Some new results on soft hypermodules*

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Abstract

We study the algebraic properties of soft sets in a hypermodule structure. The concepts of soft hypermodules and soft sub-hypermodules are introduced, and some basic properties are investigated. Furthermore, we define homomorphism and isomorphism of soft hypermodules, and derive three isomorphism theorems of soft hypermodules. By using normal fuzzy sub-hypermodules, three fuzzy isomorphism theorems of soft hypermodules are established.

Keywords: Soft sets; hypermodules; soft hypermodules; fuzzy sub-hypermodules; isomorphism theorems

1. Introduction

Due to information incompleteness, data randomness, limitations of measuring instruments, etc., uncertainties are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science [37, 52]. Several theories, such as probability theory, fuzzy set theory [54], vague set theory [20], rough set theory [46, 47] and interval mathematics [22], have been proven to be useful mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties, as pointed out by Molodtsov in [42]. One of the major reasons for these difficulties is the inadequacy of the parametrization tools of these theories. Consequently, Molodtsov proposed the soft set theory [42], as a new approach for modeling uncertainties, which is free from the difficulties existing in those theories mentioned above. Furthermore, he gave some applications of soft set theory, including

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function smootheness, Riemann integration, probability theory, measurement theory, game theory and operations research [42, 43]. At present, soft set theory has been applied to business competitive capacity evaluation [51], classification of the natural textures [44], optimization problems [30], data analysis under incomplete information [59], especially decision making. In [38], Maji et al. gave the definition of parameterization reduction on soft sets, and presented an application of soft sets in a decision making problem by using the rough sets. The application was improved by Chen et al. [9] with the help of a new definition of parameterization reduction. In [29], Kong et al. discussed the problems of suboptimal choice and added parameter set of soft sets, and introduced the normal parameter reduction of soft sets. Furthermore, Çağman and Enginoğlu [7, 8] constructed the soft decision making methods without using the rough sets, which are more practical and can be successfully applied to many problems that contain uncertainties.

Also, researches on theoretical aspect of soft sets are progressing rapidly. As a continuation of Molodtsov's pioneer work [42], Maji et al. [37] gave a detailed theoretical study on soft sets. Ali et al. [3] pointed out that several assertions in [37] are not true in general, and proposed some new operations such as restricted intersection, restricted union, restricted difference and extended intersection of two soft sets. In [21], Gong et al. presented the bijective soft sets, which are special soft sets. As an extended concept of bijective soft sets, the exclusive disjunctive soft sets [50] were introduced. In [7], Çağman and Enginoğlu defined the soft matrices, which are representative of soft sets, and demonstrated that this representation has several advantages. Furthermore, Jiang et al. [23] analyzed the existing problems of soft set theory, and presented an extended soft set theory by using the concepts of description logics to act as the parameters of soft sets. Recently, the algebraic structures of soft sets have been studied increasingly [7]. Aktaş and Çağman [2] introduced the definition of soft groups and showed that fuzzy groups can be considered a special case of the soft groups. Moreover, some basic properties of soft semirings [19] and soft rings [1] were introduced. In addition, Sun et al. [48] considered the soft modules, Jun et al. [24] presented the soft ordered semigroups and Li [35] analyzed the soft lattices. For further development, Jun et al. [25], [26], [45] considered the applications of soft sets in BCK/BCI-algebras, Hilbert algebras and subtraction algebras. Kazancı et al. [28] discussed the algebraic properties of soft sets in BCH-algebras.

On the other hand, the algebraic hyperstructures, initiated by Marty in 1934 [39], is a natural generalization of the usual algebraic structures. In recent years, algebraic hyperstructures have attracted wide attention and had applications in many domains, such as geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities [11]. Several books and many papers on hyperstructure theory have been published [4]-[6], [10]-[18], [27], [31]-[34], [36, 40, 41, 49], [55]-[58]. The book [17] was devoted especially to the study of hyperring theory and applications. Various hypermodules have been studied by many researchers [4]-[6], [27, 32, 36, 40], [55]-[58]. We concentrate on the kind of hypermodules over Krasner hyperrings. Zhan et al. [55] established three isomorphism theorems of hypermodules, and derived the Jordan-Holder theorem for hypermodules. Also, they considered the fundamental relation defined on a hypermodule and investigated some related properties. In [5], Anvariyeh and

Davvaz introduced a new strongly regular equivalence relation on hypermodules so that the quotient is module over a commutative ring. Moreover, Zhan et al. [56] applied fuzzy sets to hypermodules and introduced the concept of fuzzy sub-hypermodules of a hypermodule. Furthermore, they [57] considered the normal fuzzy sub-hypermodule of a hypermodule and investigated the fuzzy isomorphism of hypermodules by using the normal fuzzy sub-hypermodules.

In this paper, we consider the connection between soft sets and hypermodules. We define soft hypermodules and soft sub-hypermodules, and study the extended intersection, restricted intersection, extended union, $\wedge$-intersection, cartesian product of the family of soft hypermodules and soft sub-hypermodules. Also, we consider some basic properties of soft hypermodules by homomorphism between hypermodules. Moreover, we introduce homomorphism and isomorphism of soft hypermodules, and establish three (fuzzy) isomorphism theorems of soft hypermodules.

2. Preliminaries

A hyperstructure $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, i.e., a mapping $H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. If $x \in H$ and A, B are subsets of H, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$.

A hyperstructure $(H, \circ)$ is called a canonical hypergroup [41] if the following conditions hold:

(1) for every $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$;

(2) for every $x, y \in H$, $x \circ y = y \circ x$;

(3) there exists $0 \in H$ such that $0 \circ x = x$ for all $x \in H$;

(4) for every $x \in H$ there exists a unique element $x' \in H$ such that $0 \in x \circ x'$ (we shall write $-x$ for x' and we call it the opposite of x);

(5) $z \in x \circ y$ implies $y \in -x \circ z$ and $x \in z \circ -y$.

Definition 2.1([31]). A hyperring (Krasner hyperring) is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.

(3) The multiplication is distributive with respect to the hyperoperation $+$.

Definition 2.2([49]). A non-empty set M is called a left hypermodule over a hyperring R (R-hypermodule) if $(M, +)$ is a canonical hypergroup and there exists the map $\cdot$: $R \times M \to M$ by $(r, m) \mapsto r \cdot m$ such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, we have (1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$; (2) $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1$; (3) $(r_1 \cdot r_2) \cdot m_1 = r_1 \cdot (r_2 \cdot m_1)$.

A non-empty subset A of an R-hypermodule M is called a sub-hypermodule of M if A is itself a hypermodule. A sub-hypermodule A of M is normal in M if $x + A - x \subseteq A$

for all $x \in M$.

Example 2.3([49]). Let M be an R-module, where R is an unitary ring, and let A be a subgroup of the multiplicative semigroup of R, which satisfies the condition $aAbA = abA$, for every $a, b \in R$. Note that this condition is equivalent to the normality of the subgroup A only when $R \setminus \{0\}$ is a group, which appears only in the case of division rings. Define an equivalence relation $\sim$ on M by $x \sim y \Leftrightarrow x = ty, t \in A$. Let $\overline{M}$ be the set of all equivalence classes of M modulo $\sim$. A hyperoperation $\oplus$ can be endowed in M by $\overline{x} \oplus \overline{y} = \{\overline{w} \in \overline{M} \mid \overline{w} \subseteq \overline{x} + \overline{y}\}$. Then $(\overline{M}, \oplus)$ becomes a canonical hypergroup. Now, suppose that $\overline{R}$ is the quotient hyperring of R by A. Consider an external composition from $\overline{R} \times \overline{M}$ to $\overline{M}$ defined by $\overline{a}\,\overline{x} = \overline{ax}$ for every $\overline{a} \in \overline{R}, \overline{x} \in \overline{M}$. Then the above composition satisfies the conditions of the hypermodule and so $\overline{M}$ becomes a hypermodule over $\overline{R}$. Massouros [40] showed that the hypermodule is strongly related with the analytic projective geometries as well as with the Euclidean spherical geometries.

In what follows, all hypermodules mean left hypermodules. If A is a normal sub-hypermodule of an R-hypermodule M, then for all $x, y \in M$, $(A + x) + (A + y) = A + x + y = A + z$ for all $z \in x + y$, and $A + x = A + y$ for all $y \in A + x$. If A and B are sub-hypermodules of an R-hypermodule M with B normal in M, then $A \cap B$ is a normal sub-hypermodule of A, and B is a normal sub-hypermodule of $A + B$.

