2^3b_1^{0-}b_1^{1-}\rangle\langle a_3^1a_3^2a_3^3b_2^{0-}b_2^{1-}\rangle\rangle\langle b_1^0b_2^0\langle a_1^{1-}a_2^{1-}a_3^{1-}\rangle$$
$$\langle a_1^{2-}a_2^{2-}a_3^{2-}\rangle\langle a_1^{3-}a_2^{3-}a_3^{3-}\rangle\rangle.$$

After doing a series of exchangers on S,

$$S \to S' = \langle b_1^{0-}a_2^3a_2^2a_2^1b_1^{1-}\rangle\langle b_1^1a_1^1a_1^2a_1^3b_2^1\rangle\langle b_2^{1-}a_3^3a_3^2a_3^1b_2^{0-}\rangle b_2^0\langle a_3^{1-}a_1^{1-}a_2^{1-}\rangle$$
$$\langle a_2^{2-}a_1^{2-}a_3^{2-}\rangle\langle a_3^{3-}a_1^{3-}a_2^{3-}\rangle b_1^0,$$

namely an associated surface of minimum genus. Using the same method as above, all associated surfaces of minimum genus for $K_{2,2,4}$ can be obtained.

$$S_{11} = \langle b_1^{0-} a_2^3 a_2^2 a_2^1 b_1^{1-}\rangle\langle b_1^1 a_1^1 a_1^2 a_1^3 b_2^1\rangle\langle b_2^{1-} a_3^3 a_3^2 a_3^1 b_2^{0-}\rangle\langle b_2^0\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle$$

$$\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\rangle b_1^0$$

$$\sim \langle a_2^3 a_2^2 a_2^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_3^3 a_3^2 a_3^1\rangle\langle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{12} = \langle a_2^1 a_2^2 a_2^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_3^1 a_3^2 a_3^3\rangle\langle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{13} = \langle a_2^2 a_2^3 a_2^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_3^2 a_3^3 a_3^1\rangle\langle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{14} = \langle a_2^3 a_2^1 a_2^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_3^3 a_3^1 a_3^2\rangle\langle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{15} = \langle a_2^1 a_2^3 a_2^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_3^1 a_3^3 a_3^2\rangle\langle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{16} = \langle a_2^2 a_2^1 a_2^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_3^2 a_3^1 a_3^3\rangle\langle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{21} = \langle a_2^3 a_2^1 a_2^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_3^3 a_3^2 a_3^1\rangle\langle\langle a_3^{1-} a_2^{1-} a_1^{1-}\rangle\langle a_1^{2-} a_2^{2-} a_3^{2-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{22} = \langle a_2^2 a_2^1 a_2^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_3^2 a_3^3 a_3^1\rangle\langle\langle a_3^{1-} a_2^{1-} a_1^{1-}\rangle\langle a_1^{3-} a_2^{3-} a_3^{3-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{23} = \langle a_2^1 a_2^2 a_2^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_3^1 a_3^3 a_3^2\rangle\langle\langle a_3^{2-} a_2^{2-} a_1^{2-}\rangle\langle a_1^{3-} a_2^{3-} a_3^{3-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{24} = \langle a_2^3 a_2^2 a_2^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_3^3 a_3^1 a_3^2\rangle\langle\langle a_3^{2-} a_2^{2-} a_1^{2-}\rangle\langle a_1^{1-} a_2^{1-} a_3^{1-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{25} = \langle a_2^2 a_2^3 a_2^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_3^2 a_3^1 a_3^3\rangle\langle\langle a_3^{3-} a_2^{3-} a_1^{3-}\rangle\langle a_1^{1-} a_2^{1-} a_3^{1-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{26} = \langle a_2^1 a_2^3 a_2^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_3^1 a_3^2 a_3^3\rangle\langle\langle a_3^{3-} a_2^{3-} a_1^{3-}\rangle\langle a_1^{2-} a_2^{2-} a_3^{2-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{31} = \langle a_2^3 a_2^2 a_2^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_3^2 a_3^3 a_3^1\rangle\langle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{2-} a_3^{2-} a_1^{2-}\rangle\langle a_1^{3-} a_3^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{32} = \langle a_2^2 a_2^3 a_2^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_3^3 a_3^2 a_3^1\rangle\langle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{3-} a_3^{3-} a_1^{3-}\rangle\langle a_1^{2-} a_3^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{33} = \langle a_2^1 a_2^3 a_2^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_3^3 a_3^1 a_3^2\rangle\langle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{3-} a_3^{3-} a_1^{3-}\rangle\langle a_1^{1-} a_3^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{34} = \langle a_2^3 a_1^1 a_2^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_3^1 a_3^3 a_3^2\rangle\langle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{1-} a_3^{1-} a_1^{1-}\rangle\langle a_1^{3-} a_3^{3-} a_2^{3-}\rangle\rangle.$$

$$S_{35} = \langle a_2^2 a_2^1 a_2^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_3^1 a_3^2 a_3^3\rangle\langle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{1-} a_3^{1-} a_1^{1-}\rangle\langle a_1^{2-} a_3^{2-} a_2^{2-}\rangle\rangle.$$

