Gracefulness of $P_{2r,2m}$

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Abstract

The graph $P_{a,b}$ is defined as the one obtained by taking b vertex-disjoint copies of the path P_{a+1} of length a, coalescing their first vertices into one single vertex labeled u and then coalescing their last vertices into another single vertex labeled v. K M Kathiresan showed that $P_{2r,2m-1}$ is graceful and conjectured that $P_{a,b}$ is graceful except when $(a,b) = (2r+1, 4s+2)$. In this paper, an algorithm for finding another graceful labeling of $P_{2r,2}$ is provided, and $P_{2r,2(2k+1)}$ is proved to be graceful for all positives r and k.

Keywords: Graceful graph ; Vertex labeling ; Edge labeling

1 Introduction

The problem of finding a *graceful labeling* of the vertices of a graph was introduced by Rosa [1] in seminal paper in the mid-1960s and has since attracted much attention. Suppose that G is a graph with q edges. A *graceful labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \cdots, q\}$ such that when the edge uv is assigned the label $|f(u) - f(v)|$, the set of induced edge labels is $\{1, 2, \cdots, q\}$. A graph that admits a graceful labeling is called *graceful*. In [1], Rosa shows that if all trees are graceful, then the so-called Ringel-Kotzig conjecture [2] is true. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, X-ray crystallography, radar, astronomy, circuit design, and communication network addressing(see [3] and [4] for details).

Over the intervening years, a number of variations on graceful labeling have been proposed(see [5] for a comprehensive survey). In this paper, we shall consider one such variation, introduced by K M Kathiresan [6] who used the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of b

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internally disjoint paths each of length a . He conjecture $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod 4$. He has proved the conjecture for the case that a is even and b is odd. Y S Yang proved the conjecture for the case when both a and b are odd. The graphs $P_{2r,2m}$ have already been proved to be graceful by C Sekar and V Swaminathan in [7]. The specialty of this paper is to provide a new technique for proving the gracefulness of $P_{2r,2}$, from which the gracefulness of $P_{2r,2(2k+1)}$ follows.

Firstly, in Section 2, we will describe how to denote the vertices of $P_{2r,2m}$ and construct an algorithm for finding the non-trivial graceful labeling of $P_{2r,2}$, and also examples $r = 9, 300$ are given.

Secondly, from the graceful labeling of $P_{2r,2}$ given in section 2, we will prove that $P_{2r,2(2k+1)}$ is graceful in Section 3.

2 Preliminary results

Let $v_0^i, v_1^i, v_2^i, \cdots, v_{2r}^i$ denote the vertices of the ith path of length $2r$ of $P_{2r,2m}$, for all $i, v_0^i = u, v_{2r}^i = v$. For example, the vertices of $P_{6,4}$ so labeled is illustrated by Fig 1.

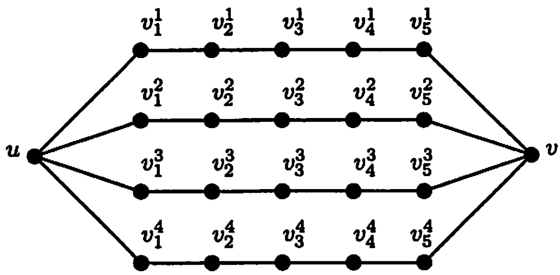

Fig.1 vertices of $P_{6,4}$

Let $r \in \mathbb{N}$, and $f_r : V(P_{2r,2}) \to \{0, 1, \cdots, 4r\}$ be the graceful labeling of graph $P_{2r,2}$. We now label the vertices $u, v, and v_{2j-1}^i \ (1 \le j \le r; i = 1, 2)$ firstly as follows:

$$f_r(u) = 0;\ f_r(v) = 2r;$$
$$f_r(v_{2j-1}^1) = 4r - 2(j-1) \quad (1 \le j \le r);$$
$$f_r(v_{2j-1}^2) = 4r - 2(j-1) - 1 \quad (1 \le j \le r);$$

We describe below how to label the vertices $v_{2j}^i \ (1 \le j \le r-1; i = 1, 2)$.

