On E_k − cordial labeling

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Abstract

Cahit and Yilmaz [15] called a graph G is E_k −cordial if it is possible to label its edges with numbers from the set $\{0,1,\ldots,k-1\}$ in such a way that, at each vertex v of G, the sum modulo k of the labels on the edges incident with v satisfies the inequalities $|m(i)-m(j)|\le 1$ and $|n(i)-n(j)|\le 1$, where $m(s)$ and $n(t)$ are, respectively, the number of edges labeled with s and the number of vertices labeled with t. In this paper, we give a necessary condition for a graph to be E_k −cordial for certain k. We also give some new families of E_k −cordial graphs and we prove Lee's conjecture about the edge-gracefulness of the disjoint union of two cycles.

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1. Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [5].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An edge labeling $f: E(G) \to \{0,1,\ldots,k-1\}$ induces a vertex labeling $f^+: V(G) \to \mathbb{Z}_k$, defined by $f^+(u) = \sum_{uv\in E(G)} f(uv) \pmod{k}$, for all vertex $u \in V(G)$. For $i \in \mathbb{Z}_k$, let $m_i(f) = |\{e \in E(G): f(e) = i\}|$ and $n_i(f) = |\{u \in V(G): f^+(u) = i\}|$. f is called an

E_k $-$cordial labeling of G, if the following conditions are satisfied for all $i, j \in \mathbb{Z}_k : \left| m_i(f) - m_j(f) \right| \leq 1$ and $\left| n_i(f) - n_j(f) \right| \leq 1$. A graph G is called E_k $-$cordial if it admits a E_k $-$cordial labeling.

The notion of E_k $-$cordial labeling was introduced by Cahit and Yilmaz [4] in 2000. Yilmaz and Cahit [15] first introduced the notion of E_2 $-$cordial labeling in 1997 under the name of E-cordial as a weaker version of edge-graceful labeling.

Lo [10] introduced the notion of edge-graceful graphs. A (p,q) graph G is said to be edge-graceful if there exists a bijection f from $E(G)$ to $\{1,2,\ldots,q\}$ so that the induced mapping f^+ from $V(G)$ to $\{0,1,\ldots,p-1\}$ defined by $f^+(u) = \sum_{uv \in E(G)} f(uv) \pmod{p}$ for every $u \in V(G)$, is a bijection.

We present some definition of labelings related to our work. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling $f : V(G) \rightarrow \mathbb{Z}_k$ induces an edge labeling $f^+ : E(G) \rightarrow \mathbb{Z}_k$, defined by $f^+(xy) = f(x) + f(y)$, for all edge $xy \in E(G)$. For $i \in \mathbb{Z}_k$, let $n_i(f) = \left| \{v \in V(G) : f(v) = i\} \right|$ and $m_i(f) = \left| \{e \in E(G) : f^+(e) = i\} \right|$. A labeling f of a graph G is called k-cordial if $\left| n_i(f) - n_j(f) \right| \leq 1$ and $\left| m_i(f) - m_j(f) \right| \leq 1$ for all $i, j \in \mathbb{Z}_k$. A graph G is called k-cordial if it admits a k-cordial labeling.

The notion of k-cordial labeling of graphs was introduced by Hovey [8] as a generalization of harmonious and cordial labelings. Harmonious labeling was introduced by Graham and Sloane [7]. They defined a graph G of q edges to be harmonious if there is an injection f from $V(G)$ to $\mathbb{Z}_q$ such that the induced function f^* from $E(G)$ to $\mathbb{Z}_q$, defined by ,

$f^*(xy) = f(x) + f(y)$ for all edge $xy \in E(G)$, is a bijection, while cordial labeling was introduced by Cahit [2] who called a graph G is cordial if there is a vertex labeling $f: V(G) \to \{0,1\}$ such that the induced labeling $f^*: E(G) \to \{0,1\}$, defined by $f^*(xy) = |f(x) - f(y)|$, for all edge $xy \in E(G)$ and the following inequalities hold: $|n_0(f) - n_1(f)| \le 1$ and $|m_0(f) - m_1(f)| \le 1$, where $n_i(f)$ (resp. $m_i(f)$) is the number of vertices (resp. edges) labeled with i.