$$S_{36} = \langle a_2^1 a_2^2 a_2^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_3^2 a_3^1 a_3^3\rangle\langle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{2-} a_3^{2-} a_1^{2-}\rangle\langle a_1^{1-} a_3^{1-} a_2^{1-}\rangle\rangle.$$

$$S_{41} = \langle a_3^3 a_3^2 a_3^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_2^3 a_2^2 a_2^1\rangle\langle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{42} = \langle a_3^2 a_3^3 a_3^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_2^2 a_2^3 a_2^1\rangle\langle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\rangle.$$

$$S_{43} = \langle a_3^1 a_3^3 a_3^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_2^1 a_2^3 a_2^2\rangle\langle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{3-} a_1^{3-} a_2^{3-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{44} = \langle a_3^3 a_3^1 a_3^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_2^3 a_2^1 a_2^2\rangle\langle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{45} = \langle a_3^2 a_3^1 a_3^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_2^2 a_2^1 a_2^3\rangle\langle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{1-} a_1^{1-} a_2^{1-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\rangle.$$

$$S_{46} = \langle a_3^1 a_3^2 a_3^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_2^1 a_2^2 a_2^3\rangle\langle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{2-} a_1^{2-} a_2^{2-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{51} = \langle a_3^3 a_3^1 a_3^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_2^3 a_2^2 a_2^1\rangle\langle\langle a_2^{1-} a_3^{1-} a_1^{1-}\rangle\langle a_1^{2-} a_3^{2-} a_2^{2-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{52} = \langle a_3^2 a_3^1 a_3^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_2^2 a_2^3 a_2^1\rangle\langle\langle a_2^{1-} a_3^{1-} a_1^{1-}\rangle\langle a_1^{3-} a_3^{3-} a_2^{3-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\rangle.$$

$$S_{53} = \langle a_3^1 a_3^2 a_3^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_2^1 a_2^3 a_2^2\rangle\langle\langle a_2^{2-} a_3^{2-} a_1^{2-}\rangle\langle a_1^{3-} a_3^{3-} a_2^{3-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{54} = \langle a_3^3 a_3^2 a_3^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_2^3 a_2^1 a_2^2\rangle\langle\langle a_2^{2-} a_3^{2-} a_1^{2-}\rangle\langle a_1^{1-} a_3^{1-} a_2^{1-}\rangle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{55} = \langle a_3^1 a_3^3 a_3^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_2^1 a_2^2 a_2^3\rangle\langle\langle a_2^{3-} a_3^{3-} a_1^{3-}\rangle\langle a_1^{2-} a_3^{2-} a_2^{2-}\rangle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{56} = \langle a_3^2 a_3^3 a_3^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_2^2 a_2^1 a_2^3\rangle\langle\langle a_2^{3-} a_3^{3-} a_1^{3-}\rangle\langle a_1^{1-} a_3^{1-} a_2^{1-}\rangle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\rangle.$$

$$S_{61} = \langle a_3^3 a_3^2 a_3^1\rangle\langle a_1^1 a_1^3 a_1^2\rangle\langle a_2^2 a_2^3 a_2^1\rangle\langle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{2-} a_2^{2-} a_1^{2-}\rangle\langle a_1^{3-} a_2^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{62} = \langle a_3^2 a_3^3 a_3^1\rangle\langle a_1^1 a_1^2 a_1^3\rangle\langle a_2^3 a_2^2 a_2^1\rangle\langle\langle a_2^{1-} a_1^{1-} a_3^{1-}\rangle\langle a_3^{3-} a_2^{3-} a_1^{3-}\rangle\langle a_1^{2-} a_2^{2-} a_3^{2-}\rangle\rangle.$$

$$S_{63} = \langle a_3^1 a_3^3 a_3^2\rangle\langle a_1^2 a_1^1 a_1^3\rangle\langle a_2^3 a_2^1 a_2^2\rangle\langle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{3-} a_2^{3-} a_1^{3-}\rangle\langle a_1^{1-} a_2^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{64} = \langle a_3^3 a_3^1 a_3^2\rangle\langle a_1^2 a_1^3 a_1^1\rangle\langle a_2^1 a_2^3 a_2^2\rangle\langle\langle a_2^{2-} a_1^{2-} a_3^{2-}\rangle\langle a_3^{1-} a_2^{1-} a_1^{1-}\rangle\langle a_1^{3-} a_2^{3-} a_3^{3-}\rangle\rangle.$$

$$S_{65} = \langle a_3^1 a_3^2 a_3^3\rangle\langle a_1^3 a_1^1 a_1^2\rangle\langle a_2^2 a_2^1 a_2^3\rangle\langle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{2-} a_2^{2-} a_1^{2-}\rangle\langle a_1^{1-} a_2^{1-} a_3^{1-}\rangle\rangle.$$

$$S_{66} = \langle a_3^2 a_3^1 a_3^3\rangle\langle a_1^3 a_1^2 a_1^1\rangle\langle a_2^1 a_2^2 a_2^3\rangle\langle\langle a_2^{3-} a_1^{3-} a_3^{3-}\rangle\langle a_3^{1-} a_2^{1-} a_1^{1-}\rangle\langle a_1^{2-} a_2^{2-} a_3^{2-}\rangle\rangle.$$

