Large sets of P_3-decompositions of complete multipartite graphs *

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Abstract. Let H, G be two graphs, where G is a simple subgraph of H. A G-decomposition of H, denoted by (H, G)-GD, is a partition of all the edges of H into subgraphs (G-blocks), each of which is isomorphic to G. A large set of (H, G)-GD, denoted by (H, G)-LGD, is a partition of all subgraphs isomorphic to G of H into (H, G)-GDs. In this paper, we determine the existence spectrums for $(\lambda K_{m,n}, P_3)$-LGD and $(\lambda K_{n,n,n}, P_3)$-LGD.

Keywords: large set; G-decomposition; path P_3, complete multipartite graph

1 Introduction

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a graph G and a positive integer λ, we use λG to denote the multigraph obtained from G by repeating each edge λ times. In this paper, K_n is the complete graph on n vertices, $K_{m,n}$ is the complete bipartite graph with parts of cardinalities m and n, and $K_m(n)$ is the complete multipartite graph with m partite sets, each of which

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has n vertices. A k-cycle $C_k = (x_1, x_2, \cdots, x_k)$ is a graph with k vertices $x_1, x_2, \cdots, x_k$ and k edges $\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{k-1}, x_k\}$, $\{x_k, x_1\}$. A k-path $P_k = [x_1, x_2, \cdots, x_k]$ is a graph with k vertices $x_1, x_2, \cdots, x_k$ and $k-1$ edges $\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{k-1}, x_k\}$. A $|V(G)|$-cycle (resp. path) of graph G is called a Hamilton cycle (resp. path) of G. A $(|V(G)|-1)$-cycle (resp. path) of graph G is called an almost Hamilton cycle (resp. path) of G.

Let H and G be two graphs, where G is a simple subgraph of H. An (H, G)-*GD* is a partition of $E(H)$ into subgraphs (called *G-blocks*), each of which is isomorphic to G. The (H, G)-*GD* is named as *G-decomposition* (or *G-design*) of H. A decomposition is said to be *simple* if it contains no repeated blocks. For $H = \lambda K_n$ and some simple graphs of G, such as cycle C_k, path P_k, star S_k, k-cube, the graphs with at most five vertices and some graphs with six vertices, the existence of these G-decompositions has been solved (see [2] for details).

A *large set* of (H, G)-*GD*, denoted by (H, G)-*LGD*, is a partition of all subgraphs isomorphic to G of H into (H, G)-*GD*s. It is easy to see that every decomposition is simple in a large set.

A *Steiner triple system* of order n, denoted by $STS(n)$, is a pair $(X, \mathcal{B})$, where X is an n-set and $\mathcal{B}$ is a collection of triples (called *blocks*) on X such that every pair from X belongs to exactly one block of $\mathcal{B}$. It is easy to see that an $STS(n)$ is actually a (K_n, C_3)-*GD*. It is well known that an $STS(v)$ exists if and only if $n \equiv 1, 3 \pmod 6$ and $n \geq 3$, see [2]. A *large set of Steiner triple systems* $LSTS(n)$ is equivalent to a (K_n, C_3)-*LGD*, existence of which have been completely solved by J. Lu and L. Teirlinck.

Lemma 1.1 [6, 7, 8] *There exists a* (K_n, C_3)-*LGD* ($LSTS(n)$) *if and only* *if* $n \equiv 1, 3$ $(mod\ 6), n \geq 3$ *and* $n \neq 7$.

There are some other results regarding the existence of (H, G)-*LGD*. In [1, 10], the necessary and sufficient conditions for the existence of $(\lambda K_n, C_n)$-*LGD* and $(\lambda K_n, P_n)$-*LGD*, i.e., *large sets of Hamilton cycle and path decompositions of* λK_n, have been given. In [3], Q. Kang and Y. Zhang solved the existence problem of $(\lambda K_n, P_3)$-*LGD*.

Lemma 1.2 [3] *There exists a* $(\lambda K_n, P_3)$-*LGD if and only if* $\lambda | 2(n-2)$ *and if* λ *odd then* $n \not\equiv 2, 3$ $(mod\ 4)$.

