GRACEFULNESS OF A CYCLE WITH PARALLEL CHORDS AND PARALLEL P_k-CHORDS OF DIFFERENT LENGTHS

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Abstract

In this paper we prove that every n-cycle ($n \geq 6$) with parallel chords is graceful for all $n \geq 6$ and every n-cycle with parallel P_k-chords of increasing lengths is graceful for $n = 2$ (mod 4) with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

Key words: Graph labeling, Graceful graphs, Cycle with parallel chords, Cycle with parallel P_k-chords.

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1 Introduction

A function f is called a *graceful labeling* of a graph G with m edges if f is an injection from the vertex set of G to the set $\{0, 1, 2, \ldots, m\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. For an excellent survey on graph labeling see [2].

A graph G is called *cycle with parallel chords* if G is obtained from a cycle $C_n : v_0v_1, \cdots, v_{n-1}v_0 (n \geq 6)$ by adding the chords v_1v_{n-1}, v_2v_{n-2}, $\ldots$, $v_\alpha v_\beta$, where $\alpha = \lfloor \frac{n}{2} \rfloor - 1$ and $\beta = \lfloor \frac{n}{2} \rfloor + 2$, if n is odd or $\beta = \lfloor \frac{n}{2} \rfloor + 1$, if n is even.

A graph G is called cycle with parallel P_k-chords of increasing lengths if G is obtained from the cycle $C_n : v_0v_1 \ldots v_{n-1}v_0$ $(n \geq 6)$ by adding disjoint path of length k, P_{k+1}, between the pair of vertices (v_k, v_{n-k}), for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. The path P_{k+1} joining the pair of vertices (v_k, v_{n-k}) is called P_{k+1}-chord, for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

In [1] Delorme et al., proved that every cycle with a chord is graceful. Koh and Yap [3] have shown that cycles with P_3-chords are graceful and conjectured that all cycles with a P_k-chords are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote [4].

In this paper we prove that every n-cycle $(n \geq 6)$ with parallel chords is graceful and every n-cycle with parallel P_k-chords of increasing lengths is graceful for $n = 2$ $(mod$ $4)$ with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

2 Gracefulness of a cycle with parallel chords and parallel P_k-chords of different lengths

In this section we prove that every n-cycle $(n \geq 6)$ with parallel chords is graceful and every n-cycle with parallel P_k-chords of increasing lengths is graceful for $n = 2$ $(mod$ $4)$ with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

Theorem 1 : *For $n \geq 6$, every n-cycle with parallel chords is graceful.*

Proof : Let G be an n-cycle with parallel chords for $n \geq 6$.
Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of an n-cycle of G. Observe that by definition, G has n vertices and $M = \frac{3n-\rho}{2}$ edges, where $\rho = 3$, if n is odd or $\rho = 2$, if n is even.

We give labels to the vertices of G in the following two cases:

Case 1: *When n is odd*

$$\begin{aligned}
\text{Define } f(v_0) &= 0 \\
f(v_{2i-1}) &= 3i - 2, && \text{for } 1 \leq i \leq \left\lceil \frac{n-1}{4} \right\rceil \\
f(v_{2i}) &= \frac{3n - 6i + 1}{2}, && \text{for } 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor \\
f(v_{n-(2i-1)}) &= \frac{3n - 6i + 3}{2}, && \text{for } 1 \leq i \leq \left\lceil \frac{n-1}{4} \right\rceil \\
f(v_{n-2i}) &= 3i, && \text{for } 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor
\end{aligned}$$

Case 2: *When n is even*

$$\begin{aligned}
\text{Define } f(v_0) &= 0 \\
f(v_1) &= 1 \\
f(v_{n-1}) &= \frac{3n - 2}{2} \\
f(v_{2i}) &= \frac{3n - 6i + 2}{2}, && \text{for } 1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor \\
f(v_{n-(2i+1)}) &= \frac{3n - 6i}{2}, && \text{for } 1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor \\
f(v_{n-2i}) &= 3i, && \text{for } 1 \leq i \leq \left\lceil \frac{n-4}{4} \right\rceil
\end{aligned}$$

$$f(v_{2i+1}) = 3i+2, \text{for } 1 \le i \le \delta,$$
$$\text{where } \delta = \left\lfloor \frac{n-4}{4} \right\rfloor - 1, \text{when } n = 4r, \text{for some } r \ge 1$$
$$\text{or} \quad \delta = \left\lfloor \frac{n-4}{4} \right\rfloor, \text{when } n = 4r+2, \text{for some } r \ge 1$$

$$f(v_{\frac{n}{2}-1}) = \begin{cases} \frac{3n}{4}, & \text{when } n = 4r, \text{for some } r \ge 1 \\ \frac{3(n+2)}{4}, & \text{when } n = 4r+2, \text{for some } r \ge 1 \end{cases}$$

$$f(v_{\frac{n}{2}}) = \begin{cases} \frac{3n-8}{4}, & \text{when } n = 4r, \text{for some } r \ge 1 \\ \frac{3(n+4)+2}{4}, & \text{when } n = 4r+2, \text{for some } r \ge 1. \end{cases}$$

It is clear that f is injective and the edge values are distinct and range from 1 to M. Thus f is graceful labeling. Hence the graph G is graceful.

