Super Spanning Connectivity of Augmented Cubes

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Abstract

A *k-container* $C(u,v)$ of G between u and v is a set of k internally disjoint paths between u and v. A k-container $C(u,v)$ of G is a k^*-*container* if it contains all nodes of G. A graph G is k^*-*connected* if there exists a k^*-container between any two distinct nodes. The *spanning connectivity* of G, $\kappa^*(G)$, is defined to be the largest integer k such that G is w^*-connected for all $1 \leq w \leq k$ if G is an 1^*-connected graph and undefined if otherwise. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. In this paper, we prove that the n-dimensional augmented cube AQ_n is super spanning connected.

Keywords: hamiltonian, hamiltonian connected, container, connectivity.

1 Introduction

For the graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. A graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two nodes u and v are *adjacent* if (u,v) is an edge of G. The set of *neighbors* of u, $N_G(u)$, is $\{v \mid (u,v) \in E\}$. The *degree* of a node u of G, $deg_G(u)$, is the number of

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edges incident with u. A graph G is *k-regular* if $deg_G(u) = k$ for every node u in G. A *path* is a sequence of nodes represented by $\langle v_0, v_1, \ldots, v_k \rangle$ with no repeated nodes and (v_i, v_{i+1}) is an edge of G for all $0 \leq i \leq k-1$ when $k \geq 1$ and $\langle v_0 \rangle$ when $k = 0$. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle$, where Q is a path from v_i to v_j. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_0 \rangle$. The *length* of a path Q, $l(Q)$, is the number of edges in Q. The *distance* of nodes u and v of G, $d_G(u, v)$, is the length of the shortest path between u and v. A path is a *hamiltonian path* if it contains all nodes of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct nodes of G. A *cycle* is a path with at least three nodes such that the first node is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every node of G. A graph is *hamiltonian* if it has a hamiltonian cycle.

A *k-container* $C(u, v)$ of G between u and v is a set of k *internally disjoint paths* between u and v. The concept of container is proposed by Hsu [4] to evaluate the performance of communication of an interconnection network. The *connectivity* of G, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's Theorem [9] that there is a k-container between any two distinct nodes of G if G is k-connected.

In this paper, we are interested in specified containers. A k-container $C(u, v)$ of G is a *k^*-container* if it contains all nodes of G. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct nodes. An 1^*-connected graph is actually a hamiltonian connected graph. Moreover, a 2^*-connected graph is a hamiltonian graph. Thus, the concept of k^*-connected graph is a hybrid concept of connectivity and hamiltonicity. We define the *spanning connectivity* of a graph G, $\kappa^*(G)$, to be the largest integer k such that G is w^*-connected for all $1 \leq w \leq k$ if G is an 1^*-connected graph. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. Obviously, the complete graph K_n is super spanning connected.

From the application point of view, k^*-containers can be used in multipath communication. Note that graph connectivity and hamiltonicity are two very interesting topics in graph theory. The concept of spanning connectivity is interesting as well. From the theoretical point of view, we should put as much emphasis as we did for hamiltonian graphs. Recently, several families of interconnection networks are proved to be super spanning connected [7, 8, 10]. In this paper, we prove that the n-dimensional augmented cube AQ_n is super spanning connected.

2 The augmented cubes

Let n be any positive integer. An n-bit binary string $\mathbf{u}$ is a sequence $u_n u_{n-1} \dots u_1$ with $u_i \in \{0,1\}$ for each $1 \le i \le n$. Let $1 \le i \le n$. We use $(\mathbf{u})_i$ to denote u_i. Moreover, we use $(\mathbf{u})^i$ to denote the sequence $u_n u_{n-1} \dots u_{i+1} \overline{u}_i u_{i-1} \dots u_1$ where $\overline{u}_i = 1 - u_i$. Furthermore, we use $(\mathbf{u})^{i\sim}$ to denote the sequence $u_n u_{n-1} \dots u_{i+1} \overline{u}_i \overline{u}_{i-1} \dots \overline{u}_1$. Obviously, $(\mathbf{u})^1 = (\mathbf{u})^{1\sim}$, $((\mathbf{u})^i)^i = \mathbf{u}$, and $((\mathbf{u})^{i\sim})^{i\sim} = \mathbf{u}$. The graph of the *$n$-dimensional augmented cube* [3], AQ_n, consists of all n-bit binary strings - its nodes. Two nodes $\mathbf{u} = u_n u_{n-1} \dots u_1$ and $\mathbf{v} = v_n v_{n-1} \dots v_1$ in AQ_n are adjacent if and only if there is $i \in \{1, 2, \dots, n\}$ such that either $(\mathbf{u})^i = \mathbf{v}$ or $(\mathbf{u})^{i\sim} = \mathbf{v}$.

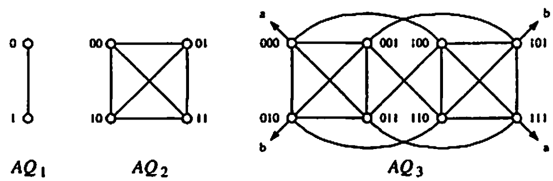

Figure 1: The augmented cubes AQ_1, AQ_2, and AQ_3

The augmented cubes AQ_1, AQ_2, and AQ_3 are illustrated in Figure 1. Some properties are discussed in [3, 5, 6, 11]. It is proved in [3] that AQ_n is node transitive, $(2n-1)$-regular, and $\kappa(AQ_n) = 2n - 1$ if and only if $n \neq 3$ and $\kappa(AQ_3) = 4$. For $i \in \{0,1\}$, let AQ_n^i denote the subgraph of AQ_n induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_n = i$. For $n \ge 2$, AQ_n can be decomposed into AQ_n^0 and AQ_n^1 such that each AQ_n^i is isomorphic to AQ_{n-1}. Thus, the augmented cube can be constructed recursively. For $0 \le i \le 1$ and $0 \le j \le 1$, let $AQ_n^{i,j}$ denote the subgraph of AQ_n induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_n = i$ and $(\mathbf{u})_{n-1} = j$. For $n \ge 3$, AQ_n can be decomposed into $AQ_n^{0,0}$, $AQ_n^{0,1}$, $AQ_n^{1,0}$, and $AQ_n^{1,1}$ such that each $AQ_n^{i,j}$ is isomorphic to AQ_{n-2}.

Theorem 1. *[5] Let F be any subset of $V(AQ_n) \cup E(AQ_n)$. Assume that n is any positive integer with $n \ge 2$ and $n \neq 3$. Then $AQ_n - F$ is 2^*-connected if $|F| \le 2n - 3$ and $AQ_n - F$ is 1^*-connected if $|F| \le 2n - 4$. Moreover, $AQ_3 - F$ is 2^*-connected if $|F| \le 2$ and $AQ_3 - F$ is 1^*-connected if $|F| \le 1$.*

Lemma 1. *Let $\mathbf{u}$ be any two adjacent vertices of AQ_n with $n \ge 3$. Then $AQ_n - \{\mathbf{u}, (\mathbf{u})^{n\sim}\}$ is hamiltonian connected.*

Proof. By Theorem 1, this statement holds for $n \ge 4$. Since AQ_3 is vertex transitive, we assume that $\mathbf{u} = 000$. By brute force, we can check the graph T in Figure 2 is a subgraph of $AQ_3 - \{\mathbf{u} = 000, (\mathbf{u})^{3\sim} = 111\}$. It is easy to check that T is vertex transitive.

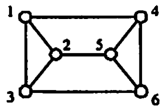

Figure 2: The subgraph T of $AQ_3 - \{000, 111\}$

Let x and y be any two distinct vertices of T. Since T is vertex transitive, we assume that $x = 1$. With the following table, T is hamiltonian connected.

$y = 2$	$\langle 1, 4, 5, 6, 3, 2 \rangle$
$y = 3$	$\langle 1, 4, 6, 5, 2, 3 \rangle$
$y = 4$	$\langle 1, 2, 3, 6, 5, 4 \rangle$
$y = 5$	$\langle 1, 2, 3, 6, 4, 5 \rangle$
$y = 6$	$\langle 1, 3, 2, 5, 4, 6 \rangle$

Since T is hamiltonian connected and T is a subgraph of $AQ_3 - \{000, 111\}$, $AQ_3 - \{000, 111\}$ is hamiltonian connected. □

Theorem 2. *[5] Let $n \geq 4$. Assume that $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{x}, \mathbf{y}\}$ are two pairs of four distinct nodes of AQ_n. There exist two node disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 joins $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans AQ_n.*

We can generalize Theorem 2 by including the case that $\mathbf{x} = \mathbf{y}$. Suppose that $\mathbf{x} = \mathbf{y}$. By Theorem 1, there is a hamiltonian path P between $\mathbf{u}$ and $\mathbf{v}$ in $AQ_n - \{\mathbf{x}\}$. Thus, we have the following theorem.

Theorem 3. *Let $n \geq 4$. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{x}$ be three distinct nodes of AQ_n and $\mathbf{y}$ be any node of $AQ_n - \{\mathbf{u}, \mathbf{v}\}$. Then there exist two node disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 joins $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans AQ_n.*

Let f be the function on $V(AQ_n)$ defined by $f(\mathbf{u}) = \mathbf{u}$ if $(\mathbf{u})_n$=0 and $f(\mathbf{u}) = (\mathbf{u})^{(n-1)\sim}$ if otherwise. The following statement can be proved easily.