If A is a normal sub-hypermodule of an R-hypermodule M, then the relation A^* defined by $x \equiv y (mod\ A)$ if and only if $(x - y) \cap A \neq \emptyset$ is an equivalence relation. Let $A^*[x]$ be the equivalence class of the element $x \in M$. Then $A + x = A^*[x]$ for all $x \in M$. Moreover, on the set of all classes $M/A = \{A^*[x] \mid x \in M\}$, define the the hyperoperations $\oplus$ and the multiplication $\odot$ by $A^*[x] \oplus A^*[y] = \{A^*[z] \mid z \in A^*[x] + A^*[y]\}$, $r \odot A^*[x] = A^*[r \cdot x]$. Then $(M/A, \oplus, \odot)$ is an R-hypermodule [55].

Let M_1 and M_2 be two R-hypermodules. A mapping $f : M_1 \to M_2$ is said to be a homomorphism if for all $a, b \in M_1$ and $r \in R$, $f(a+b) = f(a) + f(b)$, $f(r \cdot a) = r \cdot f(a)$, and $f(0) = 0$. A homomorphism f is an isomorphism if f is bijective. If f is a homomorphism from M_1 into M_2, then the kernel of f is the set $kerf = \{x \in M_1 \mid f(x) = 0\}$. It is trivial that $kerf$ is a sub-hypermodule of M_1, but in general it is not normal in M_1.

Definition 2.4([56]). A fuzzy subset μ of an R-hypermodule M is called a fuzzy sub-hypermodule of M if the conditions hold: (1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$ for all $x, y \in M$; (2) $\mu(x) \leq \mu(-x)$ for all $x \in M$; (3) $\mu(x) \leq \mu(r \cdot x)$ for all $r \in R$ and $x \in M$. A fuzzy sub-hypermodule μ of M is called normal if $\mu(y) \leq \inf_{\alpha \in x+y-x} \mu(\alpha)$ for all $x, y \in M$.

Now, we review some notions about soft sets. Let U be an initial universe set and E be a set of parameters. $\mathscr{P}(U)$ denotes the power set of U and $A \subseteq E$.

Definition 2.5([42]). A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$.

In fact, a soft set over U is a parameterized family of subsets of the universe U.

For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). There are several examples in [37].

Definition 2.6([37]). For two soft sets (F, A) and (G, B) over U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A)\tilde{\subseteq}(G, B)$, if $A \subseteq B$ and for all $e \in A$, $F(e)$ and $G(e)$ are identical approximations.

Two soft sets (F, A) and (G, B) over U are called soft equal if $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$.

Definition 2.7([3, 28]). (1) The extended intersection of two soft sets (F, A) and (G, B) over U is the soft set $(H, C) = (F, A) \cap_{\mathscr{E}} (G, B)$, where $C = A \cup B$, and for all $e \in C$, if $e \in A - B$, then $H(e) = F(e)$; if $e \in B - A$, then $H(e) = G(e)$; if $e \in A \cap B$, then $H(e) = F(e) \cap G(e)$.
(2) The extended intersection of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G, B) = (\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$, where $B = \bigcup_{i\in\Lambda} A_i$, and for all $e \in B$, $G(e) = \bigcap_{i\in\Lambda(e)} F_i(e)$ and $\Lambda(e) = \{i \in \Lambda \mid e \in A_i\}$.

Definition 2.8([3, 19]). (1) The restricted intersection (or bi-intersection) of two soft sets (F, A) and (G, B) over U is the soft set $(H, C) = (F, A) \cap_{\mathscr{R}} (G, B)$, where $C = A \cap B \neq \emptyset$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$.
(2) The restricted intersection of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G, B) = (\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$, where $B = \bigcap_{i\in\Lambda} A_i \neq \emptyset$ and for all $e \in B$, $G(e) = \bigcap_{i\in\Lambda} F_i(e)$.

Definition 2.9([37, 19]). (1) The extended union of two soft sets (F, A) and (G, B) over U is the soft set $(H, C) = (F, A) \cup_{\mathscr{E}} (G, B)$, where $C = A \cup B$, and for all $e \in C$, if $e \in A - B$, then $H(e) = F(e)$; if $e \in B - A$, then $H(e) = G(e)$; if $e \in A \cap B$, then $H(e) = F(e) \cup G(e)$.
(2) The extended union of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G, B) = (\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$, where $B = \bigcup_{i\in\Lambda} A_i$, and for all $e \in B$, $G(e) = \bigcup_{i\in\Lambda(e)} F_i(e)$ and $\Lambda(e) = \{i \in \Lambda \mid e \in A_i\}$.

Definition 2.10([3, 28]). (1) The restricted union of two soft sets (F, A) and (G, B) over U is the soft set $(H, C) = (F, A) \cup_{\mathscr{R}} (G, B)$, where $C = A \cap B \neq \emptyset$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$.
(2) The restricted union of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G, B) = (\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$, where $B = \bigcap_{i\in\Lambda} A_i \neq \emptyset$ and for all $e \in B$, $G(e) = \bigcup_{i\in\Lambda} F_i(e)$.

Definition 2.11([37, 19]). (1) The $\wedge$-intersection of two soft sets (F, A) and (G, B) over U is the soft set $(H, A \times B) = (F, A)\tilde{\wedge}(G, B)$, where $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.
(2) The $\wedge$-intersection of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G, B) = \tilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$, where $B = \prod_{i\in\Lambda} A_i$ and for all $e = (e_i)_{i\in\Lambda} \in B$, $G(e) = \bigcap_{i\in\Lambda} F_i(e_i)$.

Definition 2.12([37, 19]). (1) The $\vee$-union of two soft sets (F, A) and (G, B) over

U is the soft set $(H, A \times B) = ((F,A)\widetilde{\vee}(G,B)$, where $H(a,b) = F(a) \cup G(b)$ for all $(a,b) \in A \times B$.
(2) The $\widetilde{\vee}$-union of a non-empty family of soft sets $(F_i, A_i)_{i\in\Lambda}$ over U is the soft set $(G,B) = \widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i)$, where $B = \prod_{i\in\Lambda} A_i$ and for all $e = (e_i)_{i\in\Lambda} \in B$, $G(e) = \bigcup_{i\in\Lambda} F_i(e_i)$.

Definition 2.13([37, 28]). (1) Let (F,A) and (G,B) be two soft sets over U and V, respectively. Then the cartesian product of the two soft sets (F,A) and (G,B) is the soft set $(H, A \times B) = (F,A) \times (G,B)$, where $H(a,b) = F(a) \times G(b)$ for all $(a,b) \in A \times B$.
(2) Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft sets over U_i, $i \in \Lambda$. Then the cartesian product of these soft sets is the soft set $(G,B) = \prod_{i\in\Lambda}(F_i, A_i)$, where $B = \prod_{i\in\Lambda} A_i$ and for all $e = (e_i)_{i\in\Lambda} \in B$, $G(e) = \prod_{i\in\Lambda} F_i(e_i)$.

Definition 2.14([19]). For a soft set (F,A) over U, the set $\text{Supp}(F,A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the support of the soft set (F,A). A soft set (F,A) is non-null if $\text{Supp}(F,A) \neq \emptyset$.

3. Soft hypermodules

In what follows, let M be an R-hypermodule and A be a non-empty set. A set-valued function $F : A \to \mathcal{P}(M)$ can be defined as $F(x) = \{y \in M \mid (x,y) \in \rho\}$ for all $x \in A$, where ρ is an arbitrary binary relation between an element of A and an element of M, that is, ρ is a subset of $A \times M$. Then the pair (F,A) is a soft set over M.

Definition 3.1. Let (F,A) be a non-null soft set over M. Then (F,A) is called a soft hypermodule over M if $F(x)$ is a sub-hypermodule of M for all $x \in \text{Supp}(F,A)$.

Example 3.2. Let $R = \{0,1,2\}$ be a set with hyperoperation $+$ and operation $\cdot$ as follows:

+	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then $(R, +, \cdot)$ is a hyperring. Let $M = \{0,1,2,3,4,5,6,7,8\}$ be a set with the hyperoperation as follows:

⊕	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	1	{0,1,2}	4	4	{3,4,5}	7	7	{6,7,8}
2	2	{0,1,2}	2	5	{3,4,5}	5	8	{6,7,8}	8
3	3	4	5	3	4	5	{0,3,6}	{1,4,7}	{2,5,8}
4	4	4	{3,4,5}	4	4	{3,4,5}	{1,4,7}	{1,4,7}	M
5	5	{3,4,5}	5	5	{3,4,5}	5	{2,5,8}	M	{2,5,8}
6	6	7	8	{0,3,6}	{1,4,7}	{2,5,8}	6	7	8
7	7	7	{6,7,8}	{1,4,7}	{1,4,7}	M	7	7	{6,7,8}
8	8	{6,7,8}	8	{2,5,8}	M	{2,5,8}	8	{6,7,8}	8

Then $(M, \oplus)$ is a canonical hypergroup. Now, we define the external product from $R \times M$

to M as follows:

$\odot$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	0	1	2	3	4	5	6	7	8
2	0	1	2	3	4	5	6	7	8

Then $(M, \oplus, \odot)$ is an R-hypermodule [27].

