Some Properties of Padovan Sequence by Matrix Methods

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Abstract

It is the aim of this paper to explore some new properties of the Padovan sequence using matrix methods. We derive new recurrence relations and generating matrices for the sums of Padovan numbers and $4n$ subscripted Padovan sequences. Also, we define one type of (0,1) upper Hessenberg matrix whose permanents are Padovan numbers.

1 Introduction

The Padovan sequence is described by I. Steward, in honour of the contemporary architect Richard Padovan, for $n > 2$

$$P_n = P_{n-2} + P_{n-3}$$

where $P_0 = P_1 = P_2 = 1$ [1]. The Padovan numbers are like the Fibonacci numbers, but instead of starting with two predetermined terms, the sequence starts with skipping the previous one and each term afterward is the sum of the preceding two terms.

The sequence also can be extended to negative parametres [2] using the recurrence relation $P_{-n} = P_{-n+3} - P_{-n+1}$. The first few values of the sequence are given below:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P_n	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37
P_{-n}	1	0	1	0	0	1	-1	1	0	-1	2	-2	1	1	-3

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Generating matrix [1] of the sequence is as following:

$$Q^n = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} P_{n-2} & P_{n-1} & P_{n-3} \\ P_{n-3} & P_{n-2} & P_{n-4} \\ P_{n-4} & P_{n-3} & P_{n-5} \end{bmatrix} \quad (1)$$

The permanent of a matrix is similar to determinant but all of the signs used in the Laplace expansion of minors are positive. The permanent of an n-square matrix is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n [3].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A is *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on colummn k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a nonnegative matrix and B is a contraction of A [4], then

$$perA = perB.$$

The most important applications of permanents are in the areas of physics and chemistry. It is well-known that many mathematical identities can be expressed as special determinants.

It is known that there are a lot of relations between determinants or permanents of matrices and well-known number sequences. For example, the authors [4] derive some relations between the Pell and Perrin sequences and permanents of one type of Hessenberg matrix.

In [5], Lehmer investigate both determinant and permanent of a general tridiagonal matrix and show that the permanent of the tridiagonal matrix based on $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}$, $\{b_i\}$, $\{c_i\}$.

Kılıç [6] investigate the Tribonacci number sequence and derive new recurrence relations for the sequence. Then, he obtain explicit formulas and combinatorial representations for the sums of these sequences.

At present paper, we investigate some properties of the Padovan numbers using matrix methods. We define new matrices which help us to obtain new results for the Padovan sequence. Further on, we present type of $(0,1)$-Hessenberg matrix whose permanents are Padovan numbers.

2 Main results

In this section, we give new generating matrix for Padovan numbers and their sums. Also we investigate some properties of the sequence by matrix methods. Considering the matrix Q, we define the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } B_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{n-1} & P_{n-2} & P_{n-1} & P_{n-3} \\ S_{n-2} & P_{n-3} & P_{n-2} & P_{n-4} \\ S_{n-3} & P_{n-4} & P_{n-3} & P_{n-5} \end{bmatrix}$$

where $S_n = \sum_{k=0}^{n-2} P_k + 1$, $n \geq 2$. By definition of Padovan numbers, we can write the following matrix equation for $n \geq 2$

$$B_n = AB_{n-1}.$$

Then we have the following lemma.

Lemma 1

$$A^n = B_n, \quad n \geq 2.$$

Proof. Using the equality $B_n = AB_{n-1}$, by an inductive argument, we can write $B_n = A^{n-1}B_1$. By definition of Padovan numbers, we obtain $A = B_1$. Then $A^n = B_n$ which is desired. ■