For odd prime power $q \geq k \geq 2$, Y. Zhang gave a general construction of $((k-1)K_q, P_k)$-*LGD* in [9]. For any $n \equiv 0, 1$ mod 4 and $n \neq 5$, [11] proved the existence of $(2K_n, C_{n-1})$-*LGD*. And, [4, 12] determined the spectrums for $(\lambda K_{m,n}, C_{m+n})$-*LGD* and $(\lambda K_{m,n}, P_{m+n})$-*LGD*, i.e., *large sets of Hamilton cycle and path decompositions of* $\lambda K_{m,n}$. [13] gave the necessary and sufficient conditions for the existence of $(\lambda K_{m,n}, C_{m+n-1})$-*LGD* and $(\lambda K_{m,n}, P_{m+n-1})$-*LGD* with the possible exception $((2t-1)K_{3t,3t-1}, P_{6t-2})$-*LGD* for $t \geq 2$, i.e., *large sets of almost Hamilton cycle and path decompositions of* $\lambda K_{m,n}$. In [5], J. Lei determined the spectrum for $(K_m(n), C_3)$-*LGD*, i.e., *large sets of group divisible designs of type* n^m.

Lemma 1.3 [5] *There exists a* $(K_m(n), C_3)$-*LGD if and only if* $m(m-1)n^2 \equiv 0$ $(mod\ 6)$ *and* $(m-1)n \equiv 0$ $(mod\ 2)$ *and* $(m, n) \neq (7, 1)$.

A *r-factor* of a graph G is a r-regular spanning subgraph of G. A *r-factorization* of a graph G is a partition of $E(G)$ into r-factors of G. Obviously, a Hamilton cycle of G is a 2-factor of G. A Hamilton cycle decomposition of G is a 2-factorization of G. An almost *r-factor* of a graph

G is a r-regular spanning subgraph of $G\backslash\{x\}$ for some $x \in V(G)$. An *almost r-factorization* of a graph G is a partition of $E(G)$ into almost r-factors of G. The following results are well-known.

Lemma 1.4 [2] *For any positive integer $n \geq 1$,*

(1) *there exists a 1-factorization of K_{2n};*

(2) *there exists an almost 1-factorization of K_{2n+1};*

(3) *there exists a Hamilton cycle decomposition of K_{2n+1}.*

In this paper, we investigate the existence problems of $(\lambda K_{m,n}, P_3)$-*LGD* and $(\lambda K_m(n), P_3)$-*LGD*. Finally, we obtain the existence spectrums for $(\lambda K_{m,n}, P_3)$-*LGD* and $(\lambda K_3(n), P_3)$-*LGD*.

Example 1.5 *A* $(K_{2,3}, P_3)$-*LGD* $= \{(Z_2, \overline{Z}_3, \mathcal{A}_i) : 0 \leq i \leq 2\}$.

$\mathcal{A}_0$	$\mathcal{A}_1$	$\mathcal{A}_2$
$[\overline{0}, 0, \overline{1}]$	$[\overline{0}, 0, \overline{2}]$	$[\overline{1}, 0, \overline{2}]$
$[\overline{0}, 1, \overline{1}]$	$[\overline{0}, 1, \overline{2}]$	$[\overline{1}, 1, \overline{2}]$
$[0, \overline{2}, 1]$	$[0, \overline{1}, 1]$	$[0, \overline{0}, 1]$

2 $(\lambda K_{m,n}, P_3)$-*LGD*

For any simple graph G, the following result is trivial.

Theorem 2.1 *There exists a $(\lambda G, P_2)$-LGD if and only if $\lambda = 1$. Especially, there exists a $(\lambda K_{m,n}, P_2)$-LGD if and only if $\lambda = 1$.*

Throughout this section, let $Z_m, \overline{Z}_n$ be the two partite sets of $K_{m,n}$. Define two P_3-block families in $K_{m,n}$ as follows:

$$\mathcal{P}(m,n) = \{[a, y, b] : \ a \neq b \in Z_m, y \in \overline{Z}_n\};$$
$$\mathcal{Q}(m,n) = \{[c, x, d] : \ c \neq d \in \overline{Z}_n, x \in Z_m\}.$$

It is easy to see that $|\mathcal{P}| = \binom{m}{2}n = \frac{mn(m-1)}{2}$, $|\mathcal{Q}| = \binom{n}{2}m = \frac{mn(n-1)}{2}$. And, $|\mathcal{P}| + |\mathcal{Q}| = \frac{mn(m+n-2)}{2}$ is just the number of distinct P_3-blocks in $K_{m,n}$.