Let (F, A) be a soft set over M, where $A = M$ and $F : A \to \mathscr{P}(M)$ is a set-valued function given by $F(x) = \{y \in M \mid x\rho y \Leftrightarrow x \in x \oplus y\}$ for all $x \in A$. Then $F(0) = \{0\}, F(1) = F(2) = \{0, 1, 2\}, F(3) = F(6) = \{0, 3, 6\}$ and $F(4) = F(5) = F(7) = F(8) = M$ are sub-hypermodules of M. Thus (F, A) is a soft hypermodule over M.

Theorem 3.3. Let (F, A) be a soft hypermodule over M. If $B \subsetneq A$, then $(F\mid_B, B)$ is a soft hypermodule over M if it is non-null.
Proof. It is straightforward.

Theorem 3.4. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft hypermodules over M, then the extended intersection $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ is a soft hypermodule over M if it is non-null.
Proof. By Definition 2.7(2), we can write $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B = \bigcup_{i\in\Lambda} A_i$ and $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ for all $x \in B$. For all $x \in \text{supp}(G, B)$, we have $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x) \neq \emptyset$ which implies that $F_i(x) \neq \emptyset$ for all $i \in \Lambda(x)$. Since for all $i \in \Lambda$, (F_i, A_i) is a soft hypermodule over M, the non-empty set $F_i(x)$ is a sub-hypermodule of M. So we deduce that $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ is a sub-hypermodule of M for all $x \in \text{supp}(G, B)$. Hence, $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$ is a soft hypermodule over M.

Theorem 3.5. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft hypermodules over M, then the restricted intersection $(\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$ is a soft hypermodule over M.
Proof. According to Definition 2.8(2), we can write $(\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B = \bigcap_{i\in\Lambda} A_i \neq \emptyset$ and $G(x) = \bigcap_{i\in\Lambda} F_i(x)$ for all $x \in B$. Let $x \in \text{supp}(G, B)$, then $G(x) = \bigcap_{i\in\Lambda} F_i(x) \neq \emptyset$, i.e., $F_i(x) \neq \emptyset$ for all $i \in \Lambda$. We have $F_i(x)$ is a sub-hypermodule of M for all $i \in \Lambda$, since $(F_i, A_i)_{i\in\Lambda}$ is a non-empty family of soft hypermodules over M. It follows that $G(x) = \bigcap_{i\in\Lambda} F_i(x)$ is a sub-hypermodule of M. Therefore, $(\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i) = (G, B)$ is a soft hypermodule over M.

Theorem 3.6. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft hypermodules over M. If $A_i \cap A_j = \emptyset$ for all $i, j \in \Lambda$ and $i \neq j$, then the extended union $(\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ is a soft hypermodule over M.
Proof. From Definition 2.9(2), we can write $(\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B = \bigcup_{i\in\Lambda} A_i$, and for all $x \in B$, $G(x) = \bigcup_{i\in\Lambda(x)} F_i(x)$. Since $\{A_i \mid i \in \Lambda\}$ are pairwise disjoint, we have $\text{Supp}(G, B) = \bigcup_{i\in\Lambda} \text{Supp}(F_i, A_i) \neq \emptyset$, that is, (G, B) is non-null. For all $x \in \text{Supp}(G, B)$, $G(x) = \bigcup_{i\in\Lambda(x)} F_i(x) \neq \emptyset$. By the hypothesis, we know that $\Lambda(x)$ contains only a element and suppose the element is i_0. Furthermore, we have that $G(x) = F_{i_0}(x)$ is a sub-hypermodule of M since (F_{i_0}, A_{i_0}) is a soft hypermodule over M. Thus, $(\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$ is a soft hypermodule over M.

Theorem 3.7. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft hypermodules over M. Then the $\wedge$-intersection $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$ is a soft hypermodule over M.
Proof. Using Definition 2.11(2), we can write $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B =$

$\prod_{i\in\Lambda} A_i$ and $G(x) = \bigcap_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$. Obviously, $\mathrm{Supp}(G, B) = \prod_{i\in\Lambda} \mathrm{Supp}(F_i, A_i) \neq \emptyset$. Let $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}(G, B)$, then we have $G(x) = \bigcap_{i\in\Lambda} F_i(x_i) \neq \emptyset$. Since (F_i, A_i) is a soft hypermodules for all $i \in \Lambda$, it follows that the non-empty set $F_i(x_i)$ is a sub-hypermodule of M. So we have $G(x) = \bigcap_{i\in\Lambda} F_i(x_i)$ is a soft sub-hypermodule of M for all $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}(G, B)$. Hence, $\bigwedge_{i\in\Lambda}(F_i, A_i) = (G, B)$ is a soft hypermodule over M.

Theorem 3.8. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft hypermodules over M_i. Then the cartesian product $\prod_{i\in\Lambda}(F_i, A_i)$ is a soft hypermodule over $\prod_{i\in\Lambda} M_i$.

Proof. By Definition 2.13(2), we can write $\prod_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B = \prod_{i\in\Lambda} A_i$ and $G(x) = \prod_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$. It is clear that $\mathrm{Supp}(G, B) = \prod_{i\in\Lambda} \mathrm{Supp}(F_i, A_i) \neq \emptyset$. For all $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}(G, B)$, $G(x) = \prod_{i\in\Lambda} F_i(x_i) \neq \emptyset$ which implies that $F_i(x_i) \neq \emptyset$ for all $i \in \Lambda$. Furthermore, we have $F_i(x_i)$ is a sub-hypermodule of M_i since (F_i, A_i) is a soft hypermodule over M_i for all $i \in \Lambda$. So $G(x) = \prod_{i\in\Lambda} F_i(x_i)$ is a sub-hypermodule of $\prod_{i\in\Lambda} M_i$ for all $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}(G, B)$. Consequently, the cartesian product $\prod_{i\in\Lambda}(F_i, A_i) = (G, B)$ is a soft hypermodule over $\prod_{i\in\Lambda} M_i$.

Definition 3.9. Let (F, A) be a soft hypermodule over M, then (F, A) is called the trivial soft hypermodule over M if $F(x) = \{0\}$ for all $x \in A$; (F, A) is called the whole soft hypermodule over M if $F(x) = M$ for all $x \in A$.

Example 3.10. Consider the hypermodule M defined in Example 3.2. Let $A = \{1, 2, 3, 6\}$ and $F : A \to \mathscr{P}(M)$ be the set-valued function defined by $F(x) = \{y \in M \mid x\rho y \Leftrightarrow x \oplus y = \{x\}\} \setminus \{x\}$ for all $x \in A$. Then $F(1) = F(2) = F(3) = F(6) = \{0\}$. Hence (F, A) is the trivial soft hypermodule over M.

Let $B = \{4, 5, 7, 8\}$ and $G : B \to \mathscr{P}(M)$ be the set-valued function defined by $G(x) = \{y \in M \mid x\rho' y \Leftrightarrow x \in x \oplus y\}$ for all $x \in B$. Then $G(x) = M$ for all $x \in B$ and so (G, B) is the whole soft hypermodule over M.

Theorem 3.11. Let M_1 and M_2 be two hypermodules, and $f : M_1 \to M_2$ be a homomorphism. If (F, A) is a soft hypermodule over M_1, then $(f(F), A)$ is a soft hypermodule over M_2, where $f(F)(x) = f(F(x))$ for all $x \in A$.

Proof. It is clear that $\mathrm{Supp}(f(F), A) = \mathrm{Supp}(F, A)$. For every $x \in \mathrm{Supp}(f(F), A)$, $f(F)(x) = f(F(x))$ is a sub-hypermodule of M_2, since $F(x)$ is a sub-hypermodule of M_1 and its homomorphic image is also a sub-hypermodule of M_2. Therefore, $(f(F), A)$ is a soft hypermodule over M_2.

Theorem 3.12. Let M_1 and M_2 be two hypermodules, $f : M_1 \to M_2$ be a homomorphism, and (F, A) be a soft hypermodule over M_1.

(1) If $F(x) = ker f$ for all $x \in A$, then $(f(F), A)$ is the trivial soft hypermodule over M_2.

(2) If f is onto and (F, A) is the whole soft hypermodule over M_1, then $(f(F), A)$ is the whole soft hypermodule over M_2.