Corollary 2 *If* $n \geq 2$, *then*

$$S_n = S_{n-2} + S_{n-3} + 1.$$

Proof. It can be seen easily by induction on n. ■

As a result of Corollary 2, we also can write the following property (see [2]):

$$\sum_{k=0}^{n-5} P_k = P_n - 2$$

Corollary 3 *If* $n, m \geq 3$, *then*

$$S_{n+m-1} = S_{n-1} + P_{n-2}S_{m-1} + P_{n-1}S_{m-2} + P_{n-3}S_{m-3}.$$

Proof. By definition of matrix B_n, we can write $B_{n+m} = B_nB_m = B_mB_n$ for all $n, m \geq 3$. Using matrix multiplication, we obtain the result. ■

The characteristic equation of Padovan sequence is

$$x^3 - x - 1 = 0$$

and the roots of the equation are

$$\begin{aligned} r_1 &= \frac{w}{6}+\frac{2}{w}, \\ r_2 &= -\frac{w}{12}-\frac{1}{w}+\frac{i\sqrt{3}}{2}(\frac{w}{6}-\frac{2}{w}), \\ r_3 &= -\frac{w}{12}-\frac{1}{w}-\frac{i\sqrt{3}}{2}(\frac{w}{6}-\frac{2}{w}), \end{aligned}$$

where $w=(108+12\sqrt{69})^{1/3}$. The characteristic equation of the sequence has distinct roots and r_1 is unique real solution which is called as *plastic number*. The limit of the ratio of successive Padovan numbers approximate to value $r_1 = 1.324718$ [2].

Lemma 4 *The Binet formula for Padovan sequence is*

$$\begin{aligned} P_{n+1} &= \frac{(r_1+1)}{(r_1-r_2)(r_1-r_3)}r_1^{n+1}+\frac{(r_2+1)}{(r_2-r_1)(r_2-r_3)}r_2^{n+1} \\ &\quad +\frac{(r_3+1)}{(r_2-r_3)(r_1-r_3)}r_3^{n+1}. \end{aligned}$$

Proof. Let us define the generating function (see [7]) of the sequence as $g(x)=P_0+P_1x+P_2x^2+P_3x^3+\cdots+P_nx^n+\cdots$ and

$$\begin{aligned} xg(x) &= P_0x+P_1x^2+P_2x^3+P_3x^4+\cdots+P_{n-1}x^n+\cdots \\ x^2g(x) &= P_0x^2+P_1x^3+P_2x^4+P_3x^5+\cdots+P_{n-2}x^n+\cdots \\ x^3g(x) &= P_0x^3+P_1x^4+P_2x^5+P_3x^6+\cdots+P_{n-3}x^n+\cdots \end{aligned}$$

Then,

$$\begin{aligned} g(x)-x^2g(x)-x^3g(x) &= P_0+P_1x+(P_2-P_0)x^2+\cdots \\ &\quad +(P_n-P_{n-2}-P_{n-3})x^n \\ g(x)[1-x^2-x^3] &= x+1 \end{aligned}$$

here

$$\begin{aligned} g(x) &= \frac{x+1}{1-x^2-x^3}=\frac{a}{1-r_1x}+\frac{b}{1-r_2x}+\frac{c}{1-r_3x} \\ &= a\sum_0^\infty r_1^nx^n+b\sum_0^\infty r_2^nx^n+\sum_0^\infty r_3^nx^n \\ &= [ar_1^n+br_2^n+cr_3^n] \end{aligned}$$

using polynomial equality, we get

$$\begin{aligned} a &= \frac{(r_1+1)r_1}{(r_1-r_2)(r_1-r_3)} \\ b &= \frac{(r_2+1)r_2}{(r_2-r_1)(r_2-r_3)} \\ c &= \frac{(r_3+1)r_3}{(r_2-r_3)(r_1-r_3)} \end{aligned}$$

which is desired. ∎

The eigenvalues of the matrix A are $1, r_1, r_2, r_3$. Define the diagonal matrix D and the matrix V as below, respectively:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_3 \end{bmatrix} \text{ and } V = \begin{bmatrix} -1/2 & 0 & 0 & 0 \\ 1/2 & r_1+1 & r_2+1 & r_3+1 \\ 1/2 & r_1^2 & r_2^2 & r_3^2 \\ 1/2 & r_1 & r_2 & r_3 \end{bmatrix}$$

It is easy to see that $AV = VD$. Since the roots r_1, r_2, r_3 are distinct, $detV \neq 0$.