Theorem 4. *The function f is an isomorphism of AQ_n into itself.*

3 The k^*-container of AQ_n

Lemma 2. *Let $n \geq 4$ and $k \leq 2n-3$. Suppose that AQ_{n-1} is k^*-connected. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct nodes of AQ_n with $(\mathbf{u})_n = (\mathbf{v})_n$. Then there is a $(k+2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Without loss of generality, we assume that $\mathbf{u} \in AQ_n^0$. Hence, $\mathbf{v} \in AQ_n^0$. Let $\{P_1, P_2, \ldots, P_k\}$ be a k^*-container of AQ_n^0 between $\mathbf{u}$ and $\mathbf{v}$. Since $\lceil \frac{|V(AQ_n^0)|-2}{k} \rceil + 1 \geq 3$, one path in $\{P_1, P_2, \ldots, P_k\}$, say P_k, is of length at least three.

Case 1. $\mathbf{v} \neq (\mathbf{u})^{(n-1)\sim}$. Obviously, $\{(\mathbf{u})^n, (\mathbf{v})^{n\sim}\}$ and $\{(\mathbf{u})^{n\sim}, (\mathbf{v})^n\}$ are two pairs of four distinct nodes of AQ_n^1. By Theorem 2, there exist two node disjoint paths Q_1 and Q_2 such that (1) Q_1 joins $(\mathbf{u})^n$ to $(\mathbf{v})^{n\sim}$, (2) Q_2 joins $(\mathbf{u})^{n\sim}$ to $(\mathbf{v})^n$, and (3) $Q_1 \cup Q_2$ spans AQ_n^1. We set $P_{k+1} = \langle \mathbf{u}, (\mathbf{u})^n, Q_1, (\mathbf{v})^{n\sim}, \mathbf{v} \rangle$ and $P_{k+2} = \langle \mathbf{u}, (\mathbf{u})^{n\sim}, Q_2, (\mathbf{v})^n, \mathbf{v} \rangle$. Obviously, $\{P_1, P_2, \ldots, P_{k+2}\}$ forms a $(k+2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Case 2. $\mathbf{v} = (\mathbf{u})^{(n-1)\sim}$. Since $l(P_k) \geq 3$, we can write P_k as $\langle \mathbf{u}, \mathbf{x}, \mathbf{y}, P, \mathbf{v} \rangle$ where $\mathbf{x}$ and $\mathbf{y}$ are the internal nodes of P_k. Since $\mathbf{v} = (\mathbf{u})^{(n-1)\sim}$, $((\mathbf{u})^n, (\mathbf{v})^n) \in E(AQ_n^1)$. Note that $(\mathbf{v})^n = ((\mathbf{u})^n)^{(n-1)\sim}$. By Lemma 1, there is a hamiltonian path Q of $AQ_n^1 - \{(\mathbf{u})^n, (\mathbf{v})^n\}$ joining the nodes $(\mathbf{x})^n$ and $(\mathbf{y})^n$. We set P_k' as $\langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^n, Q, (\mathbf{y})^n, \mathbf{y}, P, \mathbf{v} \rangle$, P_{k+1} as $\langle \mathbf{u}, (\mathbf{u})^n, \mathbf{v} \rangle$, and P_{k+2} as $\langle \mathbf{u}, (\mathbf{u})^{n\sim}, \mathbf{v} \rangle$. Then $\{P_1, P_2, \ldots, P_{k-1}, P_k', P_{k+1}, P_{k+2}\}$ forms a $(k+2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. □

Lemma 3. *Let $n \geq 4$ and $k \leq 2n-3$. Suppose that AQ_{n-1} is k^*-connected. Let $\mathbf{u}$ and $\mathbf{v}$ be two adjacent nodes of AQ_n with $(\mathbf{u})_n \neq (\mathbf{v})_n$. Then there is a $(k+2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Without loss of generality, we assume that $\mathbf{u} \in AQ_n^0$. By Theorem 4, we can assume that $\mathbf{v} = (\mathbf{u})^n \in AQ_n^1$. Let $\{P_1, P_2, \ldots, P_k\}$ be a k^*-container of AQ_n^0 between $\mathbf{u}$ and $(\mathbf{v})^{n\sim}$. Without loss of generality, we can assume that $l(P_1) \geq l(P_2) \geq \cdots \geq l(P_k)$. Now, we write P_i as $\langle \mathbf{u} = \mathbf{q}_{i_1}, \mathbf{q}_{i_2}, \ldots, \mathbf{q}_{i_{l(P_i)}}, (\mathbf{v})^{n\sim} \rangle$ for every $1 \leq i \leq k$. Obviously, $l(P_i) \geq 2$ for every $1 \leq i \leq k-1$. We set P_i' as $\langle \mathbf{u} = \mathbf{q}_{i_1}, \mathbf{q}_{i_2}, \ldots, \mathbf{q}_{i_{l(P_i)}}, (\mathbf{q}_{i_{l(P_i)}})^n, \ldots, (\mathbf{q}_{i_2})^n, (\mathbf{q}_{i_1})^n = \mathbf{v} \rangle$ for every $1 \leq i \leq k-1$, P_k' as $\langle \mathbf{u} = \mathbf{q}_{k_1}, \mathbf{q}_{k_2}, \ldots, \mathbf{q}_{k_{l(P_k)}}, (\mathbf{v})^{n\sim}, \mathbf{v} \rangle$, P_{k+1}' as $\langle \mathbf{u}, (\mathbf{u})^{n\sim}, (\mathbf{q}_{k_{l(P_k)}})^n, \ldots, (\mathbf{q}_{k_2})^n, (\mathbf{q}_{k_1})^n = \mathbf{v} \rangle$, and P_{k+2}' as $\langle \mathbf{u}, \mathbf{v} \rangle$. Then $\{P_1', P_2', \ldots, P_{k+2}'\}$ forms a $(k+2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. □

Lemma 4. *Let $n \geq 5$ and k be any two positive integers with $3 \leq k \leq 2n-4$. Suppose that AQ_{n-1} is k^*-connected. Let $\mathbf{u}$ and $\mathbf{v}$ be any two nodes of AQ_n with $(\mathbf{u}, \mathbf{v}) \notin E(AQ_n)$ and $(\mathbf{u})_n \neq (\mathbf{v})_n$. Then there is a $(k+1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Without loss of generality, we assume that $\mathbf{u} \in AQ_n^0$, $\mathbf{v} \in AQ_n^1$, and $\mathbf{v} \notin \{(\mathbf{u})^n, (\mathbf{u})^{n\sim}\}$. Since AQ_n^0 is isomorphic to AQ_{n-1}, there is a k^*-container $\{Q_1, Q_2, \ldots, Q_k\}$ of AQ_n^0 between $\mathbf{u}$ and $(\mathbf{v})^n$. Assume that $l(Q_1) \leq l(Q_2) \leq \cdots \leq l(Q_k)$. Note that $l(Q_1) > 1$ if $(\mathbf{u}, (\mathbf{v})^n) \notin E(AQ_n^0)$.

For every $2 \le i \le k$, we write Q_i as $\langle \mathbf{u}, H_i, \mathbf{x}_i, (\mathbf{v})^n \rangle$ where H_i is a path joining $\mathbf{u}$ and $\mathbf{x}_i$ and $l(H_i) \ge 1$. We set $F = \{(\mathbf{x}_3)^n, (\mathbf{x}_4)^n, \ldots, (\mathbf{x}_k)^n\}$. Since $|F| = k-2 \le 2n-6$, by Theorem 1, there is a hamiltonian path R of $AQ_n^1 - F$ joining $(\mathbf{u})^n$ to $(\mathbf{x}_2)^n$. We can write R as $\langle (\mathbf{u})^n, R_1, \mathbf{v}, R_2, (\mathbf{x}_2)^n \rangle$. We set

$$\begin{aligned}
P_1 &= \langle \mathbf{u}, Q_1, (\mathbf{v})^n, \mathbf{v} \rangle, \\
P_2 &= \langle \mathbf{u}, H_2, \mathbf{x}_2, (\mathbf{x}_2)^n, R_2^{-1}, \mathbf{v} \rangle, \\
P_i &= \langle \mathbf{u}, H_i, \mathbf{x}_i, (\mathbf{x}_i)^n, \mathbf{v} \rangle \text{ for every } 3 \le i \le k, \text{ and} \\
P_{k+1} &= \langle \mathbf{u}, (\mathbf{u})^n, R_1, \mathbf{v} \rangle.
\end{aligned}$$

Then $\{P_1, P_2, \ldots, P_{k+1}\}$ forms a $(k+1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. □

Lemma 5. *Let $n \ge 4$. Suppose that* $\mathbf{x}$ *and* $\mathbf{y}$ *are two distinct nodes of AQ_n with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$. Then $\{(\mathbf{x})^{n-1}, (\mathbf{x})^{(n-1)\sim}\} \cap \{(\mathbf{y})^{n-1}, (\mathbf{y})^{(n-1)\sim}\} = \emptyset$.*

