The smallest regular graphs which are not 1-extendable

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Abstract

A graph G is 1-extendable if every ege is contained in a perfect matching of G. In this note we prove the following theorem. Let $d \geq 3$ be an integer, and let G be a d-regular graph of order n without odd components. If G is not 1-extendable, then $n \geq 2d + 4$. Examples will show that the given bound is best possible.

Keywords: Matching, 1-extendable graph, regular graph

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x in the graph G. A d-regular graph G is a graph with the property that $d(x) = d$ for all vertices $x \in V(G)$. If X is a subset of the vertex set of a graph G, then $G[X]$ is the subgraph induced by X. A *perfect matching* of a graph G, is a matching M in G with the property that every vertex is incident with an edge of M. We denote by K_n the complete graph of order n and by $K_{r,s}$ the complete bipartite graph with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G - A)$ the number of odd components in the subgraph $G - A$.

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A graph G is *p-extendable* if it contains a set of p independent edges, and every set of p independent edges can be extended to a perfect matching. In 1980, Plummer [8] studied the properties of p-extendable graphs.

As an application of Tutte's 1-factor theorem [9], Wallis [12] proved in 1981 the following result.

Theorem 1 (Wallis [12] 1981) Let $d \geq 3$ be an integer, and let G be a d-regular graph without odd components. If G has no perfect matching, then

$$|V(G)| \geq 3d + 4 \text{ when } d \geq 6 \text{ is even,}$$

$$|V(G)| \geq 3d + 7 \text{ when } d \geq 3 \text{ is odd,}$$

$$|V(G)| \geq 22 \text{ when } d = 4.$$

For extensions and generalizations of Theorem 1, we refer the reader to Zhao [13], Cacetta and Mardiyono [1], Volkmann [10] and Klinkenberg and Volkmann [4], [5] and [6]. In this note, we prove an analogue to Theorem 1 for even order graphs which are not 1-extendable. Our main tool is the following characterization of 1-extendable graphs, which follows easily from Tutte's 1-factor theorem.

Theorem 2 (Little, Grant, Holton [7] 1975) A graph G is 1-extendable if and only if for any $A \subseteq V(G)$

(1) $q(G - A) \leq |A|$ and

(2) $q(G - A) = |A|$ implies that $G[A]$ is an empty graph.

Proofs of Tutte's 1-factor theorem as well as of Theorems 1 and 2 can also be found in the book by Volkmann [11].

Theorem 3 Let $d \geq 3$ be an integer, and let G be a d-regular graph of order n without odd components. If G is not 1-extendable, then $n \geq 2d+4$.

Proof. Suppose to the contrary that there exists a d-regular graph G of order $n \leq 2d + 2$ without odd components which is not 1-extendable. Then it follows from the hypothesis and Theorem 2 that there exists a non-empty set $A \subseteq V(G)$ such that $q(G - A) \geq |A| + 1$ or $q(G - A) = |A|$ and $G[A]$ contains an edge.

We call an odd component of $G - A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G - A$, respectively. Since G is a d- regular graph without odd components, it is easy to see that there are at least d edges in G joining each small component of $G - A$ with A and at least one edge in G joining each large component of $G - A$ with A. The d-regularity of G therefore implies

$$\alpha + d\beta \leq d|A|. \tag{1}$$

Case 1. Assume that $q(G-A) \geq |A|+1$. Since n is even, the numbers $q(G - A)$ and $|A|$ are of the same parity, and we deduce that

$$\alpha + \beta = q(G - A) \geq |A| + 2. \tag{2}$$

Inequality (1) yields $\beta \leq |A|$ and thus (2) leads to $\alpha \geq 2$. Applying the assumption $n \leq 2d + 2$, and using the fact that $A \neq \emptyset$, we obtain the contradiction

$$2d + 2 \geq n \geq |A| + \alpha(d + 1) + \beta \geq |A| + 2(d + 1) > 2d + 2. \tag{3}$$

Case 2: Assume that $q(G-A) = |A|$ and $G[A]$ contains an edge. This implies that $|A| \geq 2$. If $\alpha \geq 2$, then we arrive a contradiction as in (3). In the case $\alpha = 0$, we have $\beta = |A|$. Since there are at least d edges of G joining each small component of $G - A$ with A and at least one edge in $G[A]$, there exists at least one vertex in A of deree greater than d, a contradiction to the d-regularity of G.

It remains the case that $\alpha = 1$ and thus there exists at least one small component in $G-A$. If U is a small component of minimum order in $G-A$, then we observe that

$$|V(U)| \geq d - |A| + 1 \tag{4}$$

and also

$$|V(U)| \geq d - |A| + 2 \tag{5}$$

when d and $|A|$ of different parity. Now our assumption $n \leq 2d + 2$ leads to

$$2d + 2 \geq n \geq |A| + (d + 1) + (|A| - 1)|V(U)|. \tag{6}$$

If $|A| \geq d + 1$, then (6) yields the contradiction

$$2d + 2 \geq n \geq 2d + 2 + (|A| - 1)|V(U)| > 2d + 2.$$

Hence we assume in the following that $2 \leq |A| \leq d$.