Theorem 5 *If $n \geq 2$, then $S_{n-1} = P_{n-2} + P_{n-1} + P_{n-3} - 1$.*

Proof. Since $AV = VD$ and $detV \neq 0$, we can write $V^{-1}AV = D$. Namely the matrix A is similar to the matrix D. Then $A^nV = VD^n$. By Lemma 1, we can write

$$B_nV = VD^n \tag{2}$$

by the equality of the (2,1)th elements of the equation (2), we obtain the desired. ∎

Let us define the matrices given below:

$$R = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } K_n = \begin{bmatrix} S_n & S_{n-4}+1 & -S_{n-2} & -S_{n-1} \\ S_{n-1} & S_{n-5}+1 & -S_{n-3} & -S_{n-2} \\ S_{n-2} & S_{n-6}+1 & -S_{n-4} & -S_{n-3} \\ S_{n-3} & S_{n-7}+1 & -S_{n-5} & -S_{n-4} \end{bmatrix}$$

Theorem 6 *If $n \geq 2$, then $R^n = K_n$.*

Proof. Using the equalities, $S_{n+1} = S_n + S_{n-1} - S_{n-3}$ and $S_n = S_{n-2} + S_{n-3} + 1$, we can write $K_n = RK_{n-1}$. By induction, it can be written $K_n = R^{n-1}K_1$. From definitions of the matrices R and K_n, we can write $R = K_1$. So the proof is complete. ∎

The characteristic equation of the matrix R is $x^4 - x^3 - x^2 + 1 = 0$. The roots of this equation are $1, r_1, r_2$ and r_3.

Corollary 7 *The sequence $\{S_n\}$ satisfies the following recursion, for $n > 1$*

$$S_n = S_{n-1} + S_{n-2} - S_{n-4}.$$

Define the Vandermonde matrix V_1 and diagonal matrix D_1 as following:

$$V_1 = \begin{bmatrix} r_1^3 & r_2^3 & r_3^3 & 1 \\ r_1^2 & r_2^2 & r_3^2 & 1 \\ r_1 & r_2 & r_3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let w_i be 4×1 matrix as $w_i = [r_1^{n-i+4}, r_2^{n-i+4}, r_3^{n-i+4}, 1]^T$ and $V_j^{(i)}$ be a matrix obtained from V_1 by replacing the jth column of V_1^T by w_i.

Theorem 8 *For $n > 1$, $k_{ij} = \dfrac{\det(V_j^{(i)})}{\det(V_1)}$ where $K_n = [k_{ij}]$.*

Proof. It is easy to see that $RV_1 = V_1D_1$. Due to r_1, r_2 and r_3 are distinct, the Vandermonde matrix V_1 is invertible. So we can write $V_1^{-1}RV_1 = D_1$. In other words, we can write $R^nV_1 = V_1D_1^n$. By Theorem 6, $K_nV_1 = V_1D_1^n$. Thus we have the following equations system:

$$\begin{aligned} r_1^3k_{i1} + r_1^2k_{i2} + r_1k_{i3} + k_{i4} &= r_1^{n-i+4} \\ r_2^3k_{i1} + r_2^2k_{i2} + r_2k_{i3} + k_{i4} &= r_2^{n-i+4} \\ r_3^3k_{i1} + r_3^2k_{i2} + r_3k_{i3} + k_{i4} &= r_3^{n-i+4} \\ k_{i1} + k_{i2} + k_{i3} + k_{i4} &= 1 \end{aligned}$$

where $K_n = [k_{ij}]$. Using Cramer's rule, the proof is seen. We obtained this result using similar way in [6, Theorem 4]. ∎