It is easy to see that a $(\lambda K_{m,n}, P_3)$-GD consists of $\frac{\lambda mn}{2}$ P_3-blocks, a $(\lambda K_{m,n}, P_3)$-LGD contains $\frac{m+n-2}{\lambda}$ pairwise disjoint $(\lambda K_{m,n}, P_3)$-GDs. So, we have

Lemma 2.2 *There exists a* $(\lambda K_{m,n}, P_3)$-*LGD only if* $2|\lambda mn$ *and* $\lambda|(m+n-2)$.

Therefore, in order to determine the existence spectrum for $(\lambda K_{m,n}, P_3)$-LGD, it is enough to construct $(K_{2m,2n}, P_3)$-LGD, $(K_{2m,2n+1}, P_3)$-LGD and $(2K_{2m+1,2n+1}, P_3)$-LGD for any positive integers m and n.

Lemma 2.3 *There exists a* $(K_{2m,2n}, P_3)$-*LGD for any* $m > 0$ *and* $n > 0$.

Proof. By Lemma 1.4(1), there exist 1-factorization $\{f_1, f_2, \cdots, f_{2m-1}\}$ of K_{2m} on Z_{2m} and 1-factorization $\{\overline{f}_1, \overline{f}_2, \cdots, \overline{f}_{2n-1}\}$ of K_{2n} on $\overline{Z}_{2n}$. Define

$$\mathcal{A}_i = \{[a, y, b] : \{a, b\} \in f_i, y \in \overline{Z}_{2n}\},\ 1 \le i \le 2m-1;$$
$$\mathcal{B}_j = \{[c, x, d] : \{c, d\} \in \overline{f}_j, x \in Z_{2m}\},\ 1 \le j \le 2n-1.$$

It is easy to verify that each of $(Z_{2m} \cup \overline{Z}_{2n}, \mathcal{A}_i)$ and $(Z_{2m} \cup \overline{Z}_{2n}, \mathcal{B}_j)$ is a $(K_{2m,2n}, P_3)$-GD for $1 \le i \le 2m-1$ and $1 \le j \le 2n-1$.

Furthermore, the family $\{\mathcal{A}_i : 1 \le i \le 2m-1\}$ just forms a partition of all P_3-blocks in $\mathcal{P}(2m, 2n)$, and the family $\{\mathcal{B}_j : 1 \le j \le 2n-1\}$ just forms a partition of all P_3-blocks in $\mathcal{Q}(2m, 2n)$. Therefore, $\{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{2m-1}, \mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_{2n-1}\}$ forms a $(K_{2m,2n}, P_3)$-LGD on $Z_{2m} \cup \overline{Z}_{2n}$. ■

Lemma 2.4 *There exists a* $(K_{2m,2n+1}, P_3)$-*LGD for any* $m \ge 1$, $n \ge 0$.

Proof. By Lemma 1.4(1)(2), there exist 1-factorization $\{f_1, f_2, \cdots, f_{2m-1}\}$ of K_{2m} on Z_{2m} and almost 1-factorization $\{\overline{f}_0, \overline{f}_1, \cdots, \overline{f}_{2n}\}$ of K_{2n+1} on

$\overline{Z}_{2n+1}$, where $\overline{f}_i$ not contains vertex $\overline{i}$, $0 \le i \le 2n$. Define

$\mathcal{A}_i = \{[a, y, b] : \{a, b\} \in f_i, y \in \overline{Z}_{2n+1}\}$, $1 \le i \le 2m-2$;

$\mathcal{B}_j^1 = \{[c, x, d] : \{c, d\} \in \overline{f}_j, x \in Z_{2m}\}$, $\mathcal{B}_j^2 = \{[a, \overline{j}, b] : \{a, b\} \in f_{2m-1}\}$, $0 \le j \le 2n$.

It is easy to verify that each of $(Z_{2m} \cup \overline{Z}_{2n+1}, \mathcal{A}_i)$ and $(Z_{2m} \cup \overline{Z}_{2n+1}, \mathcal{B}_j^1 \cup \mathcal{B}_j^2)$ is a $(K_{2m,2n+1}, P_3)$-GD for $1 \le i \le 2m-2$ and $0 \le j \le 2n$.