Theorem 2 : *For $n \ge 6$ and $n \equiv 2 \pmod 4$ every n-cycle with parallel P_k-chords of increasing lengths is graceful with* $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$.

Proof : Let G denote an n-cycle with parallel P_k-chords of increasing lengths with $n \equiv 2 \pmod 4$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$. By definition of G, G is obtained from the n-cycle of order n : $v_0 v_1 \cdots v_{n-1} v_0 (n \ge 6)$ by adding disjoint path of length k, P_{k+1}, between the pair of vertices (v_k, v_{n-k}), for $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$. Observe that G has $N = \frac{n^2+2n+8}{8}$ vertices and $M = \frac{n^2+6n}{8}$ edges (when $n \equiv 2 \pmod 4$), here n denote the number of vertices of an n-cycle in G).

Observe that G has a hamiltonian path containing all the P_k-chords of G, starting with v_0 of n-cycle in G and ending up with v_α of an n-cycle in G, where $\alpha = \lfloor \frac{n}{2} \rfloor$. Let $u_0 u_1 \cdots u_{N-1}$ be the hamiltonian path in G.

We give labels to the vertices $u_0, u_1, \cdots, u_{N-2}, u_{N-1}$ in the following two cases.

Case (i):

When $n = 4k + 2$, for some $k \geq 1$ (i.e., $n \equiv 2 \ (mod\ 4)$, and k even).

Then equivalently, we can consider $n = 8t + 2$, for some $t \geq 1$.

$$\begin{aligned}
\text{Define} \quad f(u_0) &= 0 \\
f(u_{2i}) &= i, \quad \text{for } 1 \leq i \leq \left(\tfrac{N-4}{2}\right) \\
f(u_{2i-1}) &= M - (i-1), \quad \text{for } 1 \leq i \leq \left(\tfrac{N-(4t+2)}{2}\right) \\
f(u_{N-(2j+1)}) &= M - \left(\tfrac{N}{2}\right) + j - 1, \quad \text{for } 1 \leq j \leq t \\
f(u_{N-(2t+2j+1)}) &= M - \left(\tfrac{N}{2}\right) + t + j, \quad \text{for } 1 \leq j \leq t \\
f(u_{N-1}) &= f(u_{N-(4t+1)}) + 1 \\
f(u_{N-2}) &= f(u_{N-4}) + 2.
\end{aligned}$$

Case (ii):

When $n = 4k + 2$, for some $k \geq 1$ (i.e., $n \equiv 2 \ (mod\ 4)$, and k odd).

Then equivalently, we can consider $n = 8t - 2$, for some $t \geq 2$.

$$\begin{aligned}
\text{Define} \quad f(u_0) &= 0 \\
f(u_{2i}) &= i, \quad \text{for } 1 \leq i \leq \left(\tfrac{N-(4t+1)}{2}\right) \\
f(u_{2i-1}) &= M - (i-1), \quad \text{for } 1 \leq i \leq \left(\tfrac{N-1}{2}\right) \\
f(u_{N-(2j+1)}) &= \left(\tfrac{N+3-2j}{2}\right), \quad \text{for } 1 \leq j \leq t \\
f(u_{N-(2t+2j+1)}) &= \left(\tfrac{N-(2t+2j-1)}{2}\right), \quad \text{for } 1 \leq j \leq t-1 \\
f(u_{N-1}) &= f(u_{N-(4t-1)}) - 1.
\end{aligned}$$

It is clear that f is injective and the edge values are distinct and range from 1 to M. Thus f is graceful labeling. Hence the graph G is graceful.

Here, in this paper we have proved that graph obtained from an n-cycle of order $n : v_0v_1 \dots v_{n-1}v_0$ $(n \geq 6)$ by adding the path P_{k+1} of length with k or 1 between a pair (v_k, v_{n-k}), for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ are graceful. Is it true that the graph obtained from an n-cycle of order $n : v_0v_1 \dots v_{n-1}$ by adding P_k-chord of an arbitrary length $k-1$ between the pairs (v_k, v_{n-k}), for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ is graceful?

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