Proof. (1) Since $F(x) = ker f$ for all $x \in A$, we have $f(F)(x) = f(F(x)) = \{0\}$. It follows from Theorem 3.11 and Definition 3.9 that $(f(F), A)$ is the trivial soft hypermodule over M_2.

(2) Since f is onto and (F, A) is the whole soft hypermodule over M_1, we have $f(F)(x) = f(F(x)) = f(M_1) = M_2$ for all $x \in A$. According to Theorem 3.11 and Definition 3.9, we have that $(f(F), A)$ is the whole soft hypermodule over M_2.

Definition 3.13. Let (F, A) and (G, B) be two soft hypermodules over M. Then (G, B) is called a soft sub-hypermodule of (F, A) if the following conditions are satisfied:

(1) $B \subseteq A$;

(2) for all $x \in \mathrm{Supp}(G, B)$, $G(x)$ is a sub-hypermodule of $F(x)$.

Example 3.14. Consider the hypermodule M and the soft hypermodule (F, A) given in Example 3.2. Let $B = \{0, 1, 2, 7, 8\} \subseteq A$ and $G : B \to \mathscr{P}(M)$ be the set-valued function defined by $G(x) = \{0\} \cup \{y \in M \backslash \{6, 7, 8\} \mid x \rho y \Leftrightarrow x \oplus y \subseteq \{6, 7, 8\}\}$ for all $x \in B$. Then $G(0) = \{0\}, G(1) = G(2) = \{0\}$ and $G(7) = G(8) = \{0, 1, 2\}$ are sub-hypermodules of $F(0) = \{0\}$, $F(1) = F(2) = \{0, 1, 2\}$ and $F(7) = F(8) = M$, respectively, so (G, B) is a soft sub-hypermodule of (F, A).

Theorem 3.15. Let (F, A) and (G, B) be soft hypermodules over M. If $B \subseteq A$ and $G(x) \subseteq F(x)$ for all $x \in \mathrm{supp}(G, B)$, then (G, B) is a soft sub-hypermodule of (F, A).
Proof. Straightforward.

Theorem 3.16. Let (F, A) be a soft hypermodule over M, and $(G_i, B_i)_{i \in \Lambda}$ be a non-empty family of soft sub-hypermodules of (F, A), then

(1) the extended intersection $(\bigcap_{\mathscr{E}})_{i \in \Lambda}(G_i, B_i)$ is a soft sub-hypermodule of (F, A) if it is non-null;

(2) the restricted intersection $(\bigcap_{\mathscr{R}})_{i \in \Lambda}(G_i, B_i)$ is a soft sub-hypermodule of (F, A);

(3) if $A_i \cap A_j = \emptyset$ for all $i, j \in \Lambda$ and $i \neq j$, then the extended union $(\bigcup_{\mathscr{E}})_{i \in \Lambda}(G_i, B_i)$ is a soft sub-hypermodule of (F, A);

(4) the $\wedge$-intersection $\widetilde{\bigwedge}_{i \in \Lambda}(G_i, B_i)$ is a soft sub-hypermodule of the soft hypermodule $\widetilde{\bigwedge}_{i \in \Lambda}(F, A)$;

(5) the cartesian product $\prod_{i \in \Lambda}(G_i, B_i)$ is a soft sub-hypermodule of the soft hypermodule $\prod_{i \in \Lambda}(F, A)$.

Proof. Similar to the proofs of Theorem 3.4-3.8.

Theorem 3.17. Let M_1 and M_2 be two hypermodules, and $f : M_1 \to M_2$ be a homomorphism. If (F, A) is a soft hypermodule over M_1, and (G, B) is a soft sub-hypermodule of (F, A), then $(f(G), B)$ is a soft sub-hypermodule of $(f(F), A)$.
Proof. Since (F, A) and (G, B) are soft hypermodule over M_1, it follows from Theorem 3.11 that $(f(F), A)$ and $(f(G), B)$ are soft hypermodules over M_2. It is clear that $\mathrm{Supp}(f(G), B) = \mathrm{Supp}(G, B)$. Since (G, B) is a soft sub-hypermodule of (F, A), we have that $B \subseteq A$ and $G(x)$ is a sub-hypermodule of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$. So for all $x \in \mathrm{Supp}(f(G), B)$, $f(G)(x) \subseteq f(F)(x)$. It follows from Theorem 3.15 that $(f(G), B)$ is a soft sub-hypermodule of $(f(F), A)$.

4. Isomorphism theorems of soft hypermodules

Definition 4.1. Let M_1 and M_2 be two hypermodules, (F, A) and (G, B) be soft hypermodules over M_1 and M_2, respectively, and $f : M_1 \to M_2$ and $g : A \to B$ be two functions. Then (f, g) is called a soft homomorphism if the following conditions hold:

(1) f is a homomorphism;

(2) g is a mapping;

(3) for all $x \in A$, $f(F(x)) = G(g(x))$.

If there is a soft homomorphism (f, g) between (F, A) and (G, B), we say that (F, A) is soft homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$. Furthermore, if f is a monomorphism (resp. epimorphism, isomorphism) and g is a injective (resp. surjective, bijective) mapping, then (f, g) is called a soft monomorphism (resp. epimorphism, isomorphism), and (F, A) is soft monomorphic (resp. epimorphic, isomorphic) to (G, B). We use $(F, A) \simeq (G, B)$ to denote that (F, A) is soft isomorphic to (G, B).

Example 4.2. Consider the hypermodule M described in Example 3.2. Let N be the set of natural numbers and (F, A) be the soft set over M, where $A = \mathbb{N}$ and $F : A \to \mathcal{P}(M)$ is the set-valued function defined by for all $x \in A$, $F(x) = M$, if $2 \mid x$; otherwise, $F(x) = \{0, 3, 6\}$, then (F, A) is a soft hypermodule over M.

Let (G, B) be the soft set over M, where $B = \mathbb{N}$ and $G : B \to \mathcal{P}(M)$ is the set-valued function defined by for all $x \in B$, $G(x) = \{0, 3, 6\}$, if $2 \mid x$; otherwise, $G(x) = \emptyset$, then (G, B) is a soft hypermodule over M.

Let $f : M \to M$ be the mapping defined by for all $x \in M$, $f(x) = 6$, if $x \in \{6, 7, 8\}$; $f(x) = 3$, if $x \in \{3, 4, 5\}$; otherwise, $f(x) = 0$, then f is a homomorphism. Define the mapping $g : A \to B$ by $g(x) = 2x$ for all $x \in A$. It is easy to check that $f(F(x)) = G(g(x))$ for all $x \in A$. Therefore, (f, g) is a soft homomorphism and $(F, A) \sim (G, B)$.

In the following theorems, if $x \in A - \text{Supp}(F, A)$, we mean that $(F/N)(x) = \emptyset$, where (F, A) is a soft hypermodule over M, and N is a normal sub-hypermodule of M.

Theorem 4.3. Let N be a normal sub-hypermodule of M, and (F, A) be a soft hypermodule over M, then (F, A) is soft epimorphic to $(F/N, A)$, where $(F/N)(x) = F(x)/N$ for all $x \in A$, and $N \subseteq F(x)$ for all $x \in \text{Supp}(F, A)$.

Proof. Clearly, Supp$(F/N, A)$ =Supp(F, A). Since $F(x)$ is a sub-hypermodule of M and $N \subseteq F(x)$ for all $x \in \text{Supp}(F, A)$, we have that $F(x)/N$ is a sub-hypermodule of M/N. Thus, $(F/N, A)$ is a soft hypermodule over M/N. Define $f : M \to M/N$ by $f(x) = N^*[x]$, for all $x \in M$. It is clear that f is an epimorphism. We define $g : A \to A$ by $g(x) = x$ for all $x \in A$. Then g is surjective. For all $x \in A$, $f(F(x)) = F(x)/N = F(g(x))/N$. Therefore, (f, g) is a soft epimorphism, and (F, A) is soft epimorphic to $(F/N, A)$.

Theorem 4.4. (*First Isomorphism Theorem*) Let M_1 and M_2 be two hypermodules, (F, A) and (G, B) be soft hypermodules over M_1 and M_2, respectively. If (f, g) is a soft epimorphism from (F, A) to (G, B) with kernel N such that N is a normal sub-hypermodule of M_1 and $N \subseteq F(x)$ for all $x \in \text{supp}(F, A)$, then

(1) $(F/N, A) \simeq (f(F), A)$;

(2) if g is bijective, then $(F/N, A) \simeq (G, B)$.