Subcase 2.1: Assume that d is odd. This implies that the large component is of order at least $d+2$, and thus our assumption $n \leq 2d+2$ and (4) yield the contradiction

$$\begin{aligned} 2d + 2 \geq n &\geq |A| + (d + 2) + (|A| - 1)|V(U)| \\ &\geq |A| + (d + 2) + |V(U)| \\ &\geq |A| + (d + 2) + d - |A| + 1 \\ &= 2d + 3. \end{aligned}$$

Subcase 2.2: Assume that d is even and $|A|$ is odd. The assumption $n \leq 2d+2$ and (5) lead to the contradiction

$$\begin{aligned} 2d+2 \geq n &\geq |A|+(d+1)+(|A|-1)|V(U)| \\ &\geq |A|+(d+1)+|V(U)| \\ &\geq |A|+(d+1)+d-|A|+2 \\ &= 2d+3. \end{aligned}$$

Subcase 2.3: Assume that d and $|A|$ are both even. If $|A| \geq 3$, then the assumption $n \leq 2d+2$ and (4) yield the contradiction

$$\begin{aligned} 2d+2 \geq n &\geq |A|+(d+1)+(|A|-1)|V(U)| \\ &\geq |A|+(d+1)+2|V(U)| \\ &\geq |A|+(d+1)+2(d-|A|+1) \\ &\geq |A|+(d+1)+d-|A|+2 \\ &= 2d+3. \end{aligned}$$

Finally, let d be even, and let $|A|=2$. Then inequality (4) shows that $|V(U)| \geq d-1$, and as d is even, we deduce that $|V(U)| = d-1$. Hence there are at least $2(d-1)$ edges in G joining U with A and at least one edge in G joining the large component of $G-A$ with A. Since the subgraph $G[A]$ contains also an edge, there exists at least one vertex in A of deree greater than d, a contradiction to the d-regularity of G. This completes the proof of Theorem 3. □

Remark 4 It is obvious that each 1-regular or 2-regular graph without odd components is 1-extendable.

Example 5 Let $d \geq 4$ be an even integer. Let H_1 be a complete graph K_2 with vertex set x, y, let H_2 be a complete graph K_{d+1} without an edge uv, and let H_3 be a complete graph K_{d+1} with vertex set $\{w_1, w_2, \dots, w_{d+1}\}$ without the edges of the path $w_1 w_2 \dots w_{d-1}$. Now we define the graph G of order $2d+4$ as the disjoint union of H_1, H_2 and H_3 together with the edges ux, vy, xw_1, yw_{d-1} and xw_i as well as yw_i for $2 \leq i \leq d-2$. The resulting graph G is d-regular, however, the edge xy is not contained in a perfect matching of G. This example shows that Theorem 3 is best possible when d is even.

Example 6 Let $d \geq 3$ be an odd integer. Let H_1 be a complete graph K_2 with vertex set x, y, let H_2 be a complete graph K_d with vertex set $\{u_1, u_2, \dots, u_d\}$, and let H_3 be a complete graph K_{d+2} with vertex set $\{w_1, w_2, \dots, w_{d+2}\}$ without the edges of the path $w_1 w_2 \dots w_d$ and without the edge $w_{d+1}w_{d+2}$. Now we define the graph G of order $2d + 4$ as the disjoint union of H_1, H_2 and H_3 together with the edges yu_i for $1 \leq i \leq d-1$, xu_d and xw_j for $2 \leq j \leq d-1$. The resulting graph G is d-regular, however, the edge xy is not contained in a perfect matching of G. This example shows that Theorem 3 is best possible when d is odd.

If $d \leq d_G(x) \leq d + k$ for each vertex x in a graph G, then we speak of a *close to regular graph* or more precisely of a *$(d, d + k)$-graph.*

Observation 7 Let $d \geq 3$ be an integer, and let G be a $(d, d+k)$-graph of even order n. If $n \leq 2d - 2$, then G is 1-extendable.

Proof. Let uv be an arbitray edge of G, and define the graph $H = G - \{u, v\}$. Then H is a $(d - 2, d + k)$ graph of even order such that $n(H) \leq 2d - 4$. By the classical theorem of Dirac [3], H has a Hamiltonian cycle. Consequently, the edge uv is contained in a perfect matching of G. This implies that G is 1-extendable. □

Example 8 Let $H = K_{d,d}$ be the complete bipartite graph. If we add a further edge e to H, then the resulting graph of order $n = 2d$ is a $(d, d + 1)$-graph, and the edge e is not contained in a perfect matching of G. This example shows that Observation 7 is best possible.

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