Let

$$\mathbf{I_r} = \left(f_r(v_2^1), f_r(v_2^2), f_r(v_4^1), f_r(v_4^2), \cdots, f_r(v_{2(r-1)}^1), f_r(v_{2(r-1)}^2)\right).$$

For example, see Fig.2, f_3 is the graceful labeling of $P_{6,2}$, and $\mathbf{I_3} = (2, 5, 1, 4)$.

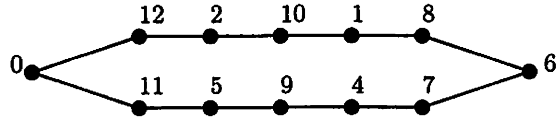

Fig.2 the graceful labeling f_3 and $\mathbf{I}_3$

In order to get the graceful labeling f_r, we must find out the proper $2(r-1)$-dimensional vector $\mathbf{I_r}$, namely we must define $f_r(v^i_{2j})$ $(i = 1,2;\ 1 \le j \le r-1)$. A useful observation is that no matter how we define $f_r(v^i_{2j})$, the $f_r(v^i_{2j-1})$ defined above satisfy (for any $i = 1,2;\ 1 \le j \le r-1$):

$$f_r(v^i_{2j-1}) - f_r(v^i_{2j}) = f_r(v^i_{2j+1}) - f_r(v^i_{2j}) + 2;$$
$$f_r(v^i_{2j-1}) > f_r(v^i_{2j});$$
$$f_r(v^i_{2j+1}) > f_r(v^i_{2j}).$$

Let D_E denote the set of induced edge labels , i.e.

$$\begin{aligned} D_E &= \Big\{ f_r(v^i_{2j-1}) - f_r(v^i_{2j}),\ f_r(v^i_{2j-1}) - f_r(v^i_{2(j-1)}) \Big| 1 \le j \le r;\ i = 1,2 \Big\}. \\ &= \Big\{ 4r, 4r-1, 1, 2 \Big\} \bigcup \Big\{ f_r(v^i_{2j+1}) - f_r(v^i_{2j}) \Big| 1 \le j \le r-1;\ i = 1,2 \Big\} \\ &\quad \bigcup \Big\{ f_r(v^i_{2j+1}) - f_r(v^i_{2j}) + 2 \Big| 1 \le j \le r-1;\ i = 1,2 \Big\}. \end{aligned}$$

It follows that an approach to constructing graceful labeling f_r of $P_{2r,2}$ is to find the proper $f_r(v^i_{2j})$ $(1 \le j \le r-1;\ i = 1,2)$ satisfying the following conditions:

$$\begin{aligned} P1: &\quad \Big\{ f_r(v^i_{2j+1}) - f_r(v^i_{2j}) \Big| 1 \le j \le r-1;\ i = 1,2 \Big\} \\ &= \Big\{ 4r - 4l, 4r - 4l - 1 \Big| 1 \le l \le r-1 \Big\} \\ &= \Big\{ 4r-4, 4r-5, 4r-8, 4r-9, \cdots, 8, 7, 4, 3 \Big\}; \\ P2: &\quad 1 \le f_r(v^i_{2j}) \le 2r-1 \quad (1 \le j \le r-1;\ i = 1,2). \end{aligned}$$

Let $\mathsf{M}_r = (m_{ij})$ be a $(2r-1) \times (2r-2)$ matrix, and $m_{ij} = (4r-1) - (i+j)$. For example, if $r = 3$, then

$$\mathsf{M}_3 = \begin{pmatrix} 9 & 8 & 7 & 6 \\ 8 & 7 & 6 & 5 \\ 7 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 \end{pmatrix}.$$