Shiu et al. [12] proved that the composition of the path P_3 and any null graph of odd order is edge-graceful. Hovey [8] has obtained the following results: odd cycles with pendant edges attached are k-cordial for all k ; cycles are k-cordial for all odd k, for k even C_{2mk+j} is k-cordial when $0 \le j < \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial for even k and he conjectured that C_{2mk+j} is k-cordial if and only if $j \ne k$, where k and j are even and $0 \le j < 2k$. This conjecture was verified by Tao [13]. This result combined with those of Hovey [8] show that for all positive integer k the n-cycle is k-cordial with the exception that k is even and $n = 2mk + k$. Terms and notations not defined in the paper can be found in ([5] and [6]). The later reference surveyed the known results to all variations of graph labelings appears in this paper.

In the next section of this paper we give a necessary condition on E_k − cordial labelings for certain k and we give new families of E_k − cordial graphs. In Section 3, we deal with cycles and we present a proof of a conjecture due to Lee (see [9]).

2. New families of E_k − cordial graphs

In this section, we give A necessary condition for a graph to be E_k - cordial in certain cases of k and give some new families of E_k − cordial graphs.

Cabaniss et al. [1] observed that if G is a (p,q) graph with $q=np$, then an edge-graceful with edges labeled 1 to q is equivalent to using each edge label $1,2,\dots,p$ exactly n times so that the induced vertex labeling yields distinct labels modulo p. We formalize this observation in the following theorem.

Theorem 1. Let G be a (p,q) graph with $q=np$, then G is edge-graceful if and only if G is E_p-cordial.

Corollary 1. C_n is edge-graceful if and only if C_n is E_n − cordial.

Yilmaz and Cahit [15] have observed that any graph of order p fails to be E-cordial when $p \equiv 2 \pmod 4$. In the following theorem we generalize the result of Yilmaz and Cahit for all even k.

Theorem 2. Let k be even. If G is a (p,q) E_k − cordial graph with p and $q \equiv 0 \pmod k$, then $p \equiv 0 \pmod{2k}$.

Proof. Let f is a E_k − cordial labeling of G, then

$$\sum_{v\in V(G)} f^{+}(v) \equiv 2 \sum_{e\in E(G)} f(e) \pmod k.$$

As p and $q \equiv 0 \pmod k$, then $m_i(f)=\frac{q}{k}$ and $n_i(f)=\frac{p}{k}$ for all $i \in \mathbb{Z}_k$ and hence

$$\sum_{i=0}^{k-1} i\frac{p}{k} \equiv 2\sum_{i=0}^{k-1} i\frac{q}{k} \pmod k \Rightarrow \frac{p(k-1)}{2} \equiv 0 \pmod k,$$ and as k is even, then $p \equiv 0 \pmod{2k}$. □

Corollary 2. (i) If G is E_2-cordial of even order p, then $p \equiv 0 \pmod 4$ [15].

(ii) Every graph of order congruent to $4(\bmod 8)$ and of size congruent to $0(\bmod 4)$ is not E_4-cordial.

Theorem 3. Let k be odd and let G be a non-null (p,q) E_k $-$cordial graph with $p \equiv 0(\bmod k)$ and H be (n,m) E_k $-$cordial graph. If q or $m \equiv 0(\bmod k)$, then $G+H$ is E_k $-$cordial graph.

Proof. Let $V(H)=\{u_1,u_2,\ldots,u_n\}$ and let g, h be the E_k $-$cordial labelings of G, H respectively. Define $f: E(G+H) \to \{0,1,\ldots,k-1\}$ as follows:

$$f\big|_{E(G)}=g,\quad f\big|_{E(H)}=h \quad \text{and for all } v \in V(G):$$

$f(uv)=i$ if $g^+(v)=i$ and for all $u \in V(H)$, in case of $n \not\equiv -1(\bmod k)$ and otherwise,

$f(uv)=i$ if $g^+(v)=i$, for all $u \in V(H)-\{u_n\}$ and $f(u_n v)=k-1-i$ if $g^+(v)=i$.

