Notes on the Support of t-Designs

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Abstract

The support of a t-design is the set of all distinct blocks in the design. The notation t-$(v, k, \lambda \mid b^*)$ is used to denote a t-design with precisely b^* distinct blocks. We present some results about the structure of support in t-designs. Some of them are about the number and the range of occurrences of i-sets $(1 \leq i \leq t)$ in the support. A new bound for the support sizes of t-designs is presented. In particular given a t-$(v, k, \lambda \mid b^*)$ design with $b > b_0$, where b and b_0 are the cardinality and the minimum cardinality of block sets in the design, respectively, then it is shown that $b^* \geq \lceil \frac{\lceil (2b/\lambda \rceil + 7}{2} \rceil$. We also show that when λ vary over all positive integers, then there is no t-$(v, k, \lambda \mid b^*)$-design with the support sizes equal to $b^*_{min}+1$, $b^*_{min}+2$ and $b^*_{min}+3$, where b^*_{min} denotes the least possible cardinality of the support sizes in this design.

1 Introduction

Let t, v, k and λ be positive integers with $v \geq k \geq t$. A t-(v, k, λ) design is a set of collection $\mathcal{D}$ of k-subsets called blocks of a v-set $V(\mid V \mid = v)$ such that every t-subset of V is contained in exactly λ blocks. It is well known that if $1 \leq i \leq t$, then every t-design is also an i-design. In other words every i-subset is contained in exactly λ_i blocks in $\mathcal{D}$. Let $\lambda = \lambda_t$ and let b be the number of all blocks in $\mathcal{D}$ $(\mid \mathcal{D} \mid = b)$, then the following are the necessary relations for the existence of a t-(v, k, λ) design:

$$\lambda \binom{v-i}{t-i} = \lambda_i \binom{k-i}{t-i}; \; b\binom{k}{i} = \lambda_i \binom{v}{i}, 1 \leq i \leq t.$$

Let b_0, λ_0 and λ_{0i} denote the least positive integers which satisfy the above relations, then these relations are hold only for $b = b_0 s, \lambda = \lambda_0 s, \lambda_i = \lambda_{0i} s,$

where $s \in N$. The support of a t-design $\mathcal{D}$ denoted by $\mathcal{D}^*$ is the set of all distinct blocks of $\mathcal{D}$. We use $b^* =| \mathcal{D}^* |$ to denote the support size. The least possible value of b^* in a given design is denoted by b^*_{min}. We sometimes denote a t-(v, k, λ) design with the support size b^*, $b = b_0 s, \lambda = \lambda_0 s$ and $\lambda_i = \lambda_{0i} s$, by a full configuration t-$(v, b_0 s, k, \lambda_{0i} s, \lambda_0 s, | b^*)$ or with a more detailed format, by t-$(v, b_0 s, k, \lambda_{01} s, \lambda_{02} s, \cdots, \lambda_0 s, | b^*)$.

For a family of t-(v, k, λ) design one of the important mathematical problems is to determine the set of all possible b^* or their existence bounds[see for example 4,5]. In this paper we give a new existence bound for the support sizes of t-designs, in general.

Let $\mathcal{D}$ be a t-$(v, k, \lambda \mid b^*)$ design. If $b^* < b$, then $\mathcal{D}$ is called a t-design with repeated blocks. Let $\mathcal{D}^* = \{B_1, B_2, \cdots, B_{b^*}\}$ be the support of $\mathcal{D}$. We denote by f_B the multiplicity or frequency of a block $B \in \mathcal{D}^*$ in $\mathcal{D}$. Consider f as the greatest common divisor of frequency of blocks or $f = gcd(f_{B_1}, \cdots, f_{B_{b^*}})$. If $gcd(f, \lambda) > 1$, then by applying the *Trade Off Method* [6], b can be reduced by dividing it over the greatest common divisor of λ and f without any change on the size of $\mathcal{D}^*$ (note that $b = f_{B_1} + \cdots + f_{B_{b^*}}$). Since the main interest in this note is the study of support sizes, we usually assume that $gcd(f, \lambda) = 1$. Now consider the following subsets of $\mathcal{D}^*$ for every $j \in N$:

$$E_j \stackrel{def}{=} \{B \in \mathcal{D}^* \mid f_B = j\}; \quad H_j \stackrel{def}{=} \{B \in \mathcal{D}^* \mid f_B \geq j\} = \bigcup_{l \geq j} E_l,$$

$$\Delta_j \stackrel{def}{=} \{B \in \mathcal{D}^* \mid f_B = nj, n \in N\} = \bigcup E_{jn}.$$

Also, the notations E_j', Δ_j' and H_j' are used for their set complements in $\mathcal{D}^*$, respectively. Let $E \subseteq \mathcal{D}^*$, where $\mathcal{D}$ is a t-design with the element set V. Let X be an i-subset of V. By $\lambda^*_{iE}(X)$, we denote the number of blocks in E containing X. By $\lambda_{iE}(X)$ we denote the sum of frequency of blocks in E containing X. When we write λ_{iE} or λ^*_{iE} without any specification of a special i-set X then a property about all i-subsets in E is considered. When $E = \mathcal{D}^*$ we usually drop E in λ_{iE} or λ^*_{iE} and write them as λ_i and λ^*_i. We use the notation $a \mid b$, when integer a divides integer b.

2 On the structure of support

The first theorem is about the number of occurrences of i-subsets in some special subsets of the support set.

Theorem 1. *Let $\mathcal{D}$ be a t-$(v, b_0 s, k, \lambda_{0i} s, \lambda_0 s, \mid b^*)$ design, for a fixed $s \in N$ and let $1 \leq i \leq t$. Then we have:*

(1) $s \mid \lambda_{i\Delta'_s}$ *and* $\lambda^*_{i\Delta'_s} \neq 1$. *If* $i < t$ *we also have* $\lambda^*_{i\Delta'_s} \neq 1, 2$.
(2) *If* $i < t$ *and* $k - i \geq 2$, *then* $\lambda^*_{i\Delta'_s} \neq 1, 2, 3$.
(3) *Either* $|\Delta'_s| = 0$, *or* $|\Delta'_s| \geq 7$.
(4) *If* $i \leq t$, *then* $\lambda^*_{iE_s} \neq (\lambda^*_{0i} - 1)$.
(5) *If* $i < t$ *and* $\lambda_0 \mid \lambda_{0i}$, *then* $\lambda^*_{iE_\lambda} \neq ((\lambda_{0i}/\lambda_0) - 1)$. *Also if for an i-set X we have* $\lambda^*_{iE_\lambda} = (\lambda_{0i}/\lambda_0)$, *then for all $(t-i)$-sets in $V \setminus X$ we have* $\lambda^*_{(t-i)E_\lambda} \neq 0$.
(6) *In case of* $\lambda_0 = 2$ *and* $1 \leq i \leq t$, *if for an i-set* $\lambda^*_{iH'_s} > 0$, *then for the same set* $\lambda^*_{iH_s} \neq \lambda_{0i} - 1$.

Proof. (1) We have $\lambda_i = \lambda_{i0} s = \lambda_{i\Delta_s} + \lambda_{i\Delta'_s}$. By definition of Δ_s, it is clear that $s \mid \lambda_{i\Delta_s}$, hence $s \mid \lambda_{i\Delta'_s}$, this implies to $\lambda^*_{i\Delta'_s} = 0$ or ≥ 2. Now let $i < t$ and X be an i-set with $\lambda^*_{i\Delta'_s}(X) = 2$. Since $i < t$ then by above, all $(i+1)$-sets, containing X in one of these two blocks of Δ'_s should occur in the second block too. This leads to the equality of two blocks in $\mathcal{D}^*$ which is impossible.