2 Main results

For complete tripartite graph $K_{n,n,l}(l \geq n \geq 1)$ with tripartition (X, Y, Z) where $X = \{u_1, u_2, \cdots, u_n\}$, $Y = \{v_1, v_2, \cdots, v_n\}$ and $Z = \{w_1, w_2, \cdots, w_l\}$, choose all edges incident with u_1 except the edges joining X to Y, and all edges incident with w_l to obtain a spanning tree. Denote cotree edge $u_i v_j$, $w_k u_s$ and $w_k v_i$ by b_j^{i-1}, a_{s-1}^k and a_{n+i-1}^k, respectively, for $1 \leq i, j \leq n$, $1 \leq k \leq l-1$ and $2 \leq s \leq n$. And all letters are distinct. Let the joint trees of $K_{n,n,l}$ have an anticlockwise rotation at each vertex. Then the associated surface can be shown as

$$S = \langle F_1(l,n)F_2(l,n)\cdots F_{n-1}(l,n)G_1(l,n)G_2(l,n)\cdots G_n(l,n)\rangle\langle E_n \underline{B}_{l-1}^{(2n-1)}\rangle,$$

where

$$G_j(l,n) = \langle a_{n-1+j}^1 a_{n-1+j}^2 \cdots a_{n-1+j}^{l-1} b_j^{0-} b_j^{1-} \cdots b_j^{(n-1)-}\rangle,\ E_n = b_1^0 b_2^0 \cdots b_n^0,$$

$$F_i(l,n) = \langle a_i^1 a_i^2 \cdots a_i^{l-1} b_1^i b_2^i \cdots b_n^i\rangle,\ \underline{B}_{l-1}^{(2n-1)} = B_1^{(2n-1)} B_2^{(2n-1)} \cdots B_{l-1}^{(2n-1)},$$

$$\underline{B}_{l-1}^{(2n-1)} - B_k^{(2n-1)} = B_1^{(2n-1)} B_2^{(2n-1)} \cdots B_{k-1}^{(2n-1)} B_{k+1}^{(2n-1)} B_{k+2}^{(2n-1)} \cdots B_{l-1}^{(2n-1)},$$

$B_k^{(2n-1)} = \langle a_1^{k-} a_2^{k-} \cdots a_{2n-1}^{k-}\rangle$ for $1 \leq i \leq n-1, 1 \leq j \leq n$ and $1 \leq k \leq l-1$.

Let $S' = G_1(l,n)F_1(l,n)G_2(l,n)F_2(l,n)\cdots G_{n-1}(l,n)$

$$F_{n-1}(l,n)G_n(l,n)\langle E_n \underline{B}_{l-1}^{(2n-1)}\rangle.$$

Then

$$S' \to b_1^0(b_1^{0-}G_1(l,n-2)b_1^{1-})(b_1^1F_1(l,n-2)b_2^1)(b_2^{1-}G_2(l,n-2)b_2^{2-})$$

$$(b_2^2 \cdots F_2(l,n-2)b_3^2)(b_{n-1}^{(n-2)-}G_{n-1}(l,n-2)b_{n-1}^{(n-1)-})(b_{n-1}^{n-1}F_{n-1}(l,n-2)$$

$$b_n^{n-1})(b_n^{(n-1)-}G_n(l,n-2)b_n^{0-})b_n^0\langle E_{n-2}\underline{B}_{l-1}^{(2n-1)}\rangle$$

$$\sim G_1(l,n-2)F_1(l,n-2)G_2(l,n-2)F_2(l,n-2)\cdots G_{n-1}(l,n-2)$$

$$F_{n-1}(l,n-2)G_n(l,n-2)\langle E_{n-2}\underline{B}_{l-1}^{(2n-1)}\rangle = S_0.$$

Let

$$S'' = G_1(l,n)F_1(l,n)G_2(l,n)\cdots F_i(l,n)G_j(l,n)F_{i+1}(l,n)\cdots F_j(l,n)$$

$$G_i(l,n)F_{j+1}(l,n)\cdots G_{n-1}(l,n)F_{n-1}(l,n)G_n(l,n)\langle E_n\underline{B}_{l-1}^{(2n-1)}\rangle.$$

Obviously,

$$\langle G_1(l,0)F_1(l,0)G_2(l,0)F_2(l,0)\cdots G_{n-1}(l,0)F_{n-1}(l,0)G_n(l,0)\rangle\langle\underline{B}_{l-1}^{(2n-1)}\rangle$$

is an associated surface for $K_{2n,l}$.

Let $\{S\}$ be meant the set of such elements obtained by doing a sequence of exchangers on associated surface S. Use $n_g(G)$ to denote the number of genus embeddings of graph G; $n_g(\{S\})$, the number of distinct surfaces of minimum genus in $\{S\}$.

Lemma 2.1. $o_{\min}(\{S'\}) = o_{\min}(\{S''\})$.