For every $u \in V(H)$, as

$$f^+(u)=(h^+(u)+\frac{p}{k}\sum_{i=0}^{k-1} i)(\bmod k)$$

$$=(h^+(u)+\frac{p(k-1)}{2})(\bmod k)$$

$$=h^+(u)(\bmod k), \qquad (\text{as } k \text{ odd})$$

we get $n_j(f\big|_{V(H)})=n_j(h)$, for all $0 \le j \le k-1$.

Again, as k odd, there exists a permutation $\sigma \in S_p$ such that $f^+(v_i)=g^+(v_{\sigma(i)})$ for all $1 \le i \le p$, that is,

$n_j(f|_{V(G)}) = n_j(g)$ for all $0 \le j \le k-1$. Now, for all $0 \le i \le k-1$, $n_i(f) = n_i(g) + n_i(h) = \frac{p}{k} + n_i(h)$ and $m_i(f) = m_i(g) + m_i(h) + \frac{pn}{k}$.

As $|n_i(f) - n_j(f)| = |n_i(h) - n_j(h)| \le 1$ and if q or $m \equiv 0 (\mathrm{mod}\, k)$, we get

$$|m_i(f) - m_j(f)| = |m_i(g) - m_j(g) + m_i(h) - m_j(h)| \le 1$$

and f is a E_k −cordial labeling of $G + H$ and this completes the proof.□

Corollary 3. If G is a non-null (p,q) E_k −cordial graph with k odd and $p \equiv 0 (\mathrm{mod}\, k)$, then $G + \bar{K}_n$ is E_k −cordial.

Corollary 4. Let k be odd, then K_n is E_k −cordial if $n \equiv 0$ or $1 (\mathrm{mod}\, k)$.

Proof. As K_k is edge-graceful for odd k [1], then by Theorem 1, K_k is E_k −cordial. Now, applying Theorem 3 recursively by taking firstly both of the graphs G and H as K_k and secondly consider one of the two graphs as K_{2k} and the other as K_k and so on, we get that K_n is E_k −cordial if $n \equiv 0 \ (\mathrm{mod}\, k)$ and then applying the same theorem taking the graph G as K_n , $n \equiv 0 \ (\mathrm{mod}\, k)$, and H as K_1, we get K_n is E_k −cordial if $n \equiv 1 \ (\mathrm{mod}\, k)$.□

Cahit and Yilmaz [4] proved that K_n is E_3 −cordial for all $n \ge 3$. Their proof is so long. However we present a simple one

Corollary 5. K_n is E_3 −cordial for all $n \geq 3$ [4].

Proof. If $n \equiv 0$ or $1 \pmod 3$, then K_n is E_3 −cordial, by Corollary 4. If $n \equiv 2 \pmod 3$, $n \geq 5$ we prove only that K_5 is E_3 −cordial and then applying Theorem 3 by taking the graph H as K_5 and firstly the graph G as K_3, secondly as K_6 and thirdly as K_9 and so on, we get the required. Now, we show that K_5 is E_3 −cordial. Let $V(K_5) = \{v_1, v_2, \ldots, v_5\}$, define $f : E(K_5) \rightarrow \{0,1,2\}$ as $f(v_1v_i) = 2$, $i = 2,3,4$, $f(v_1v_5) = 1$, $f(v_2v_i) = 1$, $i = 3,4$, $f(v_2v_5) = 0$, $f(v_3v_4) = 0$, $f(v_3v_5) = 2$ and $f(v_4v_5) = 0$. One can easily show that f is an E_3 −cordial labeling of K_5. □

3. Cycles and other labelings

We note that the notion of E_k −cordial and k-cordial labelings coincide for cycles or for the graph consisting of disjoint union of cycles. We formalize this observation in the following theorem.

Theorem 4. Let G be the disjoint union of cycles of p vertices, G is E_p −cordial if and only if G p-cordial.

Corollary 6. C_n is E_n −cordial if and only if C_n k-cordial.

Cahit and Yilmaz [4] proved that C_n is E_3 −cordial for all $n \geq 3$. However, from the work of Hovey [8] and Tao [13], we get the following:

Corollary 7. C_n is E_k −cordial for all odd k and if k is even, C_n is E_k −cordial if and only if $n \not\equiv k \pmod{2k}$.