(2) Let X be an i-set with $\lambda^*_{i\Delta'_s}(X) = 3$ number of occurrences in blocks B_1, B_2, B_3 of Δ'_s. Since $\Delta'_s \subseteq \mathcal{D}^*$, by part (1) every two blocks of these three blocks have a common point out of X, which do not appear in the third one. Since $k - i \geq 2$ there are enough points to distinct these three blocks. Now let $y \in V \setminus X$ be a point such that $y \in B_1 \cap B_2$ but $y \notin B_3$. Hence the $(i+1)$-set $(X \cup \{y\}) \subset B_1 \cap B_2$ and do not appears elsewhere in Δ'_s. By Part (1), $s|(f_{B_1} + f_{B_2})$. Also, $s|(f_{B_1} + f_{B_2} + f_{B_3})$, consequently $s|f_{B_3}$, which is impossible, since $B_3 \in \Delta'_s$.

(3) If $\Delta'_s \neq \phi$, then by part (2) we may consider a point $x \in V$ with maximum $\lambda^*_{1\Delta'_s}$, greater than or equal to 4. We study three cases $\lambda^*_{1\Delta'_s}(\{x\}) = 4, 5$ and ≥ 6 and show that in each case the claim holds. Let $\lambda^*_{1\Delta'_s}(\{x\}) = 4$, so Δ'_s has at least 4 blocks B_1, B_2, B_3, B_4 all of which contain the point x. Let $y \in V \setminus \{x\}$ and $y \in B_1 \setminus B_2$. By part (2) above y appears at least 4 times in Δ'_s. By Parts (1) and (2) the point y does not appear in both blocks B_3 and B_4, for otherwise, considering $\lambda^*_{2\Delta'_s}(\{x, y\})$ implies that $s|f_3$ or $s|f_4$, which in not possible. Hence y needs at least two other appearances in Δ'_s to satisfy the conditions of part (2). In other words Δ'_s has at least 6 blocks. Considering distinctness of the 5-th and the 6-th block, implies that $\Delta'_s \geq 7$. For the other two cases $\lambda^*_{1\Delta'_s}(\{x\}) = 5$ and $\lambda^*_{1\Delta'_s}(\{x\}) \geq 6$, a similar argument implies the claim.

(4) Let $\lambda^*{}_{iE_s}(X) = (\lambda^*_{0i} - 1)$ for an i-set $X \subset V$. Then $\lambda_{iE'_s}(X) = s$, which by part (1) implies the equality of two blocks and this is not possible.

(5) Let $X \subset V$ be an i-subset with $\lambda^*_{iE_\lambda}(X) = ((\lambda_{0i}/\lambda_0) - 1)$. Since each t-set appears in at most one block of E_λ, the number of all t-sets containing X that do not appear in E_λ is

$$\binom{v-i}{t-i} - (\frac{\lambda_{0i}}{\lambda_0} - 1)\binom{k-i}{t-i} = \binom{k-i}{t-i}$$

The i-set X has at least two occurrences in E'_λ (note that $\lambda_i - \lambda_{iE_\lambda} = \lambda$), which leads to a common occurrences of all these $\binom{k-i}{t-i}$, $(t-i)$-sets with X. This leads to the equality of at least two blocks in E'_λ which is impossible. Now, let X be an i-set with $\lambda^*{}_{iE_\lambda}(X) = (\lambda_{0i}/\lambda_0)$, so $\lambda_{iE_\lambda}(X) = \lambda_i$ and hence X does not appear in E'_λ. This implies that all $\binom{v-i}{t-i}$, $(t-i)$-sets from $V \setminus X$ appear in E_λ. In other words $\lambda^*_{(t-i)E_\lambda} \neq 0$ for all these $(t-i)$-sets.