Proof. Let $\mathbf{x} = x_n x_{n-1} \ldots x_1$ and $\mathbf{y} = y_n y_{n-1} \ldots y_1$ be two distinct nodes in AQ_n. We prove this lemma by contradiction. Assume that $(\mathbf{x})^{n-1} \in \{(\mathbf{y})^{n-1}, (\mathbf{y})^{(n-1)\sim}\}$. Suppose that $(\mathbf{x})^{n-1} = (\mathbf{y})^{n-1}$. Then $\mathbf{x} = \mathbf{y}$. Hence, $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 0$. Suppose that $(\mathbf{x})^{n-1} = (\mathbf{y})^{(n-1)\sim}$. Then $(\mathbf{x})^{(n-2)\sim} = \mathbf{y}$ and $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. We get a contradiction. Thus, $(\mathbf{x})^{n-1} \notin \{(\mathbf{y})^{n-1}, (\mathbf{y})^{(n-1)\sim}\}$. Similarly, $(\mathbf{x})^{(n-1)\sim} \notin \{(\mathbf{y})^{n-1}, (\mathbf{y})^{(n-1)\sim}\}$. Thus, $\{(\mathbf{x})^{n-1}, (\mathbf{x})^{(n-1)\sim}\} \cap \{(\mathbf{y})^{n-1}, (\mathbf{y})^{(n-1)\sim}\} = \emptyset$. □

Lemma 6. *The AQ_n is 3^*-connected if $n \ge 2$.*

Proof. Since AQ_2 is isomorphic to K_4, AQ_2 is 3^*-connected. Thus, we assume that $n \ge 3$. We need to find a 3^*-container of AQ_n between any two nodes $\mathbf{u}$ and $\mathbf{v}$ of AQ_n. Without loss of generality, we assume that $\mathbf{u} \in AQ_n^0$.

Case 1. $\mathbf{v} \in AQ_n^0$. By Theorem 1, there is a 2^*-container $\{P_1, P_2\}$ of AQ_n^0 between $\mathbf{u}$ and $\mathbf{v}$. Again, by Theorem 1, there is a hamiltonian path H of AQ_n^1 joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We set P_3 as $\langle \mathbf{u}, (\mathbf{u})^n, H, (\mathbf{v})^n, \mathbf{v} \rangle$. Then $\{P_1, P_2, P_3\}$ forms a 3^*-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Case 2. $\mathbf{v} \in AQ_n^1$. Obviously, either $(\mathbf{u})^n \ne \mathbf{v}$ or $(\mathbf{u})^{n\sim} \ne \mathbf{v}$. Without loss of generality, we assume that $(\mathbf{u})^n \ne \mathbf{v}$. Note that $(\mathbf{v})^n \ne \mathbf{u}$. Since $|V(AQ_n^0)| \ge 4$, there exists a node $\mathbf{x} \in V(AQ_n^0) - \{\mathbf{u}, (\mathbf{v})^n\}$. By Theorem 1, there is a hamiltonian path H of AQ_n^0 joining $\mathbf{x}$ to $(\mathbf{v})^n$. Moreover, there is a hamiltonian path R of AQ_n^1 joining $(\mathbf{u})^n$ to $(\mathbf{x})^n$. Obviously, we can write H as $\langle \mathbf{x}, H_1, \mathbf{u}, H_2, (\mathbf{v})^n \rangle$ and write R as $\langle (\mathbf{u})^n, R_1, \mathbf{v}, R_2, (\mathbf{x})^n \rangle$. We set

$$\begin{aligned} P_1 &= \langle \mathbf{u}, (\mathbf{u})^n, R_1, \mathbf{v} \rangle, \\ P_2 &= \langle \mathbf{u}, H_1^{-1}, \mathbf{x}, (\mathbf{x})^n, R_2^{-1}, \mathbf{v} \rangle, \text{ and} \\ P_3 &= \langle \mathbf{u}, H_2, (\mathbf{v})^n, \mathbf{v} \rangle. \end{aligned}$$

Then $\{P_1, P_2, P_3\}$ forms a 3*-container of AQ_n between **u** and **v**.

Thus, the lemma is proved. □

Lemma 7. *The AQ_3 is super spanning connected.*

Proof. By Theorem 1, AQ_3 is i^*-connected for $1 \le i \le 2$. By Lemma 6, AQ_3 is 3*-connected. Let **u** and **v** be any two distinct nodes of AQ_3. Since AQ_3 is node transitive, we can assume that $\mathbf{u} = 000$. By Theorem 4, we can assume that $\mathbf{v} \in \{001, 011, 100, 101\}$. With the following table, AQ_3 is 4*-connected.

v = 001	⟨000, 011, 001⟩ ⟨000, 010, 001⟩ ⟨000, 100, 101, 001⟩ ⟨000, 111, 110, 001⟩	v = 011	⟨000, 011⟩ ⟨000, 001, 011⟩ ⟨000, 010, 011⟩ ⟨000, 100, 101, 110, 111, 011⟩
v = 100	⟨000, 100⟩ ⟨000, 001, 101, 100⟩ ⟨000, 011, 111, 100⟩ ⟨000, 010, 110, 100⟩	v = 101	⟨000, 001, 101⟩ ⟨000, 100, 101⟩ ⟨000, 010, 101⟩ ⟨000, 011, 111, 110, 101⟩

Since $\kappa(AQ_3) = 4$, AQ_3 is super spanning connected. □

Lemma 8. *The AQ_4 is super spanning connected.*

Proof. By Theorem 1 and Lemma 6, AQ_4 is i^*-connected for $1 \le i \le 3$. Let **u** and **v** be any two distinct nodes in AQ_4. Since AQ_4 is node transitive, we assume that $\mathbf{u} = 0000$. We need to find a k^*-container of AQ_4 between **u** and **v** for every $4 \le k \le 7$.

Case 1. $v \in AQ_4^0$. By Lemma 2 and Lemma 7, there is a k^*-container of AQ_4 between **u** and **v** for $4 \le k \le 6$. The 7*-container of AQ_4 between **u** and $\mathbf{v} \in \{0001, 0010, 0011, 0100, 0101, 0110, 0111\}$ is listed below.

v = 0001	⟨0000, 0001⟩ ⟨0000, 0010, 0001⟩ ⟨0000, 0011, 0001⟩ ⟨0000, 0100, 0101, 0001⟩ ⟨0000, 0111, 0110, 0001⟩ ⟨0000, 1000, 1010, 1011, 1001, 0001⟩ ⟨0000, 1111, 1101, 1100, 1110, 0001⟩
v = 0010	⟨0000, 0010⟩ ⟨0000, 0001, 0010⟩ ⟨0000, 0011, 0010⟩ ⟨0000, 0100, 0110, 0010⟩ ⟨0000, 0101, 0111, 0010⟩ ⟨0000, 1000, 1001, 1011, 1010, 0010⟩ ⟨0000, 1111, 1110, 1100, 1101, 0010⟩
v = 0011	⟨0000, 0011⟩ ⟨0000, 0001, 0011⟩ ⟨0000, 0010, 0011⟩ ⟨0000, 0100, 0011⟩ ⟨0000, 0111, 0011⟩ ⟨0000, 1000, 1001, 1010, 1011, 0011⟩ ⟨0000, 1111, 1110, 0110, 0101, 1101, 1100, 0011⟩

v = 0100	⟨0000, 0100⟩ ⟨0000, 0111, 0100⟩ ⟨0000, 0011, 0100⟩ ⟨0000, 0010, 0110, 0100⟩ ⟨0000, 0001, 0101, 0100⟩ ⟨0000, 1000, 1001, 1010, 1011, 0100⟩ ⟨0000, 1111, 1110, 1101, 1100, 0100⟩
v = 0101	⟨0000, 0001, 0101⟩ ⟨0000, 0010, 0101⟩ ⟨0000, 0100, 0101⟩ ⟨0000, 0111, 0101⟩ ⟨0000, 1111, 1101, 0101⟩ ⟨0000, 0011, 1100, 1110, 0110, 0101⟩ ⟨0000, 1000, 1001, 1011, 1010, 0101⟩
v = 0110	⟨0000, 0001, 0110⟩ ⟨0000, 0010, 0110⟩ ⟨0000, 0100, 0110⟩ ⟨0000, 0111, 0110⟩ ⟨0000, 1000, 1001, 0110⟩ ⟨0000, 0011, 1011, 1010, 0101, 0110⟩ ⟨0000, 1111, 1101, 1100, 1110, 0110⟩
v = 0111	⟨0000, 0111⟩ ⟨0000, 0011, 0111⟩ ⟨0000, 0100, 0111⟩ ⟨0000, 1000, 0111⟩ ⟨0000, 1111, 0111⟩ ⟨0000, 0011, 1010, 1101, 0101, 0111⟩ ⟨0000, 0001, 1001, 1011, 1100, 1110, 0110, 0111⟩

Case 2. $\mathbf{v} \in AQ_4^1$ and $\mathbf{v} \in \{1000, 1111\}$. By Theorem 4, we can assume that $\mathbf{v} = 1000$. By Lemma 3 and Lemma 7, there is a k^*-container of AQ_4 between **u** and **v** for $4 \le k \le 6$. The 7*-container of AQ_4 between **u** and **v** is listed below.

v = 1000	⟨0000, 1000⟩ ⟨0000, 0111, 1000⟩ ⟨0000, 1111, 1000⟩ ⟨0000, 0100, 1100, 1000⟩ ⟨0000, 0011, 1011, 1000⟩ ⟨0000, 0010, 0110, 1110, 1010, 1000⟩ ⟨0000, 0001, 0101, 1101, 1001, 1000⟩