Proof $\forall\, S'' \in \{S''\}$, without loss of generality we may assume that

$$S'' = G_1(l,n)F_1(l,n)G_2(l,n)\cdots F_i(l,n)G_j(l,n)F_{i+1}(l,n)\cdots F_j(l,n)$$

$$G_i(l,n)F_{j+1}(l,n)\cdots G_{n-1}(l,n)F_{n-1}(l,n)G_n(l,n)\langle E_n\underline{B}_{l-1}^{(2n-1)}\rangle.$$

Then do a sequence of exchangers between b_i^0 and b_j^0, a_{n-1+i}^k and a_{n-1+j}^k for $1 \le k \le l-1$ to obtain S^0, namely $S'' \to S^0$. At this time, there must be $S' \in \{S'\}$ such that $o(S^0) = o(S')$. Then $o_{\min}(\{S'\}) \le o_{\min}(\{S''\})$, vice verse.

Let D_n be the dipole, which consists of two vertices joined by n edges and $g_m(G)$ be the number of embeddings of the graph G into the surface of genus m.

Theorem 2.2[2]. $g[D_n](x) = -\dfrac{2\cdot(n-1)!}{n(n+1)}\displaystyle\sum_{m=0}^{[\frac{n-2}{2}]} s(n+1,n-2m)x^m,$

where $s(n,k)$ is the Stirling numbers.

Corollary 2.3. *The number of such surfaces as $a_1 a_2 \cdots a_{2t}\langle a_1^- a_2^- \cdots a_{2t}^-\rangle$ of genus t is $\frac{(2t)!}{t+1}$.*

Proof An associated surface of D_{2t+1} is equivalent to $\langle a_1 a_2 \cdots a_{2t}\rangle\langle a_1^- a_2^- a_{2t}^-\rangle$. From Theorem 2.2, the corollary holds.

Theorem 2.4[1,7]. $\gamma(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$, $m, n \geq 2$.

Theorem 2.5[7]. $n_g(K_{4t+2,n}) \geq (\frac{(2t)!}{t+1})^{n-2} 2^{n-4}(4t+1)!(n-1)!$, *where* $n \geq 4t+2,\ t \geq 1$.

Theorem 2.6[7]. $n_g(K_{4t,n}) \geq (\frac{(2t-2)!}{t})^{2\lfloor\frac{n-1}{2}\rfloor} 3^{\lfloor\frac{n-1}{2}\rfloor}(4t-1)!(n-1)!$, *where* $n \geq 4t,\ t \geq 1$.

Theorem 2.7[7]. $n_g(K_{4t+3,n}) \geq ((\frac{(2t-2)!}{t})^2(\frac{(2t-4)!}{t-1})^2)^{\lfloor\frac{n-4}{4}\rfloor} 3^{\lfloor\frac{n-3}{4}\rfloor}(4t+2)!(n-1)!$, *where* $n \geq 4t+2,\ t \geq 1$.

Theorem 2.8[7]. $n_g(K_{4t+1,n}) \geq (\frac{(2t-2)!}{t})^{4\lfloor\frac{n-4}{4}\rfloor} 3^{\lfloor\frac{n-3}{4}\rfloor}(4t)!(n-1)!$, *where* $n \geq 4t+1,\ t \geq 1$.

Firstly, since $K_{2n,l}$ is a subgraph of $K_{n,n,l}$, $\gamma(K_{n,n,l}) \geq \gamma(K_{2n,l}) \geq \lceil \frac{(n-1)(l-2)}{2} \rceil$.

Theorem 2.9. $n_g(K_{n,n,l}) \geq$

$$\begin{cases} l!, & n = 1; \\ (\frac{2\cdot(n-1)!}{n+1})^{l-n} 2^{l-4} T_n, & l \geq 3n-2,\ 1 \neq n \text{ is odd}; \\ 2^{n-2}(\frac{(2m-2)!}{m})^{4\lfloor\frac{n-2}{2}\rfloor} 3^{\lfloor\frac{2n-3}{4}\rfloor} T_n, & 2 \leq l < 3n-2,\ l-n+1 = 4m; \\ 2^{n-2}(\frac{(2m)!}{m+1})^{2n-2} 2^{2n-4} T_n, & 2 \leq l < 3n-2,\ l-n = 4m; \\ 2^{n-2}(\frac{(2m-4)!}{m-1})^{\lfloor\frac{n-2}{2}\rfloor} 2^{\lfloor\frac{2n-3}{4}\rfloor} T_n, & 2 \leq l < 3n-2,\ l-n-1 = 4m; \\ 2^{n-2}(\frac{(2m-2)!}{m})^{2\lfloor\frac{2n-1}{2}\rfloor} 3^{\lfloor\frac{2n-1}{2}\rfloor} T_n, & 2 \leq l < 3n-2,\ l-n+2 = 4m. \end{cases}$$

where $T_n = (\frac{2\cdot(n-2)!}{n-1})^{n-2} 2^{n-2}(l-1)!n!(n-1)!$ *and* $l \geq n \geq 1$.