(6) Let $X \subset V$ be an i-set with $\lambda^*_{iH_s}(X) = \lambda_{0i} - 1$. Since $\lambda_0 = 2$, no t-sets appear more than two times in H_s. Let M and N be the collections of all t-sets containing X, which appear in one and two of these $\lambda_{0i} - 1$ blocks in H_s, respectively. Let $|M| = m$ and $|N| = n$. Counting the number of all these t-sets in H_s (not necessarily different) imply the equation $m + 2n = (\lambda_{0i} - 1)\binom{k-i}{t-i}$. Also all possible $\binom{v-i}{t-i}$ number of these t-sets (including X) should appear in H_s, hence clearly $m + n = \binom{v-i}{t-i}$. Solving these two equations simultaneously, yields $m = \binom{k-i}{t-i}$ and $n = \binom{v-i}{t-i} - \binom{k-i}{t-i}$. All elements of N do not appear in H'_s. Elements of M appear in H'_s (since $\lambda^*_{iH'_s}(X) > 0$), but they may make only one block there. This implies that in H_s they occur in blocks with frequency greater than s. Hence they build their unique block in H_s again, which is not possible. □

The following theorem presents a new bound for the support sizes.

Theorem 2. *Let D be a t-$(v, b_0s, k, \lambda_{0i}s, \lambda_0 s, \mid b^*)$ design, we have:*

(i) *If $b > b_0$, then $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0) \rceil + 7}{2} \rceil$.*

(ii) *If $b > b_0$ and $k - i \geq 2$, then $b^* \geq \lceil \frac{\lceil (4b_0/\lambda_{0i}) \rceil + 21}{4} \rceil$ for $1 \leq i < t$.*

(iii) *If $H_s = \phi$, then* $b^* \geq \lceil b_0/\lambda_{0i} \rceil + b_0$ for $1 \leq i \leq t$.

Proof. (i) If $b > b_0$, then clearly $s > 1$ hence $\Delta'_s \neq \phi$. For otherwise we have $\mathcal{D} = \Delta_s$ and and this contradicts our general assumption(in the introduction) that $gcd(f, \lambda) = 1$. In this situation at least $(\binom{v}{t} - \binom{k}{t}|\Delta_s|)$ number of all t-sets do not appear in Δ_s. By Part (1) of Theorem 1 these t-sets have at least 2 occurrences in Δ'_s, hence $|\Delta'_s| \geq \lceil 2\frac{\binom{v}{t}-\binom{k}{t}|\Delta_s|}{\binom{k}{t}}\rceil$ or $|\Delta'_s| \geq \lceil 2(b_0/\lambda_0)\rceil - 2|\Delta_s|$, so $b^* = |\Delta'_s| + |\Delta_s| \geq \lceil(2b_0/\lambda_0)\rceil - |\Delta_s|$. By Part (3) of Theorem 1, $|\Delta'_s| \geq 7$, therefore $b^* \geq 7 + |\Delta_s|$. Summing up these two inequalities, we get $b^* \geq \lceil\frac{\lceil(2b_0/\lambda_0)\rceil+7}{2}\rceil$.

(ii) Now let $i < t$ and $k - i \geq 2$, then as above at least $(\binom{v}{i} - \binom{k}{i}|\Delta_s|)$ number of all i-sets do not appear in Δ_s. By Part (2) of Theorem 1 these i-sets have at least 4 occurrences in Δ'_s, hence $|\Delta'_s| \geq \lceil 4\frac{\binom{v}{i}-\binom{k}{i}|\Delta_s|}{\binom{k}{i}}\rceil$ or $|\Delta'_s| \geq \lceil 4(b_0/\lambda_{0i})\rceil - 4|\Delta_s|$, so $b^* = |\Delta'_s| + |\Delta_s| \geq \lceil(4b_0/\lambda_{0i})\rceil - 3|\Delta_s|$. By Part (3) of Theorem 1, $|\Delta'_s| \geq 7$, therefore $3b^* \geq 21 + 3|\Delta_s|$. Summing up the above two inequalities for b^*, we get $b^* \geq \lceil\frac{\lceil(4b_0/\lambda_{0i})\rceil+21}{4}\rceil$.