Case 3. $\mathbf{v} \in AQ_4^1$ and $\mathbf{v} \notin \{1000, 1111\}$. By Theorem 4, we can assume that $\mathbf{v} \in \{1001, 1010, 1011\}$. Since $\mathbf{v} \notin \{(\mathbf{u})^n, (\mathbf{u})^{n\sim}\}$, $\mathbf{u} \notin \{(\mathbf{v})^n, (\mathbf{v})^{n\sim}\}$. By Theorem 1, there is a hamiltonian path R of AQ_4^0 joining $(\mathbf{v})^n$ to $(\mathbf{v})^{n\sim}$. Moreover, there is a hamiltonian path H of AQ_4^1 joining $(\mathbf{u})^n$ to $(\mathbf{u})^{n\sim}$. Without loss of generality, we write $R = \langle (\mathbf{v})^n, R_1, u, R_2, (\mathbf{v})^{n\sim} \rangle$ and $H = \langle (\mathbf{u})^n, H_1, v, H_2, (\mathbf{u})^{n\sim} \rangle$. We set $P_1 = \langle \mathbf{u}, R_1^{-1}, (\mathbf{v})^n, \mathbf{v}, \rangle$, $P_2 = \langle \mathbf{u}, R_2, (\mathbf{v})^{n\sim}, \mathbf{v} \rangle$, $P_3 = \langle \mathbf{u}, (\mathbf{u})^n, H_1, \mathbf{v} \rangle$, and $P_4 = \langle \mathbf{u}, (\mathbf{u})^{n\sim}, H_2^{-1}, \mathbf{v} \rangle$. Then $\{P_1, P_2, P_3, P_4\}$ forms a 4*-container of AQ_4 between **u** and **v**. The 5*-container, 6*-container, and the 7*-container of AQ_4 between **u** and **v** are listed below.

v	5*-container	6*-container
1001	⟨0000, 0010, 1101, 1001⟩ ⟨0000, 0100, 1011, 1001⟩ ⟨0000, 0011, 1100, 1000, 1001⟩ ⟨0000, 0001, 0101, 1010, 1001⟩ ⟨0000, 0111, 1111, 1110, 0110, 1001⟩	⟨0000, 0010, 1101, 1001⟩ ⟨0000, 0100, 1011, 1001⟩ ⟨0000, 0111, 0110, 1001⟩ ⟨0000, 1111, 1110, 1001⟩ ⟨0000, 0001, 0101, 1010, 1001⟩ ⟨0000, 0011, 1100, 1000, 1001⟩
1010	⟨0000, 1111, 1101, 1010⟩ ⟨0000, 0111, 0101, 1010⟩ ⟨0000, 0100, 1100, 1000, 1010⟩ ⟨0000, 0010, 0110, 1110, 1010⟩ ⟨0000, 0001, 0011, 1011, 1001, 1010⟩	⟨0000, 0001, 1001, 1010⟩ ⟨0000, 0011, 1011, 1010⟩ ⟨0000, 0111, 0101, 1010⟩ ⟨0000, 1111, 1101, 1010⟩ ⟨0000, 0010, 0110, 1110, 1010⟩ ⟨0000, 0100, 1100, 1000, 1010⟩
1011	⟨0000, 1000, 1011⟩ ⟨0000, 0010, 1101, 1100, 1011⟩ ⟨0000, 0100, 0101, 1010, 1011⟩ ⟨0000, 0111, 0110, 1001, 1011⟩ ⟨0000, 0011, 0001, 1110, 1111, 1011⟩	⟨0000, 0011, 1011⟩ ⟨0000, 1000, 1011⟩ ⟨0000, 0001, 1110, 1111, 1011⟩ ⟨0000, 0010, 1101, 1100, 1011⟩ ⟨0000, 0100, 0101, 1010, 1011⟩ ⟨0000, 0111, 0110, 1001, 1011⟩

v	7^*-container
1001	⟨0000, 0001, 1001⟩
	⟨0000, 1000, 1001⟩
	⟨0000, 0010, 1010, 1001⟩
	⟨0000, 1111, 1011, 1001⟩
	⟨0000, 0111, 0110, 1001⟩
	⟨0000, 0011, 1100, 1110, 1001⟩
	⟨0000, 0100, 0101, 1101, 1001⟩
1010	⟨0000, 0010, 1010⟩
	⟨0000, 1000, 1010⟩
	⟨0000, 0001, 1001, 1010⟩
	⟨0000, 0011, 1011, 1010⟩
	⟨0000, 1111, 1110, 1010⟩
	⟨0000, 0100, 1100, 1101, 1010⟩
	⟨0000, 0111, 0110, 0101, 1010⟩
1011	⟨0000, 0011, 1011⟩
	⟨0000, 0100, 1011⟩
	⟨0000, 1000, 1011⟩
	⟨0000, 1111, 1011⟩
	⟨0000, 0001, 0101, 1010, 1011⟩
	⟨0000, 0010, 1101, 1001, 1011⟩
	⟨0000, 0111, 0110, 1110, 1100, 1011⟩

Thus, this lemma is proved. □

Lemma 9. *The AQ_5 is super spanning connected.*

Proof. By Theorem 1 and Lemma 6, AQ_5 is i^*-connected for $1 \leq i \leq 3$. Let **u** and **v** be any two distinct nodes in AQ_5. We need to find a k^*-container of AQ_5 between **u** and **v** for every $4 \leq k \leq 9$. Since AQ_5 is node transitive, we assume that $\mathbf{u} = 00000 \in AQ_5^0$. We have the following cases.

Case 1. $\mathbf{v} \in AQ_5^0$. By Lemma 2 and Lemma 8, there is a k^*-container of AQ_5 between **u** and **v** for every $4 \leq k \leq 9$.

Case 2. $\mathbf{v} \in AQ_5^1$ and $\mathbf{v} \in \{10000, 11111\}$. By Lemma 3 and Lemma 8, there is a k^*-container of AQ_5 between **u** and **v** for every $4 \leq k \leq 9$.

Case 3. $\mathbf{v} \in AQ_5^1$ and $\mathbf{v} \notin \{10000, 11111\}$. By Theorem 4, we assume that $\mathbf{v} \in \{10001, 10010, 10011, 10100, 10101, 10110, 10111\}$. By Lemma 4 and Lemma 8, there is a k^*-container of AQ_5 between **u** and **v** for $4 \leq k \leq 7$. The 8^*-container and the 9^*-container of AQ_5 between **u** and **v** are listed below.

v	8^*-container
10001	⟨00000, 00001, 10001⟩
	⟨00000, 10000, 10001⟩
	⟨00000, 00010, 10010, 10001⟩
	⟨00000, 00010, 10010, 10001⟩
	⟨00000, 01111, 01101, 01100, 01110, 10001⟩
	⟨00000, 11111, 11101, 11100, 11110, 10001⟩
	⟨00000, 01000, 01001, 01011, 01010, 11010, 11000, 11011, 11001, 10001⟩
	⟨00000, 00100, 00110, 00101, 00111, 10111, 10101, 10100, 10110, 10001⟩
10010	⟨00000, 00010, 10010⟩
	⟨00000, 10000, 10010⟩
	⟨00000, 00001, 10001, 10010⟩
	⟨00000, 00011, 10011, 10010⟩
	⟨00000, 11111, 11110, 11100, 11101, 10010⟩
	⟨00000, 01111, 01110, 01100, 01101, 10010⟩
	⟨00000, 01000, 01010, 01011, 01001, 11001, 11011, 11000, 11010, 10010⟩
	⟨00000, 00100, 00110, 00111, 00101, 10101, 10111, 10100, 10110, 10010⟩
10011	⟨00000, 00011, 10011⟩
	⟨00000, 10000, 10011⟩
	⟨00000, 00001, 10001, 10011⟩
	⟨00000, 00010, 10010, 10011⟩
	⟨00000, 11111, 11101, 11110, 11100, 10011⟩
	⟨00000, 01111, 01101, 01110, 01100, 10011⟩
	⟨00000, 01000, 01001, 01011, 01010, 11010, 11000, 11001, 11011, 10011⟩
	⟨00000, 00100, 00110, 00111, 00101, 10101, 10111, 10110, 10100, 10011⟩