Proof Case1 $n_g(K_{1,1,l}) = l!$ when $n = 1$.

$$\begin{aligned} S_0 = G_1(l,0)\langle E_1 \underline{B}_l^{(1)}\rangle &= \langle a_1^1 a_1^2 \cdots a_1^{l-1} b_1^{0-}\rangle\langle b_1^0 B_1^{(1)} B_2^{(1)} \cdots B_{l-1}^{(1)}\rangle \\ &= \langle a_1^1 a_1^2 \cdots a_1^{l-1} b_1^{0-}\rangle\langle b_1^0 a_1^{1-} a_2^{1-} \cdots a_{l-1}^{1-}\rangle \end{aligned}$$

Since the minimum genus of $K_{2,l+1}$ is 0, $n_g(K_{1,1,l})$ is equal to the number of such surfaces as $\langle a_1^1 a_1^2 \cdots a_1^{l-1} b_1^{0-}\rangle\langle b_1^0 a_1^{1-} a_2^{1-} \cdots a_{l-1}^{1-}\rangle$, namely $n_g(K_{1,1,l}) = n_g(K_{2,l+1})$. Then $n_g(K_{1,1,l}) = l!$.

Case2 n is odd and greater than 1.

Case2.1 $S_0 \to G_1(l-1,n-2)a_n^1 a_1^1 F_1(l-1,n-2)G_2(l-1,n-2)a_{n+1}^1 a_2^1 F_2$

$$(l-1,n-2))\cdots G_{n-1}(l-1,n-2)a_{2n-2}^1 a_{n-1}^1 F_{n-1}(l-1,n-2)G_n(l-1,n-2)$$

$$a_{2n-1}^1\langle a_{2n-1}^{1-} a_1^{1-} a_n^{1-} a_2^{1-} a_{n+1}^{1-} \cdots a_{n-1}^{1-} a_{2n-2}^{1-}\rangle b_{n-1}^0 \langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle$$

$$\sim G_1(l-1,n-2)c_1 F_1(l-1,n-2)G_2(l-1,n-2)c_2 F_2(l-1,n-2)\cdots G_{n-1}(l-1,$$

$$n-3)b_{n-1}^{0-} c_{n-1} F_{n-1}(l-1,n-2)G_n(l-1,n-2)\langle c_1^- c_2^- \cdots c_{n-2}^-\rangle c_{n-1}^- b_{n-1}^0 \langle E_{n-3}$$

$$(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle$$

$$\sim G_1(l-1,n-2)F_2(l-1,n-2)G_3(l-1,n-2)F_4(l-1,n-2)\cdots F_{n-1}$$

$$(l-1,n-2)G_n(l-1,n-2)F_1(l-1,n-2)G_2(l-1,n-2)F_3(l-1,n-2)G_4(l-1,$$

$$n-2)\cdots F_{n-2}(l-1,n-2)G_{n-1}(l-1,n-3)\langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle O_{\frac{n-1}{2}}$$

$$\sim G_1(l-1,n-3)F_2(l-1,n-3)G_3(l-1,n-3)F_4(l-1,n-3)\cdots F_{n-1}$$

$$(l-1,n-3)G_n(l-1,n-3)F_1(l-1,n-3)G_2(l-1,n-3)F_3(l-1,n-3)G_4(l-1,$$

$$n-3)\cdots F_{n-2}(l-1,n-3)G_{n-1}(l-1,n-3)\langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle O_{\frac{n-1}{2}}.$$

Case2.2 $S_0 \to a_n^{l-1} G_1(l-1,n-2)F_1(l-1,n-2)a_1^{l-1} a_{n+1}^{l-1} G_2(l-1,n-2)$

$$F_2(l-1,n-2)a_2^{l-1} a_{n+2}^{l-1} G_3(l-1,n-2)\cdots a_{2n-2}^{l-1} G_{n-1}(l-1,n-2)$$

$$F_{n-1}(l-1,n-2)a_{n-1}^{l-1} a_{2n-1}^{l-1} G_n(l-1,n-2)\langle E_{n-2}\underline{B}_{l-2}^{(2n-1)}\rangle$$

$$\langle a_{n+1}^{(l-1)-} a_1^{(l-1)-} a_{n+2}^{(l-1)-} a_2^{(l-1)-} \cdots a_{2n-1}^{(l-1)-} a_{n-1}^{(l-1)-} a_n^{(l-1)-}\rangle$$

$$\sim G_1(l-1,n-2)F_1(l-1,n-2)c_1 G_2(l-1,n-2)F_2(l-1,n-2)c_2$$

$$G_3(l-1,n-2)\cdots c_{n-2} G_{n-1}(l-1,n-2)F_{n-1}(l-1,n-2)c_{n-1}$$

$$G_n(l-1,n-2)\langle E_{n-2}\underline{B}_{l-2}^{(2n-1)}\rangle\langle c_1^- c_2^- \cdots c_{n-1}^-\rangle$$

$$\sim G_{i_1}(l-1,n-3)F_{j_1}(l-1,n-3)G_{i_2}(l-1,n-3)F_{j_2}(l-1,n-3)\cdots G_{i_{n-1}}$$

$$(l-1,n-3)F_{j_{n-1}}(l-1,n-3)G_{i_n}(l-1,n-3)\langle E_{n-3}\underline{B}_{l-2}^{(2n-1)}\rangle O_{\frac{n-1}{2}} = S_1^{(1)}$$