Proof. (1) It is clear that $(F/N, A)$ and $(f(F), A)$ are soft hypermodules over M_1/N and M_2, respectively. Define $f' : M_1/N \to M_2$ by $f'(N^*[x]) = f(x)$, for all $x \in M_1$. If xN^*y, we have $(x - y) \cap N \neq \emptyset$, that is, there exists $z \in (x - y) \cap N$. Hence $f(z) = 0$ and $f(z) \in f(x) - f(y)$. It follows that $f(x) = f(y)$. So f' is well-defined.

Since f is surjective, it is clear that f' is surjective. To show that f' is injective, assume that $f(x) = f(y)$, then we have $0 \in f(x - y)$. Thus, there exists $z \in x - y$ such that $z \in kerf$. It follows that $(x - y) \cap N \neq \emptyset$, which implies $N^*[x] = N^*[y]$. There f' is injective. Furthermore, we have

$$f'(N^*[x] \oplus N^*[y]) = f'(\{N^*[z] \mid z \in N^*[x] + N^*[y]\}) = \{f(z) \mid z \in N^*[x] + N^*[y]\}$$

$$= f(N^*[x]) + f(N^*[y]) = f(x) + f(y) = f'([N^*[x]) + f'(N^*[y]),$$

$$f'(r \odot N^*[x]) = f'(N^*[r \cdot x]) = f(r \cdot x) = r \cdot f(x) = r \cdot f'(N^*[x])$$

and $f'(N^*[0]) = f(0) = 0$. So f' is an isomorphism.

We define $g' : A \to A$ by $g'(x) = x$ for all $x \in A$, then g' is a bijective mapping. Furthermore, $f'(F(x)/N) = f(F(x)) = f(F(g'(x)))$ for all $x \in A$. Therefore, (f', g') is a soft isomorphism, and $(F/N, A) \simeq (f(F), A)$.

(2) Since f' is an isomorphism, g is bijective and for all $x \in A$, $f'(F(x)/N) = f(F(x)) = G(g(x))$. Hence, (f', g) is a soft isomorphism. So we have $(F/N, A) \simeq (G, B)$.

Example 4.5. Let R be the hyperring described in Example 3.2, and $M_1 = \{e, a, b, c\}$ be a set with the hyperoperation as follows:

$\oplus$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Then $(M_1, \oplus)$ is a canonical hypergroup. The external product from $R \times M_1$ to M_1 is defined by $r \odot m = e$ for every $r \in R$, $m \in M_1$. Then $(M_1, \oplus, \odot)$ is a hypermodule. Clearly, $M_2 = \{e, b\}$ is a sub-hypermodule of M_1. Let (F, A) and (G, B) be the soft sets over M_1 and M_2, respectively, where $A = \mathbb{N}$, $B = 3\mathbb{N}$, $F : A \to \mathscr{P}(M_1)$ and $G : B \to \mathscr{P}(M_2)$ are the set-valued functions defined by for all $x \in A$, $F(x) = M_1$, if $2 \mid x$; otherwise, $F(x) = \{e, a\}$, and for all $x \in B$, $G(x) = M_2$, if $2 \mid x$; otherwise, $G(x) = \{e\}$, respectively. Then (F, A) is a soft hypermodule over M_1 and (G, B) is a soft hypermodule over M_2.

Let $f : M_1 \to M_2$ be the mapping defined by for all $x \in M_1$, $f(x) = b$, if $x \in \{b, c\}$; $f(x) = e$, if $x \in \{e, a\}$, and $g : A \to B$ be the mapping defined by $g(x) = 3x$ for all $x \in A$. It is easy to check that (f, g) is a soft epimorphism and g is bijective. The kernel of f is the set $N = \{e, a\}$, which is a normal sub-hypermodule of M_1, and $N \subseteq F(x)$ for all $x \in \text{supp}(F, A)$. $(F/N, A)$ is the soft set over M_1/N, where $F/N : A \to \mathscr{P}(M_1/N)$ is the set-valued function denoted by for all $x \in A$, $F(x)/N = \{\{e, a\}, \{b, c\}\}$, if $2 \mid x$; otherwise, $F(x)/N = \{\{e, a\}\}$. We can check easily that $(F/N, A)$ is a soft hypermodule over M_1/N.

Define $f' : M_1/N \to M_2$ by $f'(N^*[x]) = f(x)$, for all $x \in M_1$. Clearly, f' is an isomorphism. For $x \in A$, if $2 \mid x$, then $f'(F(x)/N) = f'(M_1/N) = f(M_1) = \{e, b\} =$

$M_2 = G(g(x))$; otherwise, $f'(F(x)/N) = f'(\{e,a\}/N) = f(\{e,a\}) = \{e\} = G(g(x))$. Therefore, (f', g) is a soft isomorphism and so $(F/N, A) \simeq (G, B)$.

Theorem 4.6. (*Second Isomorphism Theorem*) Let N and K be sub-hypermodules of M, with N normal in M. If (F, A) is a soft hypermodule of K, then $(F/(N \cap K), A) \simeq ((N + F)/N, A)$ where $N \cap K \subseteq F(x)$ for all $x \in \mathrm{supp}(F, A)$.

Proof. We can obtain easily that $(F/(N \cap K), A)$ and $((N+F)/N, A)$ are soft hypermodule over $(K/(N \cap K)$ and $(N + K)/N$, respectively. Define $f : K \to (N + K)/N$ by $f(x) = N^*[x]$ for all $x \in K$. It is easy to check that f is a homomorphism. For any $N^*[x] \in (N+K)/N$, where $x \in N+K$, that is, there exist $a \in N$ and $b \in K$ such that $x \in a+b$, we have $N^*[x] = N + x = N + a + b = N + b = N^*[b] = f(b)$. Thus, f is an epimorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$. Then g is bijective. For all $x \in A$, we have $f(F(x)) = \{N^*[a] \mid a \in F(x)\} = (N + F(x))/N = (N + F(g(x)))/N$. The proof of $\{N^*[a] \mid a \in F(x)\} = (N + F(x))/N$ is showed as follows.

It is clear that $\{N^*[a] \mid a \in F(x)\} \subseteq (N + F(x))/N$. For any $N^*[b] \in (N + F(x))/N$, where $b \in N + F(x)$, which implies that there exist $n \in N$ and $k \in F(x)$ such that $b \in n + k$, we have $N^*[b] = N + b = N + n + k = N + k = N^*[k] \in \{N^*[a] \mid a \in F(x)\}$.

Therefore, (f, g) is a soft epimorphism from (F, A) to $((N + F)/N, A)$. Since $N \cap K$ is a normal sub-hypermodule of K, if we have $kerf = N \cap K$, then $(F/(N \cap K), A) \simeq ((N + F)/N, A)$ according to Theorem 4.4 (2). For any $x \in K$, $x \in kerf \Leftrightarrow f(x) = N^*[0] = N \Leftrightarrow N^*[x] = N + x = N \Leftrightarrow x \in N$ (since $x \in K$)$\Leftrightarrow x \in N \cap K$. Hence $kerf = N \cap K$.

Theorem 4.7. (*Third Isomorphism Theorem*) Let N and K be normal sub-hypermodules of M such that $N \subseteq K$. If (F, A) is a soft hypermodule over M, and $K \subseteq F(x)$ for all $x \in \mathrm{supp}(F, A)$, then $((F/N)/(K/N), A) \simeq (F/K, A)$.

Proof. Since K and N are normal sub-hypermodules of M, and $N \subseteq K$, we know that K/N is a normal sub-hypermodule of M/N, so $(M/N)/(K/N)$ is defined. Furthermore, we can deduce easily that $(F/N, A)$, $(F/K, A)$ and $((F/N)/(K/N), A)$ are soft hypermodules over M/N, M/K and $(M/N)/(K/N)$, respectively. Define $f : M/N \to M/K$ by $f(N^*[x]) = K^*[x]$. It is clear that f is an epimorphism. We define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is a bijective mapping. Furthermore, for all $x \in A$, $f(F(x)/N) = F(x)/K = F(g(x))/K$. Consequently, (f, g) is a soft epimorphism from $(F/N, A)$ to $(F/K, A)$. If we can prove $kerf = K/N$, then $((F/N)/(K/N), A) \simeq (F/K, A)$ by using Theorem 4.4 (2). For any $N^*[x] \in M/N$, $N^*[x] \in kerf \Leftrightarrow f(N^*[x]) = K^*[0] = K \Leftrightarrow K^*[x] = K + x = K \Leftrightarrow x \in K \Leftrightarrow N^*[x] \in K/N$. Therefore, we have $kerf = K/N$.