10101	⟨00000, 10000, 10100, 10101⟩
	⟨00000, 00001, 10001, 10101⟩
	⟨00000, 00010, 10010, 10010, 10101⟩
	⟨00000, 00011, 10011, 10111, 10101⟩
	⟨00000, 11111, 11110, 11100, 11101, 10101⟩
	⟨00000, 01000, 11000, 11001, 11011, 11010, 10101⟩
	⟨00000, 00100, 00101, 00111, 00110, 10110, 10101⟩
	⟨00000, 01111, 01101, 01100, 01110, 01001, 01011, 01010, 10101⟩
10100	⟨00000, 10000, 10100⟩
	⟨00000, 00011, 10011, 10100⟩
	⟨00000, 00001, 10001, 10101, 10100⟩
	⟨00000, 00010, 10010, 10110, 10100⟩
	⟨00000, 01000, 01010, 01001, 01011, 10100⟩
	⟨00000, 01111, 01101, 01110, 01100, 11100, 10100⟩
	⟨00000, 00100, 00101, 00110, 00111, 10111, 10100⟩
	⟨00000, 11111, 11101, 11110, 11001, 11000, 11010, 11011, 10100⟩
10110	⟨00000, 00001, 10001, 10110⟩
	⟨00000, 10000, 10100, 10110⟩
	⟨00000, 00011, 10011, 10110⟩
	⟨00000, 00010, 10010, 10110⟩
	⟨00000, 11111, 11101, 11100, 11110, 10110⟩
	⟨00000, 00100, 00110, 00111, 00101, 10101, 10110⟩
	⟨00000, 01111, 01101, 01100, 01110, 01010, 01001, 10110⟩
	⟨00000, 01000, 01011, 11011, 11010, 11000, 11001, 10110⟩
10111	⟨00000, 11111, 10111⟩
	⟨00000, 01000, 10111⟩
	⟨00000, 10000, 10111⟩
	⟨00000, 00111, 10111⟩
	⟨00000, 00011, 10011, 10111⟩
	⟨00000, 00001, 00101, 00100, 00110, 10110, 10100, 10111⟩
	⟨00000, 00010, 01010, 01011, 01001, 11001, 11011, 11010, 11000, 10111⟩
	⟨00000, 01111, 01101, 01100, 01110, 11110, 11100, 11101, 10010, 10001, 10101, 10111⟩

v	$\mathcal{G}^*$-container
10001	⟨00000, 00001, 10001⟩
	⟨00000, 10000, 10001⟩
	⟨00000, 00011, 10011, 10001⟩
	⟨00000, 00010, 10010, 10001⟩
	⟨00000, 00100, 00110, 00101, 10101, 10001⟩
	⟨00000, 00111, 10111, 10100, 10110, 10001⟩
	⟨00000, 01111, 01101, 01100, 01110, 10001⟩
	⟨00000, 11111, 11101, 11100, 11110, 10001⟩
	⟨00000, 01000, 01001, 01011, 01010, 11010, 11000, 11011, 11001, 10001⟩
10010	⟨00000, 00010, 10010⟩
	⟨00000, 10000, 10010⟩
	⟨00000, 00001, 10001, 10010⟩
	⟨00000, 00011, 10011, 10010⟩
	⟨00000, 00100, 00110, 00101, 10101, 10010⟩
	⟨00000, 00111, 10111, 10100, 10110, 10010⟩
	⟨00000, 01111, 01110, 01100, 01101, 10010⟩
	⟨00000, 11111, 11110, 11100, 11101, 10010⟩
	⟨00000, 01000, 01010, 01011, 01001, 11001, 11011, 11000, 11010, 10010⟩
10011	⟨00000, 00011, 10011⟩
	⟨00000, 10000, 10011⟩
	⟨00000, 00001, 10001, 10011⟩
	⟨00000, 00010, 10010, 10011⟩
	⟨00000, 00111, 00101, 10101, 10111, 10011⟩
	⟨00000, 01111, 01101, 01110, 01100, 10011⟩
	⟨00000, 11111, 11101, 11110, 11100, 10011⟩
	⟨00000, 00100, 00110, 10110, 10100, 10011⟩
	⟨00000, 01000, 01001, 01011, 01010, 11010, 11000, 11001, 11011, 10011⟩
10101	⟨00000, 00100, 00101, 10101⟩
	⟨00000, 00111, 00110, 10101⟩
	⟨00000, 10000, 10100, 10101⟩
	⟨00000, 00001, 10001, 10101⟩
	⟨00000, 00010, 10010, 10010, 10101⟩
	⟨00000, 00011, 10011, 10111, 10101⟩
	⟨00000, 11111, 11110, 11100, 11101, 10101⟩
	⟨00000, 01000, 11000, 11001, 11011, 11010, 10101⟩
	⟨00000, 01111, 01101, 01100, 01110, 01001, 01011, 01010, 10101⟩
10100	⟨00000, 10000, 10100⟩
	⟨00000, 00100, 10100⟩
	⟨00000, 00111, 10111, 10100⟩
	⟨00000, 00011, 10011, 10100⟩
	⟨00000, 00010, 10010, 10110, 10100⟩
	⟨00000, 00001, 10001, 10101, 10100⟩
	⟨00000, 01000, 01010, 01001, 01011, 10100⟩
	⟨00000, 11111, 11101, 11110, 11001, 11000, 11010, 11011, 10100⟩
	⟨00000, 01111, 01101, 00101, 00110, 01110, 01100, 11100, 10100⟩

10110	⟨00000, 00100, 00110, 10110⟩
	⟨00000, 00011, 10011, 10110⟩
	⟨00000, 10000, 10100, 10110⟩
	⟨00000, 00001, 10001, 10110⟩
	⟨00000, 00010, 10010, 10110⟩
	⟨00000, 00111, 00101, 10101, 10110⟩
	⟨00000, 11111, 11101, 11100, 11110, 10110⟩
	⟨00000, 01111, 01101, 01100, 01110, 01010, 01001, 10110⟩
	⟨00000, 01000, 01011, 11011, 11010, 11000, 11001, 10110⟩
10111	⟨00000, 11111, 10111⟩
	⟨00000, 01000, 10111⟩
	⟨00000, 10000, 10111⟩
	⟨00000, 00111, 10111⟩
	⟨00000, 00011, 10011, 10111⟩
	⟨00000, 00100, 10100, 10111⟩
	⟨00000, 00001, 00101, 00110, 10110, 10111⟩
	⟨00000, 00010, 01010, 01011, 01001, 11001, 11011, 11010, 11000, 10111⟩
	⟨00000, 01111, 01101, 01100, 01110, 11110, 11100, 11101, 10010, 10001, 10101, 10111⟩

Hence, this lemma is proved. □

Lemma 10. *The AQ_n is super spanning connected if $n \geq 4$.*

Proof. We prove this lemma by induction. By Lemma 8 and Lemma 9, AQ_4 and AQ_5 are super spanning connected. Thus, we assume this lemma holds on AQ_t for every $4 \leq t < n$ with $n \geq 6$.

By Theorem 1 and Lemma 6, AQ_n is k^*-connected for $1 \leq k \leq 3$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes in AQ_n. We need to find a k^*-container for $4 \leq k \leq 2n-1$ in AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Suppose that $(\mathbf{u})_n = (\mathbf{v})_n$. By Lemma 2, there is a k^*-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$ for every $4 \leq k \leq 2n-1$. Suppose that $(\mathbf{u})_n \neq (\mathbf{v})_n$ and $(\mathbf{u}, \mathbf{v}) \in E(AQ_n)$. By Lemma 3, there is a k^*-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$ for every $4 \leq k \leq 2n-1$. Thus, we only need to consider the case that $(\mathbf{u})_n \neq (\mathbf{v})_n$ and $(\mathbf{u}, \mathbf{v}) \notin E(AQ_n)$. Without loss of generality, we assume that $\mathbf{u} \in AQ_n^{0,0}$. By Theorem 4, we can assume that $\mathbf{v} \in AQ_n^{1,0}$.

Suppose that $4 \leq k \leq 2n-3$. By induction hypothesis, AQ_n^0 and AQ_n^1 are i^*-connected for $3 \leq i \leq 2n-4$. By Lemma 4, there is an $(i+1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$ with $(\mathbf{u})_n \neq (\mathbf{v})_n$ and $(\mathbf{u}, \mathbf{v}) \notin E(AQ_n)$. Hence, there is a k^*-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$ with $(\mathbf{u})_n \neq (\mathbf{v})_n$ and $(\mathbf{u}, \mathbf{v}) \notin E(AQ_n)$ for $4 \leq k \leq 2n-3$.

Suppose that $2n-2 \leq k \leq 2n-1$. We have two cases: $(\mathbf{u}, (\mathbf{v})^n) \notin E(AQ_n^{0,0})$ and $(\mathbf{u}, (\mathbf{v})^n) \in E(AQ_n^{0,0})$.

Case 1. $(\mathbf{u}, (\mathbf{v})^n) \notin E(AQ_n^{0,0})$. By induction hypothesis, there is a $(2n-5)^*$-container, $\{Q_1, Q_2, \ldots, Q_{2n-5}\}$, of $AQ_n^{0,0}$ between $\mathbf{u}$ and $(\mathbf{v})^n$. Since $AQ_n^{0,0}$ is $(2n-5)$-regular, we can rearrange the indices so that Q_i can be written as $\langle \mathbf{u}, U_i, \mathbf{w}_i, (\mathbf{v})^n \rangle$ for $1 \leq i \leq 2n-5$ with $\{\mathbf{w}_1, \mathbf{w}_2\} = \{((\mathbf{v})^n)^2, ((\mathbf{v})^n)^3\}$ and $\mathbf{w}_3 = ((\mathbf{v})^n)^4$. Obviously, $d_{AQ_n^{0,0}}(\mathbf{w}_i, \mathbf{w}_j) \geq 2$ and $d_{AQ_n^{1,0}}((\mathbf{w}_i)^n, (\mathbf{w}_j)^n) \geq 2$ for $1 \leq i \neq j \leq 3$. Let $F = \{(\mathbf{w}_j)^n \mid 4 \leq j \leq 2n-5\}$. Obviously, $|F| = 2n-8$. By Theorem 1, there is a hamiltonian path H of $AQ_n^{1,0} - F$ between $(\mathbf{u})^n$ and $(\mathbf{w}_3)^n$. Since the neighbors of $\mathbf{v}$ in $AQ_n^{1,0} - (F \cup \{(\mathbf{w}_3)^n\})$ are $(\mathbf{w}_1)^n$ and $(\mathbf{w}_2)^n$, either $H =$