A sequence of such types of operations above from S_0 to $S_1^{(1)}$ are defined by a *step*. So $S_1^{(2)}$ can be obtained after a step on $S_1^{(1)}$. Then after $n-2$ steps, $S_1^{(n-2)}$ can be shown as

$$S_1^{(n-2)} = G_{k_1}(l-n+2,0)F_{l_1}(l-n+2,0)G_{k_2}(l-n+2,0)F_{l_2}(l-n+2,0)\cdots$$

$$G_{k_{n-1}}(l-n+2,0)F_{l_{n-1}}(l-n+2,0)G_{k_n}(l-n+2,0)\underline{B}_{l-n+1}^{(2n-1)}O_{\frac{(n-1)(n-2)}{2}},$$

where $i_1 i_2 \cdots i_n$ and $k_1 k_2 \cdots k_n$ are permutations on $\{1,2,\cdots,n\}$, $j_1 j_2 \cdots j_{n-1}$ and $l_1 l_2 \cdots l_{n-1}$ are permutations on $\{1,2,\cdots,n-1\}$. Then by Theorem 2.4,

$$\gamma(K_{n,n,l}) \le \frac{(n-1)(n-2)}{2} + \gamma(K_{2n,l-(n-2)}) \le \lceil \frac{(n-1)(l-2)}{2} \rceil.$$

That is to say, by doing a sequence of such exchangers as above, an associated surface of minimum genus can be got, when n is odd and greater than 1.

From S_0, according to Corollary 2.3 and the proof of Case 2, there are at least $(\frac{2\cdot(n-2)!}{n-1})^{n-2}\cdot 2^{n-2}\frac{(l-1)!}{(l-n+1)!}$ ways to obtained $S_1^{(n-2)}$.

By Lemma 2.1, when n is odd and greater than 1,

$$n_g(K_{n,n,l}) \ge (\frac{2\cdot(n-2)!}{n-1})^{n-2}2^{n-2}\frac{(l-1)!}{(l-n+1)!}n_g(\{S_1^{(n-2)}\})n!(n-1)!.$$

In this case, if $l \ge 3n-2$, applying Theorem 2.5, then

$$n_g(\{S_1^{(n-2)}\}) \ge 2^{l-n-2}(\frac{2\cdot(n-1)!}{n+1})^{l-n}(l-n+1)!.$$

Therefore

$$n_g(K_{n,n,l}) \ge (\frac{2\cdot(n-2)!}{n-1})^{n-2}(\frac{2\cdot(n-1)!}{n+1})^{l-n}2^{l-4}(l-1)!n!(n-1)!.$$

If $l < 3n-2$, according to Theorem 2.5-2.8, $n_g(K_{n,n,l}) \ge$

$$\begin{cases} (\frac{(2m-2)!}{m})^{2[\frac{2n-1}{2}]}3^{[\frac{2n-1}{2}]}T_n, & l-n+2=4m; \\ 2^{n-2}(\frac{(2m-2)!}{m})^{4[\frac{n-2}{2}]}3^{[\frac{2n-3}{4}]}T_n, & l-n+1=4m; \\ 2^{n-2}(\frac{(2m)!}{m+1})^{2n-2}2^{2n-4}T_n, & l-n=4m; \\ 2^{n-2}(\frac{(2m-4)!}{m-1})^{[\frac{n-2}{2}]}2^{[\frac{2n-3}{4}]}T_n, & l-n-1=4m. \end{cases}$$

where $T_n = (\frac{2\cdot(n-2)!}{n-1})^{n-2}2^{n-2}(l-1)!n!(n-1)!$.

Theorem 2.10. $n_g(K_{n,n,l}) \geq$

$$\begin{cases} (\frac{2(n-2)!}{n})^{2[\frac{l-n+1}{2}]}3^{[\frac{l-n+1}{2}]}H_n, & l \geq 3n-2,\ n \textit{ is even}; \\ (\frac{(2m-2)!}{m})^{2[\frac{2n-1}{2}]}3^{[\frac{2n-1}{2}]}H_n, & l-n+2=4m; \\ (\frac{(2m-2)!}{m})^{4[\frac{n-2}{2}]}3^{[\frac{2n-3}{4}]}H_n, & l-n+1=4m; \\ (\frac{(2m)!}{m+1})^{2n-2}2^{2n-4}H_n, & l-n=4m; \\ (\frac{(2m-4)!}{m-1})^{[\frac{n-2}{2}]}2^{[\frac{2n-3}{4}]}H_n, & l-n-1=4m. \end{cases}$$

where $H_n = (\frac{2\cdot(n-2)!}{n})^{n-2}(l-1)!n!(n-1)!$ *and* $l \geq n \geq 1$.