5. Fuzzy isomorphism theorems of soft hypermodules

In this scetion, we first review some related results of hypermodules [57]. Let μ be a normal fuzzy sub-hypermodule of M. Define the relation on M: $x \equiv y (mod\ \mu)$ if and only if there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$, denoted by $x\mu^* y$. The relation μ^* is an equivalence relation. If $x\mu^* y$, then $\mu(x) = \mu(y)$.

Let $\mu^*[x]$ be the equivalence class containing the element $x \in M$, and M/μ be the set of all equivalence classes, i.e., $M/\mu = \{\mu^*[x] \mid x \in M\}$. Define the two operations in

M/μ: $\mu^*[x] \oplus \mu^*[y] = \{\mu^*[z] \mid z \in \mu^*[x] + \mu^*[y]\}, r \odot \mu^*[x] = \mu^*[r \cdot x]$. Then $(M/\mu, \oplus, \odot)$ is an R-hypermodule.

Let N be a normal sub-hypermodule of M, and μ be a normal fuzzy sub-hypermodule of M. If μ is restricted to N, then μ is a normal fuzzy sub-hypermodule of N, and N/μ is a normal sub-hypermodule of M/μ. Furthermore, if μ and ν are normal fuzzy sub-hypermodules of M, then so is $\mu \cap \nu$.

If X and Y are two non-empty sets, $f : X \to Y$ is a mapping, and μ and ν are the fuzzy sets of X and Y, respectively, then the image $f(\mu)$ of μ is the fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu(x)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$. The inverse image $f^{-1}(\nu)$ of ν is the fuzzy subset of X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Let M_1 and M_2 be two hypermodules, and $f : M_1 \to M_2$ be a homomorphism. If μ and ν are (normal) fuzzy sub-hypermodules of M_1 and M_2, respectively, then (1) $f(\mu)$ is a (normal) fuzzy sub-hypermodule of M_2; (2) if f is an epimorphism, then $f^{-1}(\nu)$ is a (normal) fuzzy sub-hypermodule of M_1. If μ and ν are normal fuzzy sub-hypermodules of M_1 and M_2, respectively, then (1) if f is epimorphism, then $f(f^{-1}(\nu)) = \nu$; (2) if μ is a constant on $kerf$, then $f^{-1}(f(\mu)) = \mu$.

Let μ be a normal fuzzy sub-hypermodule of M, then $M_\mu = \{x \in M \mid \mu(x) = \mu(0)\}$ is a normal sub-hypermodule of M.

Theorem 5.1. (*First Fuzzy Isomorphism Theorem*) Let M_1 and M_2 be two hypermodules, and (F, A) and (G, B) be soft hypermodules over M_1 and M_2, respectively. If (f, g) is a soft epimorphism from (F, A) to (G, B) and μ is a normal fuzzy sub-hypermodule of M_1 with $(M_1)_\mu \supseteq kerf$, then

(1) $(F/\mu, A) \simeq (f(F)/f(\mu), A)$, where $(F/\mu)(x) = F(x)/\mu$ for all $x \in A$;

(2) if g is bijective, then $(F/\mu, A) \simeq (G/f(\mu), B)$.

Proof. (1) Since (F, A) is soft hypermodule over M_1, and μ is a normal fuzzy sub-hypermodule of M_1, $(F/\mu, A)$ is a soft hypermodule over M_1/μ. For all $x \in \text{supp}(F, A)$, $f(F(x)) = G(g(x)) \neq \emptyset$ is a sub-hypermodule of M_2. It follows that $(f(F)/f(\mu), A)$ is a soft hypermodule over $M_2/f(\mu)$. Define $f' : M_1/\mu \to M_2/f(\mu)$ by $f'(\mu^*[x]) = f(\mu)^*[f(x)]$, for all $x \in M_1$. If $\mu^*[x] = \mu^*[y]$, then $\mu(x) = \mu(y)$. Since $(M_1)_\mu \supseteq kerf$, μ is a constant on $kerf$. So we have $f^{-1}(f(\mu)) = \mu$. It follows that $f^{-1}(f(\mu))(x) = f^{-1}(f(\mu))(y)$, i.e., $f(\mu)(f(x)) = f(\mu)(f(y))$. Thus, $f(\mu)^*[(f(x))] = f(\mu)^*[(f(y))]$. So f' is well-defined. Furthermore, we have

$$f'(\mu^*[x] \oplus \mu^*[y]) = f'(\{\mu^*[z] \mid z \in \mu^*[x] + \mu^*[y]\}) = \{f(\mu)^*[f(z)] \mid z \in \mu^*[x] + \mu^*[y]\}$$
$$= f(\mu)^*(f(\mu^*[x])) \oplus f(\mu)^*(f(\mu^*[y])) = f'(\mu^*[x]) \oplus f'(\mu^*[y]),$$
$$f'(r \odot \mu^*[x]) = f'(\mu^*[r \cdot x]) = f(\mu)^*[f(r \cdot x)] = f(\mu)^*[r \cdot f(x)]$$
$$= r \odot f(\mu)^*[f(x)] = r \odot f'(\mu^*[x])$$

and $f'(\mu^*[0]) = f(\mu)^*[f(0)] = f(\mu)^*[0] = 0$. Hence, f' is a homomorphism. Clearly, f' is an epimorphism. Now, we show that f' is an monomorphism. Let $f(\mu)^*[f(x)] =$

$f(\mu)^*[f(y)]$, then we have $f(\mu)(f(x)) = f(\mu)(f(y))$, i.e., $(f^{-1}(f(\mu)))(x) = (f^{-1}(f(\mu)))(y)$. So $\mu(x) = \mu(y)$. Furthermore, we have $\mu^*[x] = \mu^*[y]$. Therefore, f' is an isomorphism.

Suppose that $g' : A \to A$ defined by $g'(x) = x$ for all $x \in A$, then g' is a bijective mapping. Furthermore, for all $x \in A$, we have $f'(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = f(F(g'(x)))/f(\mu)$. Consequently, (f', g') is a soft isomorphism. So we have $(F/\mu, A) \simeq (f(F)/f(\mu), A)$.

(2) Since f' is an isomorphism, g is bijective and for all $x \in A$, $f'(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = G(g(x))/f(\mu)$. Hence, (f', g) is a soft isomorphism. Furthermore, we have $(F/\mu, A) \simeq (G/f(\mu), B)$.

Example 5.2. Let (f, g) is the soft epimorphism described in Example 4.5. Clearly, g is bijective. Define $\mu(e) = \mu(a) = 0.8$, $\mu(b) = \mu(c) = 0.6$. We can check that μ is a normal fuzzy sub-hypermodule of M_1. Obviously, $(M_1)_\mu = \{e, a\} = kerf$. Furthermore, we have that $f(\mu)$ is a normal fuzzy sub-hypermodule of M_2 with $f(\mu)(e) = 0.8$ and $f(\mu)(b) = 0.6$. $(F/\mu, A)$ and $(G/f(\mu), B)$ are the soft sets over M_1/μ and $M_2/f(\mu)$, respectively, where $F/\mu : A \to \mathscr{P}(M_1/\mu)$ is the set-valued function denoted by for all $x \in A$, $F(x)/\mu = \{\{e, a\}, \{b, c\}\}$, if $2 \mid x$; otherwise, $F(x)/\mu = \{\{e, a\}\}$, and $G/f(\mu) : B \to \mathscr{P}(M_2/f(\mu))$ is the set-valued function denoted by for all $x \in B$, $G(x)/f(\mu) = \{\{e\}, \{b\}\}$, if $2 \mid x$; otherwise, $G(x)/f(\mu) = \{\{e\}\}$. It is easily to check that $(F/\mu, A)$ and $(G/f(\mu), B)$ are soft hypermodules over M_1/μ and $M_2/f(\mu)$, respectively.

Define $f' : M_1/\mu \to M_2/f(\mu)$ by $f'(\mu^*[x]) = f(\mu)^*[f(x)]$, for all $x \in M_1$. We can check easily that f' is an isomorphism. For $x \in A$, if $2 \mid x$, then $f'(F(x)/\mu) = f'(M_1/\mu) = \{f(\mu)^*[a] \mid a \in f(M_1)\} = \{\{e\}, \{b\}\} = G(g(x))/f(\mu)$; otherwise, $f'(F(x)/\mu) = f'(\{e, a\}/\mu) = \{f(\mu)^*[a] \mid a \in f(\{e, a\})\} = \{\{e\}\} = G(g(x))/f(\mu)$. Therefore, (f', g) is a soft isomorphism and so $(F/\mu, A) \simeq (G/f(\mu), B)$.