Furthermore, the family $\{\mathcal{A}_i : 1 \le i \le 2m-2\} \bigcup \{\mathcal{B}_j^2 : 0 \le j \le 2n\}$ just forms a partition of all P_3-blocks in $\mathcal{P}(2m, 2n+1)$, and the family $\{\mathcal{B}_j^1 : 0 \le j \le 2n\}$ just forms a partition of all P_3-blocks in $\mathcal{Q}(2m, 2n+1)$. Therefore, $\{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{2m-2}, \mathcal{B}_0, \mathcal{B}_2, \cdots, \mathcal{B}_{2n}\}$ forms a $(K_{2m,2n+1}, P_3)$-LGD on $Z_{2m} \cup \overline{Z}_{2n+1}$. ■

Lemma 2.5 *There exists a* $(2K_{2m+1,2n+1}, P_3)$*-LGD for any* $m \ge 0, n \ge 0$ *and* $m + n > 0$.

Proof. By Lemma 1.4(3), there exist Hamilton cycle decomposition $\{f_1, f_2, \cdots, f_m\}$ of K_{2m+1} on Z_{2m+1} and Hamilton cycle decomposition $\{\overline{f}_1, \overline{f}_2, \cdots, \overline{f}_n\}$ of K_{2n+1} on $\overline{Z}_{2n+1}$. Cyclically orient the edges of each Hamilton cycle so that each vertex appears once as the head of an arc and once as the tail of another arc in each Hamilton cycle. Define

$$\mathcal{A}_i = \{[a, y, b] : (a, b) \in f_i, y \in \overline{Z}_{2n+1}\},\ 1 \le i \le m;$$

$$\mathcal{B}_j = \{[c, x, d] : (c, d) \in \overline{f}_j, x \in Z_{2m+1}\},\ 1 \le j \le n.$$

It is easy to verify that each of $(Z_{2m+1} \cup \overline{Z}_{2n+1}, \mathcal{A}_i)$ and $(Z_{2m+1} \cup \overline{Z}_{2n+1}, \mathcal{B}_j)$ is a $(2K_{2m+1,2n+1}, P_3)$-GD for $1 \le i \le m$ and $1 \le j \le n$.

Furthermore, the family $\{\mathcal{A}_i : 1 \le i \le m\}$ just forms a partition of all P_3-blocks in $\mathcal{P}(2m+1, 2n+1)$, and the family $\{\mathcal{B}_j : 1 \le j \le n\}$ just forms a partition of all P_3-blocks in $\mathcal{Q}(2m+1, 2n+1)$. Therefore, $\{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m, \mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n\}$ forms a $(2K_{2m+1,2n+1}, P_3)$-LGD on $Z_{2m+1} \cup \overline{Z}_{2n+1}$. ■

Theorem 2.6 *There exists a $(\lambda K_{m,n}, P_3)$-LGD if and only if $2|\lambda mn$ and $\lambda|(m+n-2)$.*

Proof. By Lemma 2.2, we only need to prove the sufficiency.

If $2|mn$, there exists a $(K_{m,n}, P_3)$-LGD $= \{(Z_m, \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le m+n-2\}$ by Lemmas 2.3 and 2.4. Define

$$\mathcal{B}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{A}_i,\ 0 \le k \le \tfrac{m+n-2}{\lambda} - 1,$$

then $\{(Z_m, \overline{Z}_n, \mathcal{B}_k) :\ 0 \le k \le \frac{m+n-2}{\lambda} - 1\}$ is a $(\lambda K_{m,n}, P_3)$-LGD.