(iii) If $H_s = \phi$, then $D^* = H'_s$ and each i-set has at least $\lambda_{0i} + 1$ occurrences in D^*. Hence D^* should contain at least $\binom{v}{i}(\lambda_{0i} + 1)$ of i-sets. Consequently

$$b^* \geq \lceil\binom{v}{i}(\lambda_{0i} + 1)/\binom{k}{i}\rceil = \lceil b_0 + (b_0/\lambda_{0i})\rceil = b_0 + \lceil b_0/\lambda_{0i}\rceil.$$

□

Corollary 3. *Let $\mathcal{D}$ be a t-(v, k, λ) design with $\lambda_0 = 1$. Then $\mathcal{D}$ has no support sizes equal to $b_0 + 1$, $b_0 + 2$ and $b_0 + 3$. This, in particular implies that when $b_0 = b^*_{min}$ then the support sizes equal to $b^*_{min} + 1$, $b^*_{min} + 2$, $b^*_{min} + 3$ do not exist.*

Proof. Setting $b_0 = b^*_{min}$ and $\lambda_0 = 1$ in case (i) of the above Theorem, implies the claim. □

In the following we study the range and dependence of the number of occurrences of i-sets in the support of a t-design.

Proposition 4.(i) *Let λ_i^* be as defined above the number of occurrences of i-sets in the support of a t-(v,k,λ) design, then*

$$\lceil \lambda_i/h \rceil \leq \lambda_i^* \leq min\{\lambda_i, \binom{v-i}{k-i}\},$$

where $h = min\{\lambda_i,\ b/v\}$.

(ii) *Let A_j be the set of i-sets with j occurrences in D^*. Also let $a_j = |A_j|$. Then*

$$\begin{cases} \sum_{j=\lambda_i^*} ja_j &= \binom{k}{i}b^* \\ \sum_{j=\lambda_i^*} a_j &= \binom{v}{i} \end{cases},$$

where λ_i^* vary over the range given in (i).

Proof. (i) The least number of occurrences of an i-set in the support of a design is obtained if it occurs in blocks with the highest possible frequency, where by Mann's inequality [7] is at most equal to h, so clearly $\lceil \lambda_i/h \rceil \leq \lambda_i^*$. On the other hand, the least possible frequency of blocks is 1. This gives at most λ_i occurrences of an i-set in D^*. Also the maximum number of times that an i-set occur in D^* is just the maximum number of blocks that could be made by a specified i-set which is clearly equal to $\binom{v-i}{k-i}$. Therefore $\lambda_i^* \leq min\{\lambda_i,\ \binom{v-i}{k-i}\}$.

(ii) The first equation in this system is obtained by computing the total number of "occurrences" of i-sets that appear in the support. The second equation is obtained by computing the total number of i-sets that appear in the support. □

Remark. If we apply the above proposition to study a special subset of D^*, then the values of the right sides of the above equations and the ranges of λ_i^* may change.

Application.

One can build a method from the above theorems to search for the existence of support sizes near the b_0. The following are a few examples:

2-(9,3) designs. It is proved that the support sizes $b^* = 13, 14, 15, 16, 17$ and 19 do not exist. A design with the support size $b^* = 18$ is also presented [2].

2-(13,3) designs. It is shown that the support sizes $b^* = 27, 28, 29, 31$ do not exist. A design with the support size $b^* = 30$ is also given [1].

2-(15,3) designs. It is shown that the support sizes $b^* = 36, 37, 38, 40$ do not exist. A design with the support size $b^* = 39$ is also given [1].

2-(11,5) designs. It is shown that the support sizes $b^* = 12, 13, 14, 15$

do not exist [1].

4-(11,5) designs. There is no design with the support sizes $b^* = 67, 68, 69$, for we have $\lambda_0 = 1$ and $b_0 = 66$, now the corollary 3 yields the claim[3].

3-(8,4) designs. There is no design with the support sizes $b^* = 15, 16, 17$. For we have $\lambda_0 = 1$ and $b_0 = 14$, now the corollary 3 yields the claim[3].

3-(10,4) designs. There is no design with the support sizes $b^* = 31, 32, 33$. For we have $\lambda_0 = 1$ and $b_0 = 30$, now the corollary 3 yields the claim[3].

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