Theorem 5.3. Let M_1 and M_2 be two hypermodules, and (F, A) and (G, B) be soft hypermodules over M_1 and M_2 respectively. If (f, g) is a soft epimorphism from (F, A) to (G, B) and ν is a normal fuzzy sub-hypermodule of M_2, then

(1) $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$;

(2) if g is bijective, then $(F/f^{-1}(\nu), A) \simeq (G/\nu, B)$.

Proof. Since ν is a normal fuzzy sub-hypermodule of M_2 and f is an epimorphism, it follows that $f(f^{-1}(\nu)) = \nu$ and $f^{-1}(\nu)$ is a normal fuzzy sub-hypermodule of M_1. Hence, $(F/f^{-1}(\nu), A)$ and $(f(F)/\nu, A)$ are soft hypermodules over hypermodules $M_1/f^{-1}(\nu)$ and M_2/ν, respectively. For any $x \in kerf$, we have $f(x) = f(0)$. So $\nu(f(x)) = \nu(f(0))$, i.e., $f^{-1}(\nu)(x) = f^{-1}(\nu)(0)$, which implies that $x \in (M_1)_{f^{-1}(\nu)}$. So we have $(M_1)_{f^{-1}(\nu)} \supseteq kerf$. According to Theorem 5.1, we have $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$. Furthermore, if g is bijective, then we have $(F/f^{-1}(\nu), A) \simeq (G/\nu, B)$.

Theorem 5.4. (*Second Fuzzy Isomorphism Theorem*) Let (F, A) be a soft hypermodule over M. If μ and ν are two normal fuzzy sub-hypermodules with $\mu(0) = \nu(0)$, then $(F_\mu/(\mu \cap \nu), A) \simeq ((F_\mu + F_\nu)/\nu, A)$.

Proof. We can obtain easily that $\mu \cap \nu$ and ν are normal fuzzy sub-hypermodule of M_μ and $M_\mu + M_\nu$, respectively. Hence $M_\mu/(\mu \cap \nu)$ and $(M_\mu + M_\nu)/\nu$ are both R-hypermodules. Since (F, A) is a soft hypermodule over M, we can deduce that $(F_\mu/(\mu \cap \nu), A)$ and $((F_\mu + F_\nu)/\nu, A)$ are soft hypermodules over $M_\mu/(\mu \cap \nu)$ and $(M_\mu + M_\nu)/\nu$, respectively.

Define $f : M_\mu/(\mu \cap \nu) \to (M_\mu + M_\nu)/\nu$ by $f((\mu \cap \nu)^*[x]) = \nu^*[x]$ for all $x \in M_\mu$. If $(\mu \cap \nu)^*[x] = (\mu \cap \nu)^*[y]$, then $(\mu \cap \nu)(x) = (\mu \cap \nu)(y)$, that is, $\min\{(\mu(x), \nu(x)\} = \min\{(\mu(y), \nu(y)\}$. Since $x, y \in M_\mu$ and $\mu(0) = \nu(0)$, we have $\mu(x) = \mu(0) = \nu(0)$ and $\mu(y) = \mu(0) = \nu(0)$. It follows that $\nu(x) = \nu(y)$. Thus, $\nu^*(x) = \nu^*(y)$. So f is well-defined. Furthermore, we have

$$f((\mu \cap \nu)^*[x] \oplus (\mu \cap \nu)^*[y]) = f(\{(\mu \cap \nu)^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\})$$

$$= \{\nu^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\} = \nu^*((\mu \cap \nu)^*[x]) \oplus \nu^*((\mu \cap \nu)^*[y])$$

$$= f((\mu \cap \nu)^*[x]) \oplus f((\mu \cap \nu)^*[y]),$$

$$f(r \odot (\mu \cap \nu)^*[x]) = f((\mu \cap \nu)^*[r \cdot x]) = \nu^*[r \cdot x] = r \odot \nu^*[x] = r \odot f((\mu \cap \nu)^*[x])$$

and $f((\mu \cap \nu)^*[0]) = \nu^*[0] = 0$. Hence, f is a homomorphism.

For any $\nu^*[x] \in (M_\mu + M_\nu)/\nu$, where $x \in M_\mu + M_\nu$, that is, there exist $a \in M_\mu$ and $b \in M_\nu$ such that $x \in a + b$, there is $\alpha \in x - a \subseteq a + b - a \subseteq M_\nu$, i.e., $\nu(\alpha) = \nu(0)$. Hence we have $\nu^*[x] = \nu^*[a]$. So $f((\mu \cap \nu)^*[a]) = \nu^*[x]$ and f is an epimorphism. If $(\mu \cap \nu)^*[x] \neq (\mu \cap \nu)^*[y]$, then $(\mu \cap \nu)(x) \neq (\mu \cap \nu)(y)$. It follows that $\nu(x) \neq \nu(y)$. Hence, we have $\nu^*[x] \neq \nu^*[y]$. So f is a monomorphism. Consequently, f is an isomorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective. For all $x \in A$, $f(F_\mu(x)/(\mu \cap \nu)) = F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu = (F_\mu + F_\nu)(g(x))/\nu$.

Now, we prove that $F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu$. Clearly, $F_\mu(x)/\nu \subseteq (F_\mu + F_\nu)(x)/\nu$. For all $\nu^*[a] \in (F_\mu + F_\nu)(x)/\nu$, where $a \in (F_\mu + F_\nu)(x)$, which implies that there exist $m \in F_\mu(x)$ and $n \in F_\nu(x)$ such that $a \in m + n$, there is $\alpha \in a - m \subseteq m + n - m \subseteq F_\nu(x)$, i.e., $\nu(\alpha) = \nu(0)$. So we have $\nu^*[a] = \nu^*[m] \in F_\mu(x)/\nu$.

Therefore, (f, g) is a soft epimorphism and $(F_\mu/\mu \cap \nu, A) \simeq ((F_\mu + F_\nu)/\nu, A)$.

Theorem 5.5. (*Third Fuzzy Isomorphism Theorem*) Let (F, A) be a soft hypermodule over M. If μ and ν are two normal fuzzy sub-hypermodules with $\nu \leq \mu$, $\mu(0) = \nu(0)$ and $F_\mu(x) = M_\mu$ for all $x \in$Supp(F, A), then $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$.

Proof. It is clear that M_μ/ν is a normal sub-hypermodule of M/ν. So $(M/\nu)/(M_\mu/\nu)$ is defined. Since (F, A) be a soft hypermodule over M, it follows that $(F/\nu, A)$, $((F/\nu)/(F_\mu/\nu), A)$ and $(F/\mu, A)$ are soft hypermodules over M/ν, $(M/\nu)/(M_\mu/\nu)$ and M/μ, respectively. Define $f : M/\nu \to M/\mu$ by $f(\nu^*[x]) = \mu^*[x]$ for all $x \in M$. If $\nu^*[x] = \nu^*[y]$ for all $x, y \in M$, then there exists $\alpha \in x - y$ such that $\nu(\alpha) = \nu(0)$. Since $\nu \leq \mu$ and $\mu(0) = \nu(0)$, we get $\mu(\alpha) \geq \nu(\alpha) = \nu(0) = \mu(0)$, which implies that $\mu(\alpha) = \mu(0)$. So we have $\mu^*[x] = \mu^*[y]$. Hence, f is well-defined. Furthermore, we have

$$f(\nu^*[x] \oplus \nu^*[y]) = f(\{\nu^*[z] \mid z \in \nu^*[x] + \nu^*[y]\}) = \{\mu^*[z] \mid z \in \nu^*[x] + \nu^*[y]\}$$

$$= \mu^*[\nu^*[x]] \oplus \mu^*[\nu^*[y]] = f(\nu^*[x]) \oplus f(\nu^*[y]),$$

$$f(r \odot \nu^*[x]) = f(\nu^*[r \cdot x]) = \mu^*[r \cdot x] = r \odot \mu^*[x] = r \odot f(\nu^*[x])$$

and $f(\nu^*[0]) = \mu^*[0] = 0$. Hence, f is a homomorphism. Clearly, f is an epimorphism. Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective. For all $x \in A$, $f(F(x)/\nu) = F(x)/\mu = F(g(x))/\mu$. Thus, (f, g) is a soft epimorphism from $(F/\nu, A)$ to $(F/\mu, A)$.

Furthermore, $kerf = \{\nu^*[x] \in M/\nu \mid f(\nu^*[x]) = \mu^*[0]\} = \{\nu^*[x] \in M/\nu \mid \mu^*[x] = \mu^*[0]\} = \{\nu^*[x] \in M/\nu \mid \mu(x) = \mu(0)\} = \{\nu^*[x] \in M/\nu \mid x \in M_\mu\} = M_\mu/\nu$. By Theorem 4.4 (2), we have $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$.