Clearly, $2 \le m_{ij} \le 4r-3$. We consider how many elements equal to $4r-4l$ or $4r-4l-1$ are there in the matrix M_r. It is easily seen that the number of elements equal to $4r-4l$, $4r-4l-1$ is $4l-2$, $4l-1$,respectively, where $1 \le l \le r/2$. And the number of elements equal to $4r-4l$, $4r-4l-1$ is $4r-4l-1$, $4r-4l-2$,respectively, where $(r+1)/2 \le l \le r-1$. If we can pair each column $j \in \{1, \cdots, 2r-2\}$ with a distinct row $i \in \{1, \cdots, 2r-1\}$ so that the set of the (i,j)-entries of matrix M_r is the set $\Big\{4r-4, 4r-5, 4r-8, 4r-9, \cdots, 8, 7, 4, 3\Big\}$, i.e. there exist $2r-2$ distinct rows $i_1, i_2, \cdots, i_{2r-2}$, such that

$$\left\{m_{i_1 1}, m_{i_2 2}, \cdots, m_{i_{2r-2},(2r-2)}\right\} = \left\{4r-4, 4r-5, 4r-8, 4r-9, \cdots, 8, 7, 4, 3\right\}.$$

then, clearly, we define $f_r(v_{2j}^i)$ as the exact row paired with the $(2(j-1)+i)$-th column. i.e. $f_r(v_{2j}^i) = i_t$ where $t = 2(j-1)+i$.

Next we describe our algorithm (column first algorithm, CF)for finding out the indices $i_1, i_2, \cdots, i_{2r-2}$ which satisfy the required conditions.

CF algorithm:

Step1(initialize) For the given integer r, construct $(2r-1)\times(2r-2)$ matrix $\mathsf{M}_r = (m_{ij})$, where $m_{ij} := (4r-1)-(i+j)$; for $j = 1 : (2r-2)$ do $m_{rj} = 0$; $l := 1$, $\mathsf{A} = (a_{ij}) := \mathsf{M}_r$.

Step2(finding) For l from 1 to $r-1$, find the elements equal to $4r-4l$ or $4r-4l-1$ of the matrix A.

Step2.1 Firstly, find the elements equal to $4r-4l$ of the matrix A from the first column to the last column. If we find $4r-4l$ for the first time in the column j_0 and row i_0, then let $h_{j_0} = i_0$ and $a_{i_0,j} = 0$, $a_{i,j_0} = 0$ for all i and j.

Step2.2 Find the elements equal to $4r-4l-1$ of the matrix A from the first row to the last row. If we find $4r-4l-1$ for the first time in the column j_1 and row i_1, then let $h_{j_1} = i_1$ and $a_{i_1,j} = 0$, $a_{i,j_1} = 0$ for all i and j, go to Step2.3; If we can't find $4r-4l-1$ in the matrix A, go to Step2.4.

Step2.3 $i_{j_0} := i_0$, $i_{j_1} := i_1$, $\mathsf{M}_r := \mathsf{A}$, $l := l+1$. If $l < r$, go to Step2.1.

Step2.4 $\mathsf{A} = (a_{ij}) := \mathsf{M}_r$, $h_{j_0} = 0$. Firstly, find the elements equal to $4r-4l-1$ of the matrix A from the first column to the last column. If we find $4r-4l-1$ for the first time in the column j_2 and row i_2, then let $h_{j_2} = i_2$ and $a_{i_2,j} = 0$, $a_{i,j_2} = 0$ for all i and j.

Step2.5 Find the elements equal to $4r-4l$ of the matrix A from the first row to the last row. If we find $4r-4l$ for the first time in the column j_3 and row i_3, then let $h_{j_3} = i_3$ and $a_{i_3,j} = 0$, $a_{i,j_3} = 0$ for all i and j, go to Step2.6; Otherwise, go to Step2.7.

Step2.6 $i_{j_2} := i_2$, $i_{j_3} := i_3$, $\mathsf{M}_r := \mathsf{A}$, $l := l+1$. If $l < r$, go to Step2.1.

