A solution to a conjecture on two rainbow connection numbers of a graph*

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Abstract

For a graph G, Chartrand et al. defined the rainbow connection number $rc(G)$ and the strong rainbow connection number $src(G)$ in "G. Charand, G.L. John, K.A. Mckeon, P. Zhang, Rainbow connection in graphs, Mathematica Bohemica, 133(1)(2008) 85-98". They raised the following conjecture: for two given positive integers a and b, there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$". In this short note, we will show that the conjecture is true.

Keywords: edge-colored graph, (strong) rainbow coloring, (strong) rainbow connection number.

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1 Introduction

All graphs in this paper are finite, undirected, simple and connected. We follow the notation and terminology of [1]. Let c be a coloring of the edges of a graph G, i.e., $c : E(G) \longrightarrow \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$. A path is called a rainbow path if no two edges of the path have the same color. The graph G is called rainbow connected (with respect to c) if for every pair of distinct vertices of G, there exists a rainbow path connecting them in G.

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If by coloring c the graph G is rainbow connected, then the coloring c is called a rainbow coloring of G. If k colors are used in c, then c is a rainbow k-coloring of G. The minimum number k for which there exists a rainbow k-coloring of G, is called the rainbow connection number of G, denoted by $rc(G)$.

Let c is a rainbow coloring of a graph G. If for every pair u and v of distinct vertices of the graph G, the graph G contains a rainbow u-v geodesic (a shortest path in G between v and u), then G is called strongly rainbow connected. In this case, the coloring c is called a strong rainbow coloring of G. If k colors are used, then c is a strong rainbow k-coloring of G. The minimum number k satisfying that G is strongly rainbow connected, i.e., the minimum number k for which there exists a strong rainbow k-coloring of G, is called the strong rainbow connection number of G, denoted by $src(G)$. Thus for every connected graph G, $rc(G) \leq src(G)$. Recall that the diameter of G is defined as the largest distance between two vertices of G, denoted $diam(G)$. Then $diam(G) \leq rc(G) \leq src(G)$. The following results were obtained in [2] by Chartrand et al.

Proposition 1.1 *Let G be a nontrivial connected graph of size m. Then*
1. $rc(G) = 1$ if and only if $src(G) = 1$.
2. $rc(G) = 2$ if and only if $src(G) = 2$.
3. $diam(G) \leq rc(G) \leq src(G)$ for every connected graph G. ∎

Chartrand et al. also considered the problem that, given any two integers a and b, whether there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ and they got the following result.

Theorem 1.2 *Let a and b be positive integers with $a \geq 4$ and $b \geq (5a - 6)/3$. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.* ∎

Then, combining Proposition 1.1 and Theorem 1.2, they got the following result.

Corollary 1.3 *Let a and b be positive integers. If $a = b$ or $3 \leq a < b$ and $b \leq \frac{5a-6}{3}$, then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.* ∎

Finally, they thought the question that whether the condition $b \leq \frac{5a-6}{3}$ can be deleted and raised the following conjecture:

Conjecture 1.4 *Let a and b be positive integers. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1,2\}$ or $3 \leq a \leq b$.* ∎

This short note is to give a confirmative solution to this conjecture.

2 Proof of the conjecture

Proof of Conjecture 1.4: From Proposition 1.1 one can see that the condition is necessary. For the sufficiency, when $a = b \in \{1,2\}$, from Corollory 1.3 the conjecture is true. So, we just need to consider the situation when $3 \leq a \leq b$.

Let $n = 3b(b-a+2)$, and let H_n be the graph consisting of an n-cycle $C_n: v_1, v_2, \cdots, v_n$ and another two vertices w and v, each of which joins to every vertex of C_n. Let G be the graph constructed from H_n of order $n+2$ and the path $P_{a-1}: u_1, u_2, \cdots, u_{a-1}$ on $a-1$ vertices by identifying v and u_{a-1}.