$\langle (\mathbf{u})^n, R_1, \mathbf{x}, R_2, \mathbf{y}, \mathbf{v}, (\mathbf{w}_3)^n \rangle$ or $H = \langle (\mathbf{u})^n, S_1, \mathbf{x}, \mathbf{v}, \mathbf{y}, S_2, (\mathbf{w}_3)^n \rangle$ where $\{\mathbf{x}, \mathbf{y}\} = \{(\mathbf{w}_1)^n, (\mathbf{w}_2)^n\}$. Without loss of generality, we set $\mathbf{x} = (\mathbf{w}_1)^n$ and $\mathbf{y} = (\mathbf{w}_2)^n$. We set

$$\begin{aligned} P_1 &= \langle \mathbf{u}, U_1, \mathbf{w}_1, (\mathbf{v})^n, \mathbf{v} \rangle \text{ and} \\ P_i &= \langle \mathbf{u}, U_i, \mathbf{w}_i, (\mathbf{w}_i)^n, \mathbf{v} \rangle \text{ for every } 2 \le i \le 2n-5. \end{aligned}$$

Suppose that $H = \langle (\mathbf{u})^n, R_1, (\mathbf{w}_1)^n, R_2, (\mathbf{w}_2)^n, \mathbf{v}, (\mathbf{w}_3)^n \rangle$. Let j be the length of R_2. We can write R_2 as $\langle \mathbf{r}_0 = (\mathbf{w}_1)^n, \mathbf{r}_1, \dots, \mathbf{r}_j = (\mathbf{w}_2)^n \rangle$. Since $d_{AQ_n^{1,0}}((\mathbf{w}_1)^n, (\mathbf{w}_2)^n) \ge 2$, $j \ge 2$. Since the neighbors of $\mathbf{v}$ in $AQ_n^{1,0} - F$ are $(\mathbf{w}_1)^n$, $(\mathbf{w}_2)^n$, and $(\mathbf{w}_3)^n$, $d_{AQ_n^{1,0}}(\mathbf{r}_{j-1}, \mathbf{v}) = 2$. By Lemma 5, $\{(\mathbf{r}_{j-1})^{n-1}, (\mathbf{r}_{j-1})^{(n-1)\sim}\} \cap \{(\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\} = \emptyset$. Thus, there exists a node $\mathbf{q}$ in $\{(\mathbf{r}_{j-1})^{n-1}, (\mathbf{r}_{j-1})^{(n-1)\sim}\} - \{(\mathbf{u})^{n\sim}, (\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\}$. By Theorem 1, there exists a hamiltonian path T of $AQ_n^{1,1}$ between $\mathbf{q}$ and $(\mathbf{v})^{n-1}$. Since $\mathbf{r}_1 \ne \mathbf{v}$, $(\mathbf{r}_1)^{n\sim} \ne (\mathbf{v})^{n\sim}$. We set $\mathbf{g}_1 = (\mathbf{u})^{n-1}$ and $\mathbf{g}_2 = (\mathbf{u})^{(n-1)\sim}$ if $(\mathbf{r}_1)^{n\sim} = (\mathbf{u})^{(n-1)\sim}$, and we set $\mathbf{g}_1 = (\mathbf{u})^{(n-1)\sim}$ and $\mathbf{g}_2 = (\mathbf{u})^{n-1}$ if otherwise. Thus, $(\mathbf{v})^{n\sim}$, $\mathbf{g}_1$, and $\mathbf{g}_2$ are three distinct nodes of $AQ_n^{0,1}$ and $(\mathbf{r}_1)^{n\sim}$ is a node of $AQ_n^{0,1} - \{(\mathbf{v})^{n\sim}, \mathbf{g}_1\}$. By Theorem 3, there exist two node disjoint paths X_1 and X_2 of $AQ_n^{0,1}$ such that (1) X_1 joins $\mathbf{g}_1$ to $(\mathbf{v})^{n\sim}$, (2) X_2 joins $\mathbf{g}_2$ to $(\mathbf{r}_1)^{n\sim}$, and (3) $X_1 \cup X_2$ spans $AQ_n^{0,1}$. We set

$$\begin{aligned} P_{2n-4} &= \langle \mathbf{u}, (\mathbf{u})^n, R_1, (\mathbf{w}_1)^n, \mathbf{v} \rangle, \\ P_{2n-3} &= \langle \mathbf{u}, \mathbf{g}_1, X_1, (\mathbf{v})^{n\sim}, \mathbf{v} \rangle, \text{ and} \\ P_{2n-2} &= \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{r}_1)^{n\sim}, \mathbf{r}_1, \dots, \mathbf{r}_{j-1}, \mathbf{q}, T, (\mathbf{v})^{n-1}, \mathbf{v} \rangle. \end{aligned}$$

Then $\{P_1, P_2, \dots, P_{2n-2}\}$ forms a $(2n-2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Obviously, $\{\mathbf{q}, (\mathbf{v})^{n-1}\}$ and $\{(\mathbf{u})^{n\sim}, (\mathbf{v})^{(n-1)\sim}\}$ are two pairs of four distinct nodes of $AQ_n^{1,1}$. By Theorem 2, there exist two node disjoint paths T_1 and T_2 of $AQ_n^{1,1}$ such that (1) T_1 joins $\mathbf{q}$ to $(\mathbf{v})^{n-1}$, (2) T_2 joins $(\mathbf{u})^{n\sim}$ to $(\mathbf{v})^{(n-1)\sim}$, and (3) $T_1 \cup T_2$ spans $AQ_n^{1,1}$. We set

$$\begin{aligned} P'_{2n-2} &= \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{r}_1)^{n\sim}, \mathbf{r}_1, \dots, \mathbf{r}_{j-1}, \mathbf{q}, T_1, (\mathbf{v})^{n-1}, \mathbf{v} \rangle \text{ and} \\ P'_{2n-1} &= \langle \mathbf{u}, (\mathbf{u})^{n\sim}, T_2, (\mathbf{v})^{(n-1)\sim}, \mathbf{v} \rangle. \end{aligned}$$

Therefore, $\{P_1, P_2, \dots, P_{2n-3}, P'_{2n-2}, P'_{2n-1}\}$ forms a $(2n-1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. See Figure 2 for an illustration.

Suppose that $H = \langle (\mathbf{u})^n, S_1, (\mathbf{w}_1)^n, \mathbf{v}, (\mathbf{w}_2)^n, S_2, (\mathbf{w}_3)^n \rangle$. Let j be the length of S_2. We can write S_2 as $\langle \mathbf{s}_0 = (\mathbf{w}_2)^n, \mathbf{s}_1, \dots, \mathbf{s}_j = (\mathbf{w}_3)^n \rangle$. Since $d_{AQ_n^{1,0}}((\mathbf{w}_2)^n, (\mathbf{w}_3)^n) \ge 2$, $j \ge 2$. Since the neighbors of $\mathbf{v}$ in

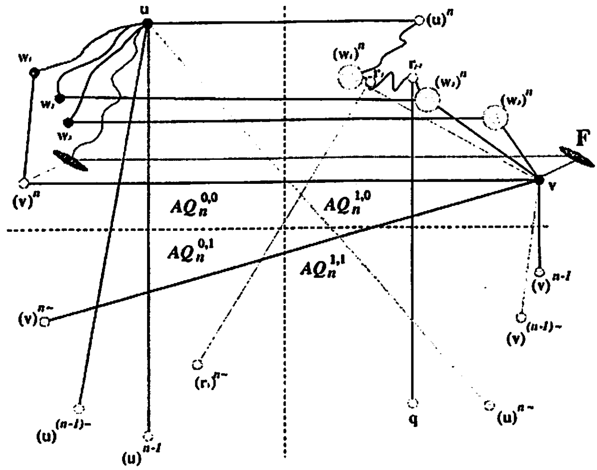

Figure 3: Illustration for Lemma 10, Case 1

$AQ_n^{1,0} - F$ are $(\mathrm{w}_1)^n$, $(\mathrm{w}_2)^n$, and $(\mathrm{w}_3)^n$, $d_{AQ_n^{1,0}}(\mathrm{s}_{j-1}, \mathrm{v}) = 2$. By Lemma 5, $\{(\mathrm{s}_{j-1})^{n-1}, (\mathrm{s}_{j-1})^{(n-1)\sim}\} \cap \{(\mathrm{v})^{n-1}, (\mathrm{v})^{(n-1)\sim}\} = \emptyset$. Thus, there exists a node q in $\{(\mathrm{s}_{j-1})^{n-1}, (\mathrm{s}_{j-1})^{(n-1)\sim}\} - \{(\mathrm{u})^{n\sim}, (\mathrm{v})^{n-1}, (\mathrm{v})^{(n-1)\sim}\}$. By Theorem 1, there exists a hamiltonian path T of $AQ_n^{1,1}$ between q and $(\mathrm{v})^{n-1}$. Since $\mathrm{s}_1 \neq \mathrm{v}$, $(\mathrm{s}_1)^{n\sim} \neq (\mathrm{v})^{n\sim}$. We set $\mathrm{g}_1 = (\mathrm{u})^{n-1}$ and $\mathrm{g}_2 = (\mathrm{u})^{(n-1)\sim}$ if $(\mathrm{s}_1)^{n\sim} = (\mathrm{u})^{(n-1)\sim}$, and we set $\mathrm{g}_1 = (\mathrm{u})^{(n-1)\sim}$ and $\mathrm{g}_2 = (\mathrm{u})^{n-1}$ if otherwise. Thus, $(\mathrm{v})^{n\sim}$, g_1, and g_2 are three distinct nodes of $AQ_n^{0,1}$ and $(\mathrm{s}_1)^{n\sim}$ is a node of $AQ_n^{0,1} - \{(\mathrm{v})^{n\sim}, \mathrm{g}_1\}$. By Theorem 3, there exist two node disjoint paths X_1 and X_2 of $AQ_n^{0,1}$ such that (1) X_1 joins g_1 to $(\mathrm{v})^{n\sim}$, (2) X_2 joins g_2 to $(\mathrm{s}_1)^{n\sim}$, and (3) $X_1 \cup X_2$ spans $AQ_n^{0,1}$. Now, we set