If $2 \not| mn$, then $2|\lambda$ and there exists a $(2K_{m,n}, P_3)$-$LGD = \{(Z_m, \overline{Z}_n, \mathcal{A}_i) : 1 \le i \le \frac{m+n-2}{2}\}$ by Lemma 2.5. Define

$$\mathcal{B}_k = \bigcup_{i=k\frac{\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{A}_i,\ 0 \le k \le \tfrac{m+n-2}{\lambda} - 1,$$

then $\{(Z_m \cup \overline{Z}_n, \mathcal{B}_k) :\ 0 \le k \le \frac{m+n-2}{\lambda} - 1\}$ is a $(\lambda K_{m,n}, P_3)$-LGD. ■

3 $(\lambda K_3(n), P_3)$-LGD

Throughout this section, three partite sets of $K_3(n) = K_{n,n,n}$ are

$X_0 = \{\infty_0, \infty_1, \cdots, \infty_{n-1}\}$,

$X_1 = \{2i : 0 \le i \le n-1\}, X_2 = \{2i+1 : 0 \le i \le n-1\}$.

Obviously, $X_0 \cup X_1 \cup X_2 = Z_{2n} \cup \{\infty_0, \infty_1, \cdots, \infty_{n-1}\}$. The P_3-block $[x, y, z]$ in $K_3(n)$ belongs to two types respectively:

A-type — x, y, z belong to three different partite sets;

B-type — x, z belong to the same partite set.

Define six P_3-block families in $K_3(n)$:

$\mathcal{P}_i^1 = \{[a,y,b] : a \neq b \in X_i,\ y \in X_{i+1}\}$, $\mathcal{P}_i^2 = \{[a,y,b] : a \neq b \in X_{i+1},\ y \in X_i\}$, $i \in Z_3$.

It is easy to see that $\mathcal{P}_0^1, \mathcal{P}_0^2, \mathcal{P}_1^1, \mathcal{P}_1^2, \mathcal{P}_2^1, \mathcal{P}_2^2$ form just a partition of all B-type's P_3-blocks, and each $|\mathcal{P}_i^j| = \binom{n}{2}n,\ i \in Z_3, j = 1,2$. Further, for even n, define two P_3-block families in $K_3(n)$:

$$\mathcal{P}(n)=\{[\infty_i, y+i, -y-1+i] : 0 \leq i, y \leq n-1\};$$

$$\mathcal{Q}(n)=\{[z+n+i, \infty_i, 2n-1-z+i] : 0 \leq i \leq n-1, 0 \leq z \leq \tfrac{n}{2}-1\}.$$

Since $(y+i)+(-y-1+i) = 2i-1$ is odd, so $y+i$ and $-y-1+i$ belong to X_1 and X_2 respectively. Since $(z+n+i)+(2n-1-z+i) = 3n+2i-1$ is odd for even n, so $z+n+i$ and $2n-1-z+i$ belong to X_1 and X_2 respectively. This means that the P_3-blocks in $\mathcal{P}(n) \cup \mathcal{Q}(n)$ are all A-type's. Under the action of the automorphic group Z_{2n}, all A-type's P_3-blocks in $K_3(n)$ are separated into $(2n^3+n^3)/(2n) = \frac{3n^2}{2}$ equivalent classes, named A-orbits, each with length $2n$. Obviously, $|\mathcal{P}(n)| + |\mathcal{Q}(n)| = n^2 + n^2/2 = \frac{3n^2}{2}$. We have

Lemma 3.1 *The P_3-blocks in $\mathcal{P}(n) \cup \mathcal{Q}(n)$ belong to distinct A-orbits in $K_3(n)$.*

Proof. For given $i \in Z_n$ and $0 \leq y \neq y' \leq n-1,\ 0 \leq z \neq z' \leq \frac{n}{2}-1$, let

$$B = [\infty_i, y+i, -y-1+i],\ B' = [\infty_i, y'+i, -y'-1+i];$$

$$C = [z+n+i, \infty_i, 2n-1-z+i],\ C' = [z'+n+i, \infty_i, 2n-1-z'+i].$$

If B and B' belong to the same orbit, then $y'-y \equiv y-y'$, i.e., $2y' \equiv 2y \bmod 2n$, which is impossible since $0 \leq y \neq y' \leq n-1$. And, if C and C' belong to the same orbit, then

$$z'-z \equiv z-z' \ \text{ or } \ z'+z-n+1 \equiv n-1-z'-z \bmod 2n.$$

The former is impossible. The latter implies $2(z'+z) \equiv 2n-2 \bmod 2n$, i.e., $z + z' \equiv n - 1 \mod n$, a contradiction by $0 \le z \ne z' \le \frac{n}{2} - 1$. ■