Proof When n is even,

$$S_0 \to G_1(l-1,n-2)a_n^1a_1^1F_1(l-1,n-2)G_2(l-1,n-2)a_{n+1}^1a_2^1$$
$$F_2(l-1,n-2)G_3(l-1,n-2)a_{n+2}^1a_3^1F_3(l-1,n-2)\cdots G_{n-1}(l-1,$$
$$n-2)a_{2n-2}^1a_{n-1}^1F_{n-1}(l-1,n-2)G_n((l-1,n-2)a_{2n-1}^1)a_{2n-1}^{1-}$$
$$\langle a_1^{1-}a_n^{1-}a_2^{1-}a_{n+1}^{1-}\cdots a_{n-1}^{1-}a_{2n-2}^{1-}\rangle b_{n-1}^0\langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle$$
$$\sim G_1(l-1,n-2)c_1F_1(l-1,n-2)G_2(l-1,n-2)c_2F_2(l-1,n-2)$$
$$G_3(l-1,n-2)c_3F_3(l-1,n-2)\cdots G_{n-1}(l-1,n-3)b_{n-1}^{0-}c_{n-1}F_{n-1}(l-1,$$
$$n-2)G_n(l-1,n-2)\langle c_1^-c_2^-\cdots c_{n-2}^-\rangle c_{n-1}^-b_{n-1}^0\langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle$$
$$\sim G_1(l-1,n-2)F_2(l-1,n-2)G_3(l-1,n-2)F_4(l-1,n-2)G_5(l-1,n-2)$$
$$\cdots F_{n-2}(l-1,n-2)G_{n-1}(l-1,n-3)c_{n-1}F_{n-1}(l-1,n-2)G_n(l-1,n-2)$$
$$F_1(l-1,n-2)G_2(l-1,n-2)F_3(l-1,n-2)G_4(l-1,n-2)\cdots F_{n-3}(l-1,n-2)$$
$$G_{n-2}(l-1,n-2)c_{n-1}^-\langle E_{n-3}(\underline{B}_{l-1}^{(2n-1)} - B_1^{(2n-1)})\rangle O_{\frac{n-2}{2}}$$
$$\to b_{n-2}^0\langle a_{2n-3}^{(l-1)-}a_{n-3}^{(l-1)-}a_{n+2}^{(l-1)-}a_2^{(l-1)-}\cdots a_{2n-2}^{(l-1)-}a_{n-2}^{(l-1)-}a_{2n-1}^{(l-1)-}a_{n-1}^{(l-1)-}$$
$$a_{n+1}^{(l-1)-}a_1^{(l-1)-}a_{n+3}^{(l-1)-}a_3^{(l-1)-}\cdots a_{2n-5}^{(l-1)-}a_{n-5}^{(l-1)-}a_n^{(l-1)-}\rangle(a_n^{l-1}G_1(l-2,n-3)$$
$$b_1^{2-})(b_1^2F_2(l-2,n-3)a_2^{l-1})(a_{n+2}^{l-1}G_3(l-2,n-3)b_3^{4-})(b_3^4F_4(l-2,n-3)a_4^{l-1})$$
$$(a_{n+4}^{l-1}G_5(l-2,n-3)b_5^{6-})(b_5^6F_3(l-2,n-3)a_6^{l-1})\cdots(b_{n-3}^{n-2}F_{n-2}(l-2,n-3)$$
$$a_{n-2}^{l-1})(a_{2n-2}^{l-1}G_{n-1}(l-2,n-3))c_{n-1}(b_{n-2}^{n-1}F_{n-1}(l-2,n-3)a_{n-1}^{l-1})(a_{2n-1}^{l-1}$$
$$G_n(l-2,n-3)b_n^{1-})(b_n^1F_1(l-2,n-3)a_1^{l-1})(a_{n+1}^{l-1}G_2(l-2,n-3)b_2^{3-})(b_2^3$$
$$F_3(l-2,n-3)a_3^{l-1})(a_{n+3}^{l-1}G_4(l-2,n-3)b_4^{5-})\cdots(b_{n-4}^{n-3}F_{n-3}(l-2,n-3)a_{n-3}^{l-1})$$

$$(a_{2n-3}^{l-1}b_{n-2}^{0-}G_{n-2}(l-2,n-4)b_{n-2}^{(n-1)-})c_{n-1}^{-}\langle E_{n-4}(\underline{B}_{l-2}^{(2n-1)}-B_1^{(2n-1)})\rangle O_{\frac{n-2}{2}}$$

$$\sim d_{n-1}^{-}\langle d_1^{-}d_2^{-}\cdots d_{n-2}^{-}\rangle G_1(l-2,n-3)F_2(l-2,n-3)d_1G_3(l-2,n-3)F_4(l-2,n-3)d_2G_5(l-2,n-3)F_3(l-2,n-3)d_3\cdots F_{n-2}(l-2,n-3)d_{\frac{n-2}{2}}G_{n-1}(l-2,n-3)c_{n-1}F_{n-1}(l-2,n-3)d_{\frac{n}{2}}G_n(l-2,n-3)F_1(l-2,n-3)d_{\frac{n+2}{2}}G_2(l-2,n-3)F_3(l-2,n-3)d_{\frac{n+4}{2}}G_4(l-2,n-3)\cdots F_{n-3}(l-2,n-3)d_{n-1}G_{n-2}(l-2,n-4)c_{n-1}^{-}\langle E_{n-4}(\underline{B}_{l-2}^{(2n-1)}-B_1^{(2n-1)})\rangle O_{\frac{n-2}{2}}$$

$$\sim G_{i_1}(l-2,n-4)F_{j_1}(l-2,n-4)G_{i_2}(l-2,n-4)F_{j_2}(l-2,n-4)\cdots G_{i_{n-1}}(l-2,n-4)F_{j_{n-1}}(l-2,n-4)G_{i_n}(l-2,n-4)\langle E_{n-4}(\underline{B}_{l-2}^{(2n-1)}-B_1^{(2n-1)})\rangle O_{n-1}=S_2^{(1)}$$