Corollary 9 *For $n > 0$;*

$$\begin{aligned} S_n = & \frac{[r_1^{n+3}(r_2-1)(r_2-r_3)(r_3-1) + r_2^{n+3}(r_1-1)(r_1-r_3)(r_3-1)}{[(r_1-1)(r_1-r_3)(r_1-r_2)(r_2-1)(r_2-r_3)(r_3-1)]} \\ & + \frac{-r_3^{n+3}(r_1-1)(r_1-r_2)(r_2-1) + r_1^2(r_2-r_3) - r_2^2(r_1-r_3)}{[(r_1-1)(r_1-r_3)(r_1-r_2)(r_2-1)(r_2-r_3)(r_3-1)]} \\ & + \frac{r_3^2(r_1-r_2)]}{[(r_1-1)(r_1-r_3)(r_1-r_2)(r_2-1)(r_2-r_3)(r_3-1)]} - 2. \end{aligned}$$

Proof. Taking $i = 1$ and $j = 1$ in Theorem 8, $k_{11} = S_n$, computing $\det(V_1)$ and $\det(V_1^{(1)})$, we obtain

$$|V_1| = (r_1-1)(r_2-1)(r_3-1)(r_1-r_2)(r_1-r_3)(r_2-r_3)$$

and

$$\left|V_1^{(1)}\right| = r_1^{n+3}(r_3 - r_2)(r_3 - 1)(1 - r_2) + r_2^{n+3}(r_1 - r_3)(r_1 - 1)(1 - r_3) + r_3^{n+3}(r_2 - r_1)(r_2 - 1)(1 - r_1) + r_1^2(r_3 - r_2) + r_2^2(r_1 - r_3) + r_3^2(r_2 - r_1)$$

respectively. So the proof is complete. ■

3 On the Padovan sequence subscripted $\{4n\}$

In this section, we consider $4n$ subscripted Padovan numbers. Initially we define a new third order recurrence relation for $4n$ subscripted Padovan sequence. Then, we give a new generating matrix for these numbers.

Lemma 10 *For* $n > 1$

$$P_{4(n+1)} = 2P_{4n} + 3P_{4(n-1)} + P_{4(n-2)}$$

where $P_0 = 1,\ P_4 = 2,\ P_8 = 7.$

Proof. By mathematical induction principle, proof can be seen easily. ■

Let us define the following matrices

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$H_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_n & P_{4k} & 3P_{4(k-1)} + P_{4(k-2)} & P_{4(k-1)} \\ s_{n-1} & P_{4(k-1)} & 3P_{4(k-2)} + P_{4(k-3)} & P_{4(k-2)} \\ s_{n-2} & P_{4(k-2)} & 3P_{4(k-3)} + P_{4(k-4)} & P_{4(k-3)} \end{bmatrix}$$

where $s_n = \sum_{k=0}^{n-1} P_{4k}$. Using matrix multiplication property, we can write $H_n = WH_{n-1}$. Then we have the following corollary.

Corollary 11 *For* $n \geq 1$, $W^n = H_n$.

It can be seen that the eigenvalues of the matrix W are r_1^4, r_2^4, r_3^4 and 1. Define the matrices E and Λ as below:

$$E = \begin{bmatrix} -5/16 & 0 & 0 & 0 \\ 1/16 & r_1^8 & r_2^8 & r_3^8 \\ 1/16 & r_1^4 & r_2^4 & r_3^4 \\ 1/16 & 1 & 1 & 1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1^4 & 0 & 0 \\ 0 & 0 & r_2^4 & 0 \\ 0 & 0 & 0 & r_3^4 \end{bmatrix}$$

Theorem 12 *For $n > 1$,* $s_n = \dfrac{P_{4n} + 4P_{4(n-1)} + P_{4(n-2)} - 1}{5}$.