$$\begin{aligned} P_{2n-4} &= \langle \mathrm{u}, (\mathrm{u})^n, S_1, (\mathrm{w}_1)^n, \mathrm{v} \rangle, \\ P_{2n-3} &= \langle \mathrm{u}, \mathrm{g}_1, X_1, (\mathrm{v})^{n\sim}, \mathrm{v} \rangle, \text{ and} \\ P_{2n-2} &= \langle \mathrm{u}, \mathrm{g}_2, X_2, (\mathrm{s}_1)^{n\sim}, \mathrm{s}_1, \ldots, \mathrm{s}_{j-1}, \mathrm{q}, T, (\mathrm{v})^{n-1}, \mathrm{v} \rangle. \end{aligned}$$

Then $\{P_1, P_2, \ldots, P_{2n-2}\}$ forms a $(2n-2)^*$-container of AQ_n between u and v.

Obviously, $\{\mathrm{q}, (\mathrm{v})^{n-1}\}$ and $\{(\mathrm{u})^{n\sim}, (\mathrm{v})^{(n-1)\sim}\}$ are two pairs of four distinct nodes of $AQ_n^{1,1}$. By Theorem 2, there exist two node disjoint paths T_1 and T_2 of $AQ_n^{1,1}$ such that (1) T_1 joins q to $(\mathrm{v})^{n-1}$, (2) T_2 joins $(\mathrm{u})^{n\sim}$

to $(\mathbf{v})^{(n-1)\sim}$, and (3) $T_1 \cup T_2$ spans $AQ_n^{1,1}$. We set

$$\begin{aligned} P'_{2n-2} &= \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{s}_1)^{n\sim}, \mathbf{s}_1, \ldots, \mathbf{s}_{j-1}, \mathbf{q}, T_1, (\mathbf{v})^{n-1}, \mathbf{v}\rangle \text{ and} \\ P'_{2n-1} &= \langle \mathbf{u}, (\mathbf{u})^{n\sim}, T_2, (\mathbf{v})^{(n-1)\sim}, \mathbf{v}\rangle. \end{aligned}$$

Therefore, $\{P_1, P_2, \ldots, P_{2n-3}, P'_{2n-2}, P'_{2n-1}\}$ forms a $(2n-1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Case 2. $(\mathbf{u}, (\mathbf{v})^n) \in E(AQ_n^{0,0})$. By induction hypothesis, there is a $(2n-5)^*$-container, $\{Q_1, Q_2, \ldots, Q_{2n-5}\}$, of $AQ_n^{0,0}$ between $\mathbf{u}$ and $(\mathbf{v})^n$. Let $\mathbf{w}_2$ and $\mathbf{w}_3$ be two nodes in $\{((\mathbf{v})^n)^2, ((\mathbf{v})^n)^3, ((\mathbf{v})^n)^4\} - \{\mathbf{u}\}$. Since $AQ_n^{0,0}$ is $(2n-5)$-regular, we can rearrange the indices so that $Q_1 = \langle \mathbf{u}, (\mathbf{v})^n\rangle$ and Q_i can be written as $\langle \mathbf{u}, U_i, \mathbf{w}_i, (\mathbf{v})^n\rangle$ for $2 \le i \le 2n-5$. Obviously, $d_{AQ_n^{0,0}}(\mathbf{w}_2, \mathbf{w}_3) \ge 2$ and $d_{AQ_n^{1,0}}((\mathbf{w}_2)^n, (\mathbf{w}_3)^n) \ge 2$. Let $F = \{(\mathbf{w}_j)^n \mid 4 \le j \le 2n-5\}$. Obviously, $|F| = 2n-8$. By Theorem 1, there is a hamiltonian path H of $AQ_n^{1,0} - F$ between $(\mathbf{u})^n$ and $(\mathbf{w}_3)^n$. Since the neighbors of $\mathbf{v}$ in $AQ_n^{1,0} - (F \cup \{(\mathbf{w}_3)^n\})$ are $(\mathbf{u})^n$ and $(\mathbf{w}_2)^n$, either $H = \langle (\mathbf{u})^n, \mathbf{v}, (\mathbf{w}_2)^n, R, (\mathbf{w}_3)^n\rangle$ or $H = \langle (\mathbf{u})^n, S, (\mathbf{w}_2)^n, \mathbf{v}, (\mathbf{w}_3)^n\rangle$. We set

$$\begin{aligned} P_1 &= \langle \mathbf{u}, (\mathbf{v})^n, \mathbf{v}\rangle, \\ P_i &= \langle \mathbf{u}, U_i, \mathbf{w}_i, (\mathbf{w}_i)^n, \mathbf{v}\rangle \text{ for every } 2 \le i \le 2n-5, \text{ and} \\ P_{2n-4} &= \langle \mathbf{u}, (\mathbf{u})^n, \mathbf{v}\rangle. \end{aligned}$$

Suppose that $H = \langle (\mathbf{u})^n, \mathbf{v}, (\mathbf{w}_2)^n, R, (\mathbf{w}_3)^n\rangle$. Let j be the length of R. We can write R as $\langle \mathbf{r}_0 = (\mathbf{w}_2)^n, \ldots, \mathbf{r}_j = (\mathbf{w}_3)^n\rangle$. Since $|V(AQ_n^{1,0}) - \{(\mathbf{u})^n, (\mathbf{w}_2)^n, \ldots, (\mathbf{w}_{2n-5})^n\}| \ge 2$ when $n \ge 6$, $j \ge 3$. Since the neighbors of $\mathbf{v}$ in $AQ_n^{1,0}(\mathbf{v}) - F$ are $(\mathbf{u})^n$, $(\mathbf{w}_2)^n$, and $(\mathbf{w}_3)^n$, $d_{AQ_n^{1,0}}(\mathbf{r}_{j-1}, \mathbf{v}) = 2$. By Lemma 5, $\{(\mathbf{r}_{j-1})^{n-1}, (\mathbf{r}_{j-1})^{(n-1)\sim}\} \cap \{(\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\} = \emptyset$. Thus, there exists a node $\mathbf{q}$ in $\{(\mathbf{r}_{j-1})^{n-1}, (\mathbf{r}_{j-1})^{(n-1)\sim}\} - \{(\mathbf{u})^{n\sim}, (\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\}$. By Theorem 1, there exists a hamiltonian path T of $AQ_n^{1,1}$ between $\mathbf{q}$ and $(\mathbf{v})^{n-1}$. Since $\mathbf{r}_1 \ne \mathbf{v}$, $(\mathbf{r}_1)^{n\sim} \ne (\mathbf{v})^{n\sim}$. We set $\mathbf{g}_1 = (\mathbf{u})^{n-1}$ and $\mathbf{g}_2 = (\mathbf{u})^{(n-1)\sim}$ if $(\mathbf{r}_1)^{n\sim} = (\mathbf{u})^{(n-1)\sim}$, and we set $\mathbf{g}_1 = (\mathbf{u})^{(n-1)\sim}$ and $\mathbf{g}_2 = (\mathbf{u})^{n-1}$ if otherwise. Thus, $(\mathbf{v})^{n\sim}$, $\mathbf{g}_1$, and $\mathbf{g}_2$ are three distinct nodes of $AQ_n^{0,1}$ and $(\mathbf{r}_1)^{n\sim}$ is a node of $AQ_n^{0,1} - \{(\mathbf{v})^{n\sim}, \mathbf{g}_1\}$. By Theorem 3, there exist two node disjoint paths X_1 and X_2 of $AQ_n^{0,1}$ such that (1) X_1 joins $\mathbf{g}_1$ to $(\mathbf{v})^{n\sim}$, (2) X_2 joins $\mathbf{g}_2$ to $(\mathbf{r}_1)^{n\sim}$, and (3) $X_1 \cup X_2$ spans $AQ_n^{0,1}$. We set

$$\begin{aligned} P_{2n-3} &= \langle \mathbf{u}, \mathbf{g}_1, X_1, (\mathbf{v})^{n\sim}, \mathbf{v}\rangle \text{ and} \\ P_{2n-2} &= \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{r}_1)^{n\sim}, \mathbf{r}_1, \ldots, \mathbf{r}_{j-1}, \mathbf{q}, T, (\mathbf{v})^{n-1}, \mathbf{v}\rangle. \end{aligned}$$

Then $\{P_1, P_2, \ldots, P_{2n-2}\}$ forms a $(2n-2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Obviously, $\{\mathbf{q}, (\mathbf{v})^{n-1}\}$ and $\{(\mathbf{u})^{n\sim}, (\mathbf{v})^{(n-1)\sim}\}$ are two pairs of four distinct nodes of $AQ_n^{1,1}$. By Theorem 2, there exist two node disjoint paths T_1 and T_2 of $AQ_n^{1,1}$ such that (1) T_1 joins $\mathbf{q}$ to $(\mathbf{v})^{n-1}$, (2) T_2 joins $(\mathbf{u})^{n\sim}$ to $(\mathbf{v})^{(n-1)\sim}$, and (3) $T_1 \cup T_2$ spans $AQ_n^{1,1}$. We set

$$\begin{aligned} P'_{2n-2} &= \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{r}_1)^{n\sim}, \mathbf{r}_1, \ldots, \mathbf{r}_{j-1}, \mathbf{q}, T_1, (\mathbf{v})^{n-1}, \mathbf{v}\rangle \text{ and} \\ P'_{2n-1} &= \langle \mathbf{u}, (\mathbf{u})^{n\sim}, T_2, (\mathbf{v})^{(n-1)\sim}, \mathbf{v}\rangle. \end{aligned}$$

Therefore, $\{P_1, P_2, \ldots, P_{2n-3}, P'_{2n-2}, P'_{2n-1}\}$ forms a $(2n-1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. See Figure 3 for an illustration.