Note that $|\bigcup_{i=0}^{2}(\mathcal{P}_i^1 \cup \mathcal{P}_i^2)| + 2n(|\mathcal{P}| + |\mathcal{Q}|) = 6\binom{n}{2}n + 2n\frac{3n^2}{2} = 3n^2(n-1) + 3n^3 = 3n^2(2n-1)$ is just the number of all P_3-blocks in $K_3(n)$. It is easy to see that a $(\lambda K_3(n), P_3)$-GD consists of $\frac{3\lambda n^2}{2}$ P_3-blocks, a $(\lambda K_3(n), P_3)$-LGD contains $\frac{2(2n-1)}{\lambda}$ disjoint $(\lambda K_3(n), P_3)$-GDs. We have

Lemma 3.2 *There exists a* $(\lambda K_3(n), P_3)$-LGD *only if* $2|\lambda n^2$, $\lambda|2(2n-1)$.

Therefore, in order to determine the existence spectrum for $(\lambda K_3(n), P_3)$-LGD, it is enough to construct $(K_3(2n), P_3)$-LGD and $(2K_3(2n+1), P_3)$-LGD for any positive integer n.

Lemma 3.3 *There exists a* $(K_3(2n), P_3)$-LGD *for any* $n > 0$.

Proof. By Theorem 2.6, for each $i \in Z_3$, there exists a $(K_{2n,2n}, P_3)$-LGD $= \{(X_i, X_{i+1}, \mathcal{B}_i^j) : 1 \le j \le 2(2n-1)\}$. Obviously, the family $\{\mathcal{B}_i^j : i \in Z_3, 1 \le j \le 4n-2\}$ just forms a partition of all B-type's P_3-block in $K_3(2n)$. Let $\mathcal{A}_{j+4n-1} = \mathcal{B}_0^j \cup \mathcal{B}_1^j \cup \mathcal{B}_2^j$, then each $(X_0, X_1, X_2, \mathcal{A}_{j+4n-1})$ is a $(K_3(2n), P_3)$-GD for $1 \le j \le 4n-2$.

Furthermore, define $\mathcal{A}_0 = \mathcal{P}(2n) \cup \mathcal{Q}(2n)$ and $\mathcal{A}_x = \mathcal{A}_0 + x$ for $x \in Z_{4n}$. By Lemma 3.1, the family $\{\mathcal{A}_x : x \in Z_{4n}\}$ just forms a partition of all A-type's P_3-block in $K_3(2n)$. Finally, we assert that $\mathcal{A}_0$ is a $(K_3(2n), P_3)$-GD, so each $\mathcal{A}_x$ is also for $x \in Z_{4n}$. In fact, we have

(1) $|\mathcal{A}_0| = |\mathcal{P}| + |\mathcal{Q}| = 3n^2/2$ is just the number of P_3-blocks in a $(K_3(2n), P_3)$-GD.

(2) $\{y : 0 \le y \le 2n-1\} \cup \{z+2n, 4n-1-z : 0 \le z \le n-1\} = Z_{4n}$ implies $\{y+i : 0 \le y \le 2n-1\} \cup \{z+2n+i, 4n-1-z+i : 0 \le z \le n-1\} = Z_{4n}$ for each $i \in Z_{2n}$. Thus, P_3-blocks in $\mathcal{A}_0$ covers all edges between X_0 and X_s for $s = 1, 2$.

(3) When y runs over Z_{2n}, the directed differences $y-(-y-1)=2y+1$ takes all odd numbers of Z_{4n}. Thus, $\{\{y+i,-y-1+i\}: 0 \le i \le 2n-1, 0 \le y \le 2n-1\}$ covers all edges between X_1 and X_2.

Therefore, $\{(X_0, X_1, X_2, \mathcal{A}_i): 0 \le i \le 8n-3\}$ is a $(K_3(2n), P_3)$-*LGD*. For convenience, we denote $(K_3(2n), P_3)$-$LGD^* = \{(X_0, X_1, X_2, \mathcal{A}_i): i \in Z_{4n}\}$. ■

Example 3.4 *A* $(K_3(2), P_3)$-$LGD = \{(\{\infty_0, \infty_1\}, \{0,2\}, \{1,3\}, \mathcal{A}_i): 0 \le i \le 5\}$.