6. Conclusion

In this paper, we introduce soft hypermodules and soft sub-hypermodules, and investigate several basic properties. Furthermore, we define homomorphism and isomorphism of soft hypermodules, and establish the first, second and three (fuzzy) isomorphism theorems of soft hypermodules. In the following work, we will apply the notion of soft sets to other algebraic hyperstructures.

References

[1] U. Acar, F. Koyuncu, B. Tanay, Soft sets and soft rings, Comput. Math. Appl. 59 (2010) 3458-3463.

[2] H. Aktaş N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (13) (2007) 2726-2735.

[3] M. I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabira, On some new operations in soft set theory, Comput. Math. Appl. 57 (9) (2009) 1547-1553.

[4] R. Ameri, On categories of hypergroups and hypermodules, J. Discrete Math. Sci. Cryptography, 6(2) (2003) 121-132.

[5] S. M. Anvariyeh, B. Davvaz, Strongly transitive geometric spaces associated to hypermodules, J. Algebra 322 (2009) 1340-1359.

[6] S. M. Anvariyeh, S. Mirvakili, B. Davvaz, θ^*-relation on hypermodules and fundamental modules over commutative fundamental rings, Commun. Algebra 36 (2008) 622-631.

[7] N. Çağman, S. Enginoğlu, Soft matrix theory and its decision making, Comput. Math. Appl. 59 (2010) 3308-3314.

[8] N. Çağman, S. Enginoğlu, Soft set theory and uni-int decision making, Eur. J. Oper. Res. 207 (2010) 848-855.

[9] D. Chen, E. C. C. Tsang, D. S. Yeung, X. Wang, The parameterization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757-763.

[10] P. Corsini, Prolegomena of Hypergroup Theory, Second edition. Aviani Editor, 1993.

[11] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematics, Kluwer Academic Publishers, 2003.

[12] I. Cristea, B. Davvaz, Atanassov's intuitionistic fuzzy grade of hypergroups, Inform. Sci. 180 (8) (2010) 1506-1517.

[13] I. Cristea, M. Jafarpour, S. S. Mousavi, A. Soleymani, Enumeration of Rosenberg hypergroups, Comput. Math. Appl. 60 (10) (2010) 2753-2763.

[14] I. Cristea, M. Ştefănescu, Carmen Angheţă, About the fundamental relations defined on the hypergroupoids associated with binary relations, Eur. J. Combin. 32 (2011) 72-81.

[15] B. Davvaz, Fuzzy H_v-submodules, Fuzzy Sets Syst. 117 (2001) 477-484.

[16] B. Davvaz, P. Corsini, Redefined fuzzy H_v-submodules and many valued implications, Inform. Sci. 177 (2007) 865-875.

[17] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, International Academic Press, USA, 2007.

[18] B. Davvaz, J. Zhan, K. P. Shum, Generalized fuzzy Hv-submodules endowed with interval valued membership functions, Inform. Sci. 178 (2008) 3147-3159.
[19] F. Feng, Y. B. Jun, X. Zhao, Soft semirings, Comput. Math. Appl. 56 (10) (2008) 2621-2628.
[20] W. L. Gau, D. J. Buehrer, Vague sets, IEEE Trans. Syst. Man and Cybernet. 23 (2) (1993) 610-614.
[21] K. Gong, Z. Xiao, X. Zhang, The bijective soft set with its operations, Comput. Math. Appl. 60 (2010) 2270-2278.
[22] M. B. Gorzalzany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets Syst. 21 (1987) 1-17.
[23] Y. Jiang, Y. Tang, Q. Chen, J. Wang, S. Tang, Extending soft sets with description logics, Comput. Math. Appl. 59 (2010) 2087-2096.
[24] Y. B. Jun, K. J. Lee, A. Khan, Soft ordered semigroups, Math. Log. Quart. 56 (1) (2010) 42-50.
[25] Y. B. Jun, C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466-2475.
[26] Y. B. Jun, C. H. Park, Applications of soft sets in Hilbert algebras, Iran. J. of Fuzzy Syst. 6 (2) (2009) 75-86.
[27] O. Kazancı, S. Yamak, B. Davvaz, The lower and upper approximations in a quotient hypermodule with respect to fuzzy sets, Inform. Sci. 178 (2008) 2349-2359.
[28] O. Kazancı, Ş Yılmaz and S. Yamak, Soft sets and soft BCH-algebras, Hacet. J. Math. Stat. 39 (2) (2010) 205-217.
[29] Z. Kong, L. Gao, L. Wang, S. Li, The normal parameter reduction of soft sets and its algorithm, Comput. Math. Appl. 56 (2008) 3029-3037.
[30] D. V. Kovkov, V.M. Kolbanov, D.A. Molodtsov, Soft sets theory-based optimization, J. Comput. Syst. Sci. Int. 46 (6) (2007) 872-880.
[31] M. Krasner, A class of hyperrings and hyperfields, Int. J. Math. Sci. 6(2) (1983) 307-312.
[32] V. Leoreanu-Fotea, Fuzzy hypermodules, Comput. Math. Appl. 57 (2009) 466-475.
[33] V. Leoreanu-Fotea, Fuzzy join n-ary spaces and fuzzy canonical n-ary hypergroups, Fuzzy Sets Syst. 161 (24) (2010) 3166-3173.
[34] V. Leoreanu-Fotea, I. G. Rosenberg, Hypergroupoids determined by lattices, Eur. J. Combin. 31 (3) (2010) 925-931.
[35] F. Li, Soft Lattices, Glob. J. Sci. Front. Res. 10 (4) (2010) 56-58.
[36] A. Madanshekaf, Exact category of hypermodules, Int. J. of Math. and Math. Sci. 2006 (2006) 1-8.
[37] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555-562.
[38] P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077-1083.
[39] F. Marty, Sur une generalization de la notion de group, Proc. 8th Congress Math. Scandenaves, Stockholm, 1934, pp. 45-49.

[40] C. G. Massouros, Free and cyclic hypermodules, Ann. Mat. Pure Appl. 150(5) (1998) 153-166.

[41] J. Mittas, Hypergroupes canoniques, Math. Balk. 2 (1979) 165-179.

[42] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19-31.

[43] D. Molodtsov, The Theory of Soft Sets, URSS Publishers, Moscow, 2004 (in Russian).

[44] M. M. Mushrif, S. Sengupta, A.K. Ray, Texture classification using a novel, soft-set theory based classification algorithm, Lecture Notes in Computer Science 3851 (2006) 246-254.

[45] C. H. Park, Y. B. Jun, M.A. Öztürk, Soft WS-algebras, Commun. Korean Math. Soc. 23 (3) (2008) 313-24.

[46] Z. Pawlak, Rough sets, Int. J. Inform. Comput. Sci. 11 (1982) 341-356.

[47] Z. Pawlak, Rough Sets: Theoretical aspects of reasoning about data, Kluwer Academic, Boston, MA, 1991.

[48] Q. Sun, Z. Zhang, and J. Liu, Soft Sets and Soft Modules, Lecture Notes in Computer Science 5009 (2008) 403-409.

[49] T. Vougiouklis, Hyperstructures and Their Representations. Hadronic Press Inc, Palm Harber, USA, 1994.

[50] Z. Xiao, K. Gong, S. S. Xia, Y. Zou, Exclusive disjunctive soft sets, Comput. Math. Appl. 59 (2010) 2128-2137.

[51] Z. Xiao, Y. Li, B. Zhong, X. Yang, Research on synthetically evaluating method for business competitive capacity based on soft set, Stat. Res. (2003) 52-54.

[52] W. Xu, Jian Ma, S. Wang, Gang Hao, Vague soft sets and their properties, Comput. Math. Appl. 59 (2010) 787-794.

[53] Y. Yin, J. Zhan, D. Xu, J. Wang, The L-fuzzy hypermodules, Comput. Math. Appl. 59(2) (2010) 953-963.

[54] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338-353.

[55] J. Zhan, B. Davvaz, K. P. Shum, Isomorphism theorems of hypermodules. Acta Math. Sin. Chin. Ser. 50(4) (2007) 909-914.

[56] J. Zhan, B. Davvaz, K. P. Shum, A new view of fuzzy hypermodules. Acta Math. Sin. Engl. Ser. 23(8) (2007) 1345-1356.

[57] J. Zhan, B. Davvaz, K. P. Shum, On fuzzy isomorphism theorems of hypermodules, Soft Comput. 11 (2007) 1053-1057.

[58] J. Zhan, B. Davvaz, K. P. Shum, On probabilistic n-ary hypergroups, Inform. Sci. 180 (7) (2010) 1159-1166.

[59] Y. Zou, Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowl.-Based Syst. 21 (8) (2008) 941-945.