Step2.7 Stop.

Step3 $\mathbf{I}_r := (i_1, i_2, \cdots, i_{2r-2})$. Stop.

Theorem 1 *CF algorithm can find out all the distinct rows $i_1, i_2, \cdots, i_{2r-2}$ of the matrix M_r satisfy*

$$\left\{m_{i_1 1}, m_{i_2 2}, \cdots, m_{i_{2r-2},(2r-2)}\right\} = \left\{4r-4, 4r-5, 4r-8, 4r-9, \cdots, 8, 7, 4, 3\right\}.$$

Proof When $1 \le l \le r/2$, we know that there are $4l-2$ elements of the matrix M_r equal to $4r-4l$, and $4l-1$ elements of the matrix M_r equal to $4r-4l-1$. It is easily seen that we get $i_1 = 2$ and $i_3 = 1$ at the very beginning of the CF algorithm when we find $4r-4$ and $4r-5$. Suppose we have already found $4r-4, 4r-5, \cdots, 4r-4(l-1), 4r-4(l-1)-1$, now we find $4r-4l$ and $4r-4l-1$. Clearly, here, there are at less $(4l-2)-4(l-1)-1 = 1$ elements of the matrix A equal to $4r-4l$ left. So we can find $4r-4l$ for any $1 \le l \le r-1$. The problem is that whether we can find $4r-4l-1$ all the time. The only case we cannot find $4r-4l-1$ is when all the columns and rows which contain the elements equal to $4r-4l-1$ have been endowed with **0**. Since there are $4l-1$ elements equal to $4r-4l-1$ in the matrix M_r, this case occurs if and only if the following cases occurs:

(a) each of the preceding numbers $4r-4, 4r-5, \cdots, 4r-4(l-1), 4r-4(l-1)-1$ have changed two of the elements equal to $4r-4l-1$ into 0. Namely, altogether there are $4(l-1)$ elements equal to $4r-4l-1$ of the matrix M_r changed into 0. i.e. there are only $(4l-1)-(4l-4)-1=2$ elements equal to $4r-4l-1$ of the matrix A have been left when we find the elements $4r-4l$.

(b) each of the preceding numbers $4r-4, 4r-5, \cdots, 4r-4(l-1), 4r-4(l-1)-1$ has changed two of the elements equal to $4r-4l$ into 0. i.e. there is only one $4r-4l$ left when we find $4r-4l$.

(c) when we find $4r-4l$, we changed two of the elements equal to $4r-4l-1$ into 0.

Fig.3 illustrate this badly case, ♠ represents the elements equal to $4r-4l$ and ♣ represents the elements equal to $4r-4l-1$.

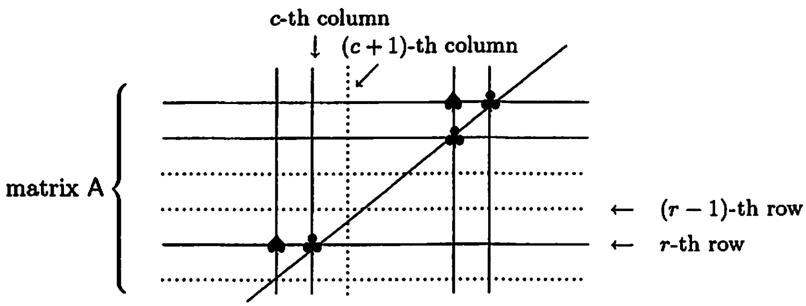

Fig.3 the only case in which one cannot find the $4r-4l-1$

Suppose the bad case (see Fig.3) occurs, we consider the $(r-1)$-th row. It is easily seen that the cell in row $r-1$ and column c contains the elements $4r-4l$. By the case (b) above, all of the elements lie in the $(r-1)$-th row must have been changed into 0, i.e. $a_{(r-1),j}=0$ $(1 \le j \le (2r-2))$. And by the case (a), the $(c+1)$-column must not have been changed into 0. Hence the $(r-2)$-th row must have been changed into 0. So consider the $(r-1)$-th row,$(r-2)$-th row,$\cdots$ in turn, we know all the rows in front of the r-row should have been changed into 0. This contradicts the case (c). Thus, the bad case does not occur. We can find the elements equal to $4r-4l$, or $4r-4l-1$ for any $1 \le l \le r/2$.