First, we will show $rc(G) = a$. Because diam(G)=a, by Proposition 1.1 we have $rc(G) \geq a$. It remains to show $rc(G) \leq a$. Note that $n = 3b(b-a+2) \geq 18$. Define a coloring c for the graph G by the following rules:

$$c(e) = \begin{cases} i & if\ e = u_iu_{i+1}\ for\ 1 \leq i \leq a-2, \\ a-1 & if\ e = v_iv\ and\ i\ is\ odd, \\ a & if\ e = v_iv\ and\ i\ is\ even, \\ a & if\ e = v_iw\ and\ 1 \leq i \leq n \\ 1 & otherwise. \end{cases}$$

Since c is a rainbow a-coloring of the edges of G, it follows that $rc(G) \leq a$. This implies $rc(G) = a$.

Next, we will show $src(G) = b$. We first show $src(G) \leq b$, by giving a strong rainbow b-coloring c for the graph G as follows:

$$c(e) = \begin{cases} i & if\ e = u_iu_{i+1}\ for\ 1 \leq i \leq a-2, \\ a-2+i & if\ e = v_{3b(i-1)+j}v\ for\ 1 \leq i \leq b-a+2\ and\ 1 \leq j \leq 3b, \\ i & if\ e = v_{3(j-1)b+3(i-1)+k}w\ for\ 1 \leq j \leq b-a+2\ and\ 1 \leq i \leq b \\ & and\ 1 \leq k \leq 3, \\ 1 & if\ e = v_{3(i-1)+1}v_{3(i-1)+2}\ for\ 1 \leq i \leq b(b-a+2), \\ 2 & if\ e = v_{3(i-1)+2}v_{3(i-1)+3}\ for\ 1 \leq i \leq b(b-a+2), \\ 3 & otherwise \end{cases}$$

It remains to show $src(G) \geq b$. By contradiction, suppose $src(G) <$

b. Then there exists a strong rainbow $(b-1)$-coloring $c: E(G) \to \{1,2,\cdots,b-1\}$. For every v_i $(1 \le i \le n)$, $d(v_i,u_1) = a-1$, and the path $v_i, v, u_{a-2}, \cdots, u_1$ is the only path of length $a-1$ connecting v_i and u_1, and so $v_i, v, u_{a-2}, \cdots, u_1$ is a rainbow path. Without loss of generality, suppose $c(u_2u_1) = 1$, $c(u_3u_2) = 2, \cdots$, $c(u_{a-1}u_{a-2}) = a-2$. Then $c(v_iv) \in \{a-1, a, \cdots, b\}$, for $1 \le i \le n$. We first consider the set of edges $A = \{v_iv, 1 \le i \le n\}$, and so $|A| = n$. Thus there exist at least $\lceil \frac{n}{b-a+1} \rceil \ge 3b+1$ edges in A colored the same. Suppose there exist m edges $v_{j_1}v, \cdots, v_{j_m}v, (1 \le j_1 < j_2 < \cdots < j_m \le n)$ colored the same and $m \ge \lceil \frac{n}{b-a+1} \rceil \ge 3b+1$. Second, we consider the set of edges $B = \{v_{j_1}w, \cdots, v_{j_m}w\}$. Since $c(v_{j_i}w) \in \{1,2,\cdots,b-1\}$, for $1 \le i \le m$, there exist at least $\lceil \frac{m}{b-1} \rceil \ge \lceil \frac{3b+1}{b-1} \rceil \ge 4$ edges colored the same. Thus from B we can choose 4 edges of the same color. Since $n \ge 18$, from the corresponding vertices on the cycle C_n of the four edges chosen above, we can get two vertices such that their distance on the cycle C_n is more than 3. Without loss of generality, we assume that the two vertices are v_1', v_2' and their distance in graph G is 2. Then the geodesic between v_1' and v_2' in graph G is either v_1', w, v_2' or v_1', v, v_2'. However, neither v_1', w, v_2' nor v_1', v, v_2' is a rainbow path. Thus the coloring c is not a strong rainbow coloring of G, a contradiction. Therefore $src(G) \le b$ and so $src(G) = b$. The proof is thus complete. ∎

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References

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[2] G. Chartrand, G.L. Johns, K.A. MeKeon, P. Zhang, Rainbow connection in graphs. *Math.Bohem.*, 133(1)(2008) 85-98.