A sequence of such types of operations above from S_0 to $S_2^{(1)}$ are defined by a *step*. So $S_2^{(2)}$ can be obtained after a step on $S_2^{(1)}$. Then after $\frac{n-2}{2}$ steps, $S_2^{(n-2)}$ can be shown as

$$S_2^{(n-2)}=G_{k_1}(l-n+2,0)F_{l_1}(l-n+2,0)G_{k_2}(l-n+2,0)F_{l_2}(l-n+2,0)\cdots G_{k_{n-1}}(l-n+2,0)F_{l_{n-1}}(l-n+2,0)G_{k_n}(l-n+2,0)\underline{B}_{l-n+1}^{(2n-1)}O_{\frac{(n-1)(n-2)}{2}},$$

where $i_1i_2\cdots i_n$ and $k_1k_2\cdots k_n$ are permutations on $\{1,2,\cdots,n\}$, $j_1j_2\cdots j_{n-1}$ and $l_1l_2\cdots l_{n-1}$ are permutations on $\{1,2,\cdots,n-1\}$. Then by Theorem 2.4,

$$\gamma(K_{n,n,l})\le\frac{(n-1)(n-2)}{2}+\gamma(K_{2n,l-(n-2)})\le\lceil\frac{(n-1)(l-2)}{2}\rceil.$$

That is to say, when n is even, an associated surface of minimum genus can be got by doing a sequence of such exchangers as above.

From S_0, according to Corollary 2.3, there are at least $(\frac{2\cdot(n-2)!}{n})^{n-2}\cdot\frac{(l-1)!}{(l-n+1)!}$ ways to obtained $S_2^{(n-2)}$.

Therefore, by Lemma 2.1, when n is even,

$$n_g(K_{n,n,l})\ge(\frac{2\cdot(n-2)!}{n})^{n-2}\frac{(l-1)!}{(l-n+1)!}n_g(\{S_2^{(n-2)}\})n!(n-1)!.$$

In this case, if $l\ge 3n-2$, according to Theorem 2.6, then

$$n_g(\{S_2^{(n-2)}\})\ge(\frac{2(n-2)!}{n})^{2[\frac{l-n+1}{2}]}3^{[\frac{l-n+1}{2}]}(l-n+1)!.$$

Therefore

$$n_g(K_{n,n,l}) \geq [\frac{2(n-2)!}{n}]^{n-2+2[\frac{l-n+1}{2}]}3^{[\frac{l-n+1}{2}]}(l-1)!n!(n-1)!.$$

If $l < 3n-2$, according to Theorem 2.5-2.8, $n_g(K_{n,n,l}) \geq$

$$\begin{cases} (\frac{2\cdot(n-2)!}{n})^{n-2}(\frac{(2m-2)!}{m})^{2[\frac{2n-1}{2}]}3^{[\frac{2n-1}{2}]}(l-1)!n!(n-1)!, & l-n+2=4m; \\ (\frac{2\cdot(n-2)!}{n})^{n-2}(\frac{(2m-2)!}{m})^{4[\frac{n-2}{2}]}3^{[\frac{2n-3}{4}]}(l-1)!n!(n-1)!, & l-n+1=4m; \\ (\frac{2\cdot(n-2)!}{n})^{n-2}(\frac{(2m)!}{m+1})^{2n-2}2^{2n-4}(l-1)!n!(n-1)!, & l-n=4m; \\ (\frac{2\cdot(n-2)!}{n})^{n-2}(\frac{(2m-4)!}{m-1})^{[\frac{n-2}{2}]}2^{[\frac{2n-3}{4}]}(l-1)!n!(n-1)!, & l-n-1=4m. \end{cases}$$

Since $n! \sim \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}$, we extend Theorem 2.9-2.10 as follows:

Theorem 2.11. *For integers $l \geq n \geq 1$, $K_{n,n,l}$ has about $\sqrt{l}f(l)$ when $l = 1$, and at least $C_1^{C_2n+C_3}f^{C_5n+C_6}(\frac{l-n-C_4}{2})\sqrt{n(n-1)(l-1)(n-2)^{n-2}}f(n)$ $f(n-1)f(l-1)f^{n-2}(n-2)$ distinct genus embeddings when $l > 1$, where $C_i(1 \leq i \leq 6)$ is a constant depending on the residual of $l-n$ modular 4 and $f(n) = \sqrt{2\pi}(\frac{n}{e})^n$.*

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GRACEFULNESS OF A CYCLE WITH PARALLEL CHORDS AND PARALLEL P_k-CHORDS OF DIFFERENT LENGTHS

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Abstract

In this paper we prove that every n-cycle ($n \geq 6$) with parallel chords is graceful for all $n \geq 6$ and every n-cycle with parallel P_k-chords of increasing lengths is graceful for $n = 2$ (mod 4) with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

Key words: Graph labeling, Graceful graphs, Cycle with parallel chords, Cycle with parallel P_k-chords.

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