Proof. Since r_1, r_2 and r_3 are distinct, we know that $\det E \neq 0$. Also we have the following equality $WE = E\Lambda$. In other words $W^n E = E\Lambda^n$. By Corollary 11, $H_n E = E\Lambda^n$. By matrix equality, the (2,1)th of the equation gives us s_n. So the theorem is proven. ■

Let us define the following matrices:

$$K = \begin{bmatrix} 3 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$U_n = \begin{bmatrix} s_{n+1} & (s_n - 2s_{n-1} - s_{n-2}) & -(2s_n - s_{n-1}) & -s_n \\ s_n & (s_{n-1} - 2s_{n-2} - s_{n-3}) & -(2s_{n-1} - s_{n-2}) & -s_{n-1} \\ s_{n-1} & (s_{n-2} - 2s_{n-3} - s_{n-4}) & -(2s_{n-2} - s_{n-3}) & -s_{n-2} \\ s_{n-2} & (s_{n-3} - 2s_{n-4} - s_{n-5}) & -(2s_{n-3} - s_{n-4}) & -s_{n-3} \end{bmatrix}.$$

Theorem 13 *For $n > 4$, $K^n = U_n$.*

Proof. By mathematical induction principle, it can be seen. ■

The characteristic polynomial of the matrix K is $x^4 - 3x^3 - x^2 + 2x + 1$. The roots of the characteristic equation of the matrix are $1, r_1$, r_2 and r_3. Define the vandermonde matrix E_1 and diagonal matrix Λ_1 as below:

$$E_1 = \begin{bmatrix} r_1^{12} & r_2^{12} & r_3^{12} & 1 \\ r_1^8 & r_2^8 & r_3^8 & 1 \\ r_1^4 & r_2^4 & r_3^4 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \Lambda_1 = \begin{bmatrix} r_1^4 & 0 & 0 & 0 \\ 0 & r_2^4 & 0 & 0 \\ 0 & 0 & r_3^4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since E_1 is a Vandermonde matrix, $\det(E_1) \neq 0$.

Theorem 14 *For $n > 1$,*

$$\begin{aligned} s_n &= \frac{r_1^{4n+8}}{[(r_1^4 - 1)(r_1^4 - r_3^4)(r_1^4 - r_2^4)]} - \frac{r_2^{4n+8}}{[(r_2^4 - 1)(r_1^4 - r_2^4)(r_2^4 - r_3^4)]} \\ &\quad + \frac{r_3^{4n+8}}{[(r_3^4 - 1)(r_1^4 - r_3^4)(r_2^4 - r_3^4)]}. \end{aligned}$$

Proof. It can be written that $KE_1 = E_1\Lambda_1$. Since $\det(E_1) \neq 0$, E_1 is invertible. Then $E_1^{-1}KE_1 = \Lambda_1$. So $K^n E_1 = E_1\Lambda_1^n$. By Theorem 13, we can rewrite $U_nE_1 = E_1\Lambda_1^n$. In other words, we have the following linear equations system:

$$\begin{aligned} r_1^{12}u_{i1}+r_1^8u_{i2}+r_1^4u_{i3}+u_{i4} &= r_1^{4(n-i)+16} \\ r_2^{12}u_{i1}+r_2^8u_{i2}+r_2^4u_{i3}+u_{i4} &= r_2^{4(n-i)+16} \\ r_3^{12}u_{i1}+r_3^8u_{i2}+r_3^4u_{i3}+u_{i4} &= r_3^{4(n-i)+16} \\ u_{i1}+u_{i2}+u_{i3}+u_{i4} &= 1 \end{aligned}$$

where $U_n = [u_{ij}]$. Let v_i be a 4×1 matrix as below:

$$v_i = [r_1^{4(n-i)+16} \quad r_2^{4(n-i)+16} \quad r_3^{4(n-i)+16} \quad 1]^T$$