$\mathcal{A}_0$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$
$[\infty_0, 0, 3]$	$[\infty_0, 1, 0]$	$[\infty_0, 2, 1]$	$[\infty_0, 3, 2]$	$[\infty_0, 0, \infty_1]$	$[0, \infty_0, 2]$
$[\infty_0, 1, 2]$	$[\infty_0, 2, 3]$	$[\infty_0, 3, 0]$	$[\infty_0, 0, 1]$	$[\infty_0, 2, \infty_1]$	$[0, \infty_1, 2]$
$[2, \infty_0, 3]$	$[3, \infty_0, 0]$	$[0, \infty_0, 1]$	$[1, \infty_0, 2]$	$[0, 1, 2]$	$[1, 0, 3]$
$[\infty_1, 1, 0]$	$[\infty_1, 2, 1]$	$[\infty_1, 3, 2]$	$[\infty_1, 0, 3]$	$[0, 3, 2]$	$[1, 2, 3]$
$[\infty_1, 2, 3]$	$[\infty_1, 3, 0]$	$[\infty_1, 0, 1]$	$[\infty_1, 1, 2]$	$[1, \infty_0, 3]$	$[\infty_0, 1, \infty_1]$
$[3, \infty_1, 0]$	$[0, \infty_1, 1]$	$[1, \infty_1, 2]$	$[2, \infty_1, 3]$	$[1, \infty_1, 3]$	$[\infty_0, 3, \infty_1]$

Lemma 3.5 *If there exists a* $(K_3(n), C_3)$-*LGD, then there exists a* $(2K_3(n), P_3)$-*LGD.*

Proof. It is easy to see that a $(2K_3(n), P_3)$-*GD* consists of $3n^2$ P_3-blocks, a $(2K_3(n), P_3)$-*LGD* contains $2n-1$ pairwise disjoint $(2K_3(n), P_3)$-*GD*s.

By Theorem 2.6, for each $i \in Z_3$, there exists a $(2K_{n,n}, P_3)$-$LGD = \{(X_i, X_{i+1}, \mathcal{B}_i^j): 0 \le j \le n-2\}$. The family $\{\mathcal{B}_i^j : i \in Z_3, 0 \le j \le n-2\}$ forms just a partition of all B-type's P_3-blocks in $K_3(n)$. For $0 \le j \le n-2$, let $\mathcal{A}_j = \mathcal{B}_0^j \cup \mathcal{B}_1^j \cup \mathcal{B}_2^j$, then $(X_0, X_1, X_2, \mathcal{A}_j)$ is a $(2K_3(n), P_3)$-*GD*. The $n-1$ *GD*s are pairwise disjoint.

By assumption, there exists a $(K_3(n), C_3)$-LGD=$\{(X_0, X_1, X_2, \mathcal{B}_i):$ $0 \le i \le n-1\}$. For $0 \le i \le n-1$, define $\mathcal{A}_{i+n-1} = \{[x,y,z],[y,z,x],[z,x,y]:$ $(x,y,z) \in \mathcal{B}_i\}$, then $(X_0, X_1, X_2, \mathcal{A}_{i+n-1})$ forms a $(2K_3(n), P_3)$-*GD*. The

n GDs are pairwise disjoint. The family $\{\mathcal{A}_{i+n-1} : 0 \le i \le n-1\}$ forms just a partition of all A-type's P_3-blocks in $K_3(n)$.

So, $\{(X_0 \cup X_1 \cup X_2, \mathcal{A}_i) : i \in Z_{2n-1}\}$ forms a $(2K_3(n), P_3)$-*LGD*. ■

Lemma 3.6 *There exists a* $(2K_3(n), P_3)$*-LGD for any* $n > 0$.

Proof. By Lemma 3.5 and Lemma 1.3, where $m = 3$. ■

Theorem 3.7 *There exists a* $(\lambda K_3(n), P_3)$*-LGD if and only if* $2|\lambda n^2$ *and* $\lambda|2(2n-1)$.

Proof. By Lemmas 3.2, 3.3 and 3.6, the proof is similar to that of Theorem 2.6 ■

Remark. For $(\lambda K_m(n), P_3)$-*LGD* $m > 3$, there are many problems to research.

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