Similarly to prove the case $(r+1)/2 \le l \le (r-1)$.

On all accounts, we can find all the distinct indices $i_1, i_2, \cdots, i_{2r-2}$ satisfying the required conditions by the CF algorithm. □

For example, we get some $\mathbf{I_r}$ by the CF algorithm as following:

$\mathbf{I_9}$= (2, 5, 1, 7, 3, 13, 8, 4, 11, 6, 12, 16, 14, 10, 17, 15);

I_{300}=(2 , 5, 1, 7, 3, 17, 8, 4, 10, 6, 9, 27, 11, 13, 16, 12, 18, 14, 48, 20, 15, 21, 28, 23, 19, 22, 32, 24, 26, 30, 25, 31, 86, 34, 29, 35, 42, 37, 33, 36, 54, 38, 40, 43, 39, 45, 41, 44, 50, 46, 60, 51, 47, 53, 49, 52, 55, 57, 64, 56, 62, 58, 72, 63, 59, 65, 61, 64, 82, 66, 68, 71, 67, 73, 69, 99, 75, 70, 76, 83, 78, 74, 77, 87, 79, 81, 85, 80, 90, 93, 89, 84, 106, 97, 92, 88, 91, 94, 96, 103, 95, 98, 112, 100, 102, 105, 101, 104, 107, 109, 316, 108, 114, 110, 124, 115, 111, 117, 113, 116, 134, 118, 120, 123, 119, 125, 121, 155, 127, 122, 128, 135, 130, 126, 129, 139, 131, 133, 137, 132, 138, 189, 141, 136, 142, 149, 144, 140, 143, 161, 145, 147, 150, 146, 152, 148, 151, 157, 153, 167, 158, 154, 160, 156, 159, 162, 168, 175, 163, 169, 165, 195, 171, 166, 172, 179, 174, 170, 173, 176, 178, 181, 177, 180, 194, 182, 184, 187, 183, 209, 185, 199, 190, 186, 192, 188, 191, 197, 193, 196, 202, 198, 204, 200, 214, 205, 201, 207, 203, 206, 212, 208, 466, 210, 224, 215, 211, 217, 213, 216, 234, 218, 220, 223, 219, 225, 221, 255, 227, 222, 228, 235, 230, 226, 229, 239, 231, 233, 237, 232, 238, 293, 241, 236, 242, 249, 244, 240, 243, 261, 245, 247, 250, 246, 252, 248, 251, 257, 253, 267, 258, 254, 260, 256, 259, 262, 264, 331, 263, 269, 265, 279, 270, 266, 272, 268, 271, 289, 273, 275, 278, 274, 280, 276, 306, 282, 277, 283, 290, 285, 281, 284, 294, 286, 288, 292, 287, 297, 336, 296, 291, 313, 304, 299, 295, 298, 301, 303, 310, 302, 305, 319, 307, 309, 312, 308, 311, 314, 320, 327, 315, 321, 317, 335, 323, 318, 324, 343, 326, 322, 325, 328, 330, 333, 329, 332, 338, 334, 337, 355, 339, 341, 344, 340, 346, 342, 345, 371, 347, 349, 352, 348, 354, 350, 356, 359, 351, 358, 353, 407, 366, 361, 357, 360, 378, 362, 364 , 367, 363, 369, 365, 368, 374, 370, 376, 372, 382, 377, 373, 380, 375, 381, 476, 384, 379, 385, 392, 387, 383, 386, 404, 388, 390, 393, 389, 395, 391, 394, 420, 396, 398, 401, 397, 403, 399, 405, 409, 400, 406, 402, 412, 408, 414, 410, 424, 415, 411, 417, 413, 416, 422, 418, 428, 423, 419, 426, 421, 427, 482, 430, 425, 431, 438, 433, 429, 432, 450, 434, 436, 439, 435, 441, 437, 440, 526, 442, 444, 447, 443, 449, 445, 451, 454, 446, 453, 448, 498, 461, 456, 452, 455, 473, 457, 459, 462, 458, 464, 460, 463, 469, 465, 471, 467, 477, 472, 468, 475, 470, 480, 483, 479, 474, 484, 507, 487, 478, 481, 495, 486, 489, 485, 488, 546, 490, 492, 511, 491, 497, 493, 499, 502, 494, 501, 496, 514, 509, 504, 500, 503, 521, 505, 515, 510, 506, 513, 508, 518, 553, 517, 512, 534, 525, 520, 516, 519, 522, 524, 528, 523, 529, 536, 531, 527, 530, 548, 532, 566, 537, 533, 539, 535, 538, 556, 540, 542, 545, 541, 547, 543, 573, 549, 544, 562, 550, 552, 555, 551, 554, 557, 559, 582, 558, 564, 560, 570, 565, 561, 568, 563, 569, 576, 572, 567, 589, 580, 575, 571, 574, 577, 579, 586, 578, 581, 595, 583, 585, 588, 584, 587, 590, 592, 596, 591, 593, 597, 599, 594, 598).