and $E_{1,j}^{(i)}$ be a matrix obtained by replacing the jth column of E_1^T by v_i. By Cramer's rule, the solution of the above system is:

$$u_{ij} = \frac{\det(E_{1,j}^{(i)})}{\det(E_1)}$$

since $u_{21} = s_n$,

$$s_n = \frac{\det(E_{1,1}^{(2)})}{\det(E_1)}.$$

In other words

$$\begin{aligned} \det(E_{1,1}^{(2)}) &= r_1^{(4n+8)}(r_2^4-1)(r_3^4-1)(r_2^4-r_3^4) - r_2^{(4n+8)}(r_3^4-1)(r_1^4-1)(r_1^4-r_3^4) \\ &\quad + r_3^{(4n+8)}(r_1^4-1)(r_2^4-1)(r_1^4-r_2^4) \end{aligned}$$

and

$$\det(E_1) = (r_1^4-1)(r_2^4-1)(r_3^4-1)(r_1^4-r_2^4)(r_1^4-r_3^4)(r_2^4-r_3^4).$$

So the proof is complete. We obtained this result using similar way in [6, Theorem 7]. ■

4 Padovan numbers by Hessenberg matrices

In this section, we define one type of $(0,1)$ upper Hessenberg matrix family and show that the permanent of these type of matrices are Padovan numbers. Let us define a Hessenberg matrix as below:

$$H(n) = \begin{bmatrix} 1 & 1 & 1 & & & & \\ 1 & 0 & 1 & 1 & & 0 & \\ & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 & 1 \\ & 0 & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix} \tag{3}$$

Theorem 15 *Let $H(n)$ be an n-square matrix as in (3), then*

$$perH(n) = perH^{(n-2)}(n) = P_{n-2}$$

where P_n is the nth Padovan number.

Proof. By definition of the matrix $H(n)$, it can be contracted on colunm 1. That is:

$$H^{(1)}(n) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & & 1 & 0 & 1 & 1 \\ 0 & 0 & & & 1 & 0 & 1 \\ 0 & 0 & & & & 1 & 0 \end{bmatrix}$$

$H^{(1)}(n)$ also can be contracted on first column:

$$H^{(2)}(n) = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & & 1 & 0 & 1 & 1 \\ 0 & 0 & & & 1 & 0 & 1 \\ 0 & 0 & & & & 1 & 0 \end{bmatrix}$$

Contracting $H^{(2)}(n)$ according to first column:

$$H^{(3)}(n) = \begin{bmatrix} 2 & 3 & 2 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & & 1 & 0 & 1 & 1 \\ 0 & 0 & & & 1 & 0 & 1 \\ 0 & 0 & & & & 1 & 0 \end{bmatrix}$$

Continuing this method, we obtain the rth contraction:

$$H^{(r)}(n) = \begin{bmatrix} P_{r+1} & P_{r+2} & P_r & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & & 1 & 0 & 1 & 1 \\ 0 & 0 & & & 1 & 0 & 1 \\ 0 & 0 & & & & 1 & 0 \end{bmatrix}$$

The $(n-3)$th contraction is:

$$H^{(n-3)}(n) = \begin{bmatrix} P_{n-2} & P_{n-1} & P_{n-3} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and contracting $H^{(n-3)}(n)$ according to first column:

$$H^{(n-2)}(n) = \begin{bmatrix} P_{n-1} & P_{n-2} \\ 1 & 0 \end{bmatrix}$$

which is desired. ■

Corollary 16 *Let $M(n) = [m_{ij}]_{n\times n}$ be an n-square $(0,-1,1)$ Hessenberg matrix as below:*

$$M(n) = \begin{cases} m_{j,j+2} = m_{i+1,i} = 1 \\ m_{i,i+1} = -1 \\ 0, \quad \textit{otherwise} \end{cases}$$

where $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n-2$.

$$\det M(n) = P_{n-2}.$$

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