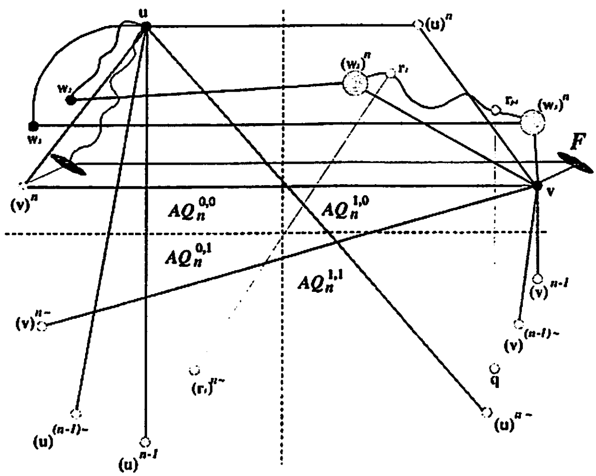

Figure 4: Illustration for Lemma 10, Case 2

Suppose that $H = \langle (\mathbf{u})^n, S, (\mathbf{w}_2)^n, \mathbf{v}, (\mathbf{w}_3)^n\rangle$. Let j be the length of S. We can write S as $\langle \mathbf{s}_0 = (\mathbf{u})^n, \ldots, \mathbf{s}_j = (\mathbf{w}_2)^n\rangle$. Since $|V(AQ_n^{1,0}) - \{(\mathbf{u})^n, (\mathbf{w}_2)^n, \ldots, (\mathbf{w}_{2n-5})^n\}| \geq 2$ when $n \geq 6$, $j \geq 3$. Since the neighbors of $\mathbf{v}$ in $AQ_n^{1,0}(\mathbf{v}) - F$ are $(\mathbf{u})^n$, $(\mathbf{w}_2)^n$, and $(\mathbf{w}_3)^n$, $d_{AQ_n^{1,0}}(\mathbf{s}_1, \mathbf{v}) = 2$. By Lemma 5, $\{(\mathbf{s}_1)^{n-1}, (\mathbf{s}_1)^{(n-1)\sim}\} \cap \{(\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\} = \emptyset$. Thus, there exists a node $\mathbf{q}$ in $\{(\mathbf{s}_1)^{n-1}, (\mathbf{s}_1)^{(n-1)\sim}\} - \{(\mathbf{u})^{n\sim}, (\mathbf{v})^{n-1}, (\mathbf{v})^{(n-1)\sim}\}$. By Theorem 1, there exists a hamiltonian path T of $AQ_n^{1,1}$ between $\mathbf{q}$ and $(\mathbf{v})^{n-1}$. Since $\mathbf{s}_{j-1} \neq \mathbf{v}$, $(\mathbf{s}_{j-1})^{n\sim} \neq (\mathbf{v})^{n\sim}$. We set $\mathbf{g}_1 = (\mathbf{u})^{n-1}$ and $\mathbf{g}_2 = (\mathbf{u})^{(n-1)\sim}$ if $(\mathbf{s}_{j-1})^{n\sim} = (\mathbf{u})^{(n-1)\sim}$, and we set $\mathbf{g}_1 = (\mathbf{u})^{(n-1)\sim}$ and $\mathbf{g}_2 = (\mathbf{u})^{n-1}$ if otherwise. Thus, $(\mathbf{v})^{n\sim}$, $\mathbf{g}_1$, and $\mathbf{g}_2$ are three distinct nodes of $AQ_n^{0,1}$ and $(\mathbf{s}_{j-1})^{n\sim}$ is a node of $AQ_n^{0,1} - \{(\mathbf{v})^{n\sim}, \mathbf{g}_1\}$. By Theorem 3, there exist two node disjoint paths X_1 and X_2 of $AQ_n^{0,1}$ such that (1) X_1 joins $\mathbf{g}_1$ to $(\mathbf{v})^{n\sim}$, (2) X_2 joins $\mathbf{g}_2$ to $(\mathbf{s}_{j-1})^{n\sim}$, and (3) $X_1 \cup X_2$ spans

$AQ_n^{0,1}$. We set

$$P_{2n-3} = \langle \mathbf{u}, \mathbf{g}_1, X_1, (\mathbf{v})^{n\sim}, \mathbf{v} \rangle \text{ and}$$
$$P_{2n-2} = \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{s}_{j-1})^{n\sim}, \mathbf{s}_{j-1}, \ldots, \mathbf{s}_1, \mathbf{q}, T, (\mathbf{v})^{n-1}, \mathbf{v} \rangle.$$

Then $\{P_1, P_2, \ldots, P_{2n-2}\}$ forms a $(2n-2)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$.

Obviously, $\{\mathbf{q}, (\mathbf{v})^{n-1}\}$ and $\{(\mathbf{u})^{n\sim}, (\mathbf{v})^{(n-1)\sim}\}$ are two pairs of four distinct nodes of $AQ_n^{1,1}$. By Theorem 2, there exist two node disjoint paths T_1 and T_2 of $AQ_n^{1,1}$ such that (1) T_1 joins $\mathbf{q}$ to $(\mathbf{v})^{n-1}$, (2) T_2 joins $(\mathbf{u})^{n\sim}$ to $(\mathbf{v})^{(n-1)\sim}$, and (3) $T_1 \cup T_2$ spans $AQ_n^{1,1}$. We set

$$P'_{2n-2} = \langle \mathbf{u}, \mathbf{g}_2, X_2, (\mathbf{s}_{j-1})^{n\sim}, \mathbf{s}_{j-1}, \ldots, \mathbf{s}_1, \mathbf{q}, T_1, (\mathbf{v})^{n-1}, \mathbf{v} \rangle \text{ and}$$
$$P'_{2n-1} = \langle \mathbf{u}, (\mathbf{u})^{n\sim}, T_2, (\mathbf{v})^{(n-1)\sim}, \mathbf{v} \rangle.$$

Therefore, $\{P_1, P_2, \ldots, P_{2n-3}, P'_{2n-2}, P'_{2n-1}\}$ forms a $(2n-1)^*$-container of AQ_n between $\mathbf{u}$ and $\mathbf{v}$. □

It is easy to check that AQ_1 and AQ_2 are super spanning connected. With Lemma 7, Lemma 8, Lemma 9, and Lemma 10, we have the following theorem.

Theorem 5. *The n-dimensional augmented cube AQ_n is super spanning connected.*

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References

[1] M. Albert, R. E. L. Aldred, and D. Holton, On 3*-connected graphs, Australasian Journal of Combinatorics, 24 (2001) 193–208.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, New York, 1980.

[3] S. A. Choudum and V. Sunitha, Augmented cubes, Networks, 40 (2002) 71–84.

[4] D. F. Hsu, On container width and length in graphs, groups, and networks, IEICE Transactions Fundamentals, E77-A (1994) 668–680.

[5] H. C. Hsu, L. C. Chiang, J. J. M. Tan, and L. H. Hsu, Fault hamiltonicity of augmented cubes, Parallel Computing, 31 (2005) 131–145.

[6] H. C. Hsu, P. L. Lai, and C. H. Tsai, Geodesic pancyclicity and balanced pancyclicity of augmented cubes, Information Processing Letters, (2007) to appear.

[7] H. C. Hsu, C. K. Lin, H. M. Hung, and L. H. Hsu, The spanning connectivity of the (n, k)-star graphs, International Journal of Foundations of Computer Science, 17 (2006) 415–434

[8] C. K. Lin, H. M. Huang, and L. H. Hsu, The super connectivity of the pancake graphs and star graphs, Theoretical Computer Science, 339 (2005) 257–271.

[9] K. Menger, Zur allgemeinen kurventheorie, Fundamenta Mathematicae, 10 (1927) 95–115.

[10] C. H. Tsai, J. J. M. Tan, and L. H. Hsu, The super connected property of recursive circulant graphs, Information Processing Letters, 91 (2004) 293–298.

[11] M. Xu, and J. M. Xu, The forwarding indices of augmented cubes, Information Processing Letters, 101 (2007) 185–189.