Graham and Sloane [7] showed that C_n is harmonious if and only if n is odd. Again as C_n is n-cordial if and only if C_n is harmonious [8]. Lo

[10] and Cabaniss et. Al [1] proved that C_n is edge-graceful when n is odd. However, from Corollary 1 and Corollary 6, we get the following result:

Corollary 8. C_n is edge-graceful if and only if n is odd.

In 1987, Lee (see [9]) conjectured that $C_{2m} \cup C_{2n+1}$ is edge-graceful for all $m \geq 2$ and $n \geq 1$ except for $(m,n)=(2,1)$. Lee, Seah and Lo [9] have proved this conjecture for the case $m=n$ and m is odd. They also prove: mC_n is edge-graceful if m and n are odd, $C_n \cup C_{2n+2}$ is edge-graceful when n is odd and $C_n \cup C_{4n}$ is edge-graceful for n odd. Cahit [3] proved that mC_n is harmonious for all m and n are odd. Same result have done by Youssef [16] who proved that if G is a harmonious graph of odd order then mG is harmonious for all odd m. Seoud, Abdel Maqsoud and Sheehan [11] proved that when m or n is even, mC_n is not harmonious. They also proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geq 4$ and conjectured that $C_3 \cup C_{2n}$ is harmonious when $n \geq 3$. This conjectured proved by Yang, Lu and Zeng [14]. They proved that all graphs of the form $C_{2j+1} \cup C_{2n}$ are harmonious except for $(n,j)=(2,1)$. Combining the above results with our results in this paper enable us to prove the conjecture of Lee in the following corollary:

Corollary 9. $C_{2m} \cup C_{2n+1}$ is edge-graceful for all $m \geq 2$ and $n \geq 1$ except for $(m,n)=(2,1)$.

Again as, mC_n is harmonious if and only if m and n are odd, we get

Corollary 10. mC_n is edge-graceful if and only if m and n are odd.

References

[1] **S. Cabaniss, R. Low and J. Mitchem,** On edge-graceful regular graphs and trees, *Ars Combin.*, **34** (1992) 129-142.

[2] **I. Cahit,** Cordial graphs : a weaker version of graceful and harmonious graphs, *Ars Combin.*, **23** (1987) 201-207.

[3] **I. Cahit,** On harmonious tree labelings, *Ars Combin.*, **41** (1995) 311-317.

[4] **I. Cahit and R. Yilmaz,** E_3 —cordial graphs, *Ars Combin.*, **54** (2000)119-127.

[5] **G. Chartrand and L. Lesniak-Foster**, Graphs and Digraphs (3nd Edition) CRC Press, 1996.

[6] **J. A. Gallian,** A dynamic survey of graph labeling, *The Electronic J. of Combin.***14** (2007), # DS6, 1-180.

[7] **R. L. Graham and N. J. A. Sloane,** On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Method,* **1** (1980) 382-404.

[8] **M. Hovey**, A-cordial graphs, *Discrete Math.*, **93** (1991) 183-194.

[9] **S. M. Lee, E. Seah and S.-P Lo,** On edge-graceful of 2-regular graphs, preprint.

[10] **S. Lo,** On edge-graceful labelings of graphs, *Congr. Numer.*, **50** (1985) 231-241.

[11] **M. A. Seoud, A. I. Abdel Maqsoud and J. Sheehan**, Harmonious graphs, *Utilitas Math.*, **47** (1995) 225-233.

[12] **W. C. Shiu, P. C. Lam and H. L. Cheng**, Edge-gracefulness of the composition of paths with the null graphs, *Discrete Math.*, **253** (2002) 63-76.

[13] **R. Tao**, On k-cordiality of cycles, crowns and wheels, *System Sci. Math. Sci.* **11** (1998) 227-229.

[14] **Y. C. Yang, W. Lu and Q. Zeng,** Harmonious graphs $C_{2k} \cup C_{2j+1}$, *Utilitas Math.*, **62** (2002) 191-198.

[15] **R. Yilmaz and Cahit**, E-cordial graphs, *Ars Combin.*, **46** (1997) 251-266.

[16] **M. Z. Youssef,** Two general results on harmonious labelings, *Ars Combin.*, **68** (2003)225-230.