The preceding discussion is summarized in the following theorem.

Theorem 2 *For any given positive integer* r*,* $P_{2r,2}$ *is graceful with the special graceful labeling given by the CF algorithm.* □

3 Generalization

In order to prove $P_{2r,2m}$ is graceful graph, we show the following lemmas first.

Lemma 1 *Let f_r be the graceful labeling of $P_{2r,2}$ and satisfying:*

$$f_r(v_{2j-1}^i) > f_r(v_{2j}^i), \quad (1 \le j \le r; i = 1, 2)$$

$$f_r(v_{2j+1}^i) > f_r(v_{2j}^i), \quad (0 \le j \le r-1; i = 1, 2)$$

then $P_{2r,\,2(2k+1)}$ is a graceful graph.

Proof We construct the graceful labeling g_r of graph $P_{2r,\,2(2k+1)}$ as follows.

$g_r(u) = 0$; $g_r(v) = (2k+1)f_r(v)$;

$$g_r(v_{2j-1}^i) = \begin{cases} (2k+1)f_r(v_{2j-1}^1) - (i-1), & 1 \le i \le 2k+1; \\ (2k+1)f_r(v_{2j-1}^2) - (i-(2k+2)), & 2k+2 \le i \le 4k+2. \end{cases}$$

$$g_r(v_{2j}^i) = \begin{cases} (2k+1)f_r(v_{2j}^1) + i, & 1 \le i \le k; \\ (2k+1)f_r(v_{2j}^1) - ((2k+1)-i), & k+1 \le i \le 2k+1; \\ (2k+1)f_r(v_{2j}^2) + (i-(2k+1)), & 2k+2 \le i \le 3k+1; \\ (2k+1)f_r(v_{2j}^2) - ((4k+2)-i), & 3k+2 \le i \le 4k+2. \end{cases}$$

It is easily seen that the set of labels induced on the edge set of graph $P_{2r,\,2(2k+1)}$ is $\{1, 2, \cdots, 4r(2k+1)\}$. i.e. g_r is a graceful labeling of $P_{2r,\,2(2k+1)}$, and thus $P_{2r,\,2(2k+1)}$ is graceful graph. □

Taken with theorem 2 and lemma 1, the discussion above establishes our main result.

Theorem 3 *For all positive integers r and k, $P_{2r,\,2(2k+1)}$ is graceful graph.* □

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