On the Packing Number of Generalized Petersen Graphs $P(n, 2)$*

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Abstract

Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V(G)$ is a packing if for any two vertices of u and v in S we have $d(u, v) \geq 3$. That is, S is a packing if and only if for any vertex $v \in V(G)$, $|N[v] \cap S| \leq 1$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G. In this paper, we study the packing number of generalized Petersen graphs $P(n, 2)$ and prove that $\rho(P(n, 2)) = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \geq 5)$.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges.

A graph $G = (V(G), E(G))$ is a set $V(G)$ of vertices and a subset $E(G)$ of the unordered pairs of vertices, called edges.

The open neighborhood and the closed neighborhood of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The maximum degree of vertices in $V(G)$ is denoted by $\Delta(G)$. The distance of two distinct vertices u and v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v.

A set $S \subseteq V(G)$ is a packing if for any two vertices of u and v in S we have $d(u, v) \geq 3$. That is, S is a packing if and only if for any vertex $v \in V(G)$, $|N[v] \cap S| \leq 1$. The *packing number* $\rho(G)$, is the maximum cardinality of a packing in G.

In 1993, David C. Fisher [1] studied the packing number of complete grid graph $P_{m,n}$ and gave the exact values.

In 1998, David C. Fisher and Sarah R. Beel [2] studied the packing number of 3-dimensional grids and proved the following result.

Theorem 1.1. Let $G_{l,m,n}$ be an $l \times m \times n$ grid graph. If the packing number of $G_{l,m,n}$ is $P_2(G_{l,m,n})$, then $lmn/7 \leq P_2(G_{l,m,n}) \leq lmn + 2(P_2(G_{l,m,1}) + P_2(G_{l,1,n}) + P_2(G_{1,m,n}))/7$

In 2000, B.L. Hartnell [3] studied the packing and domination numbers of the Cartesian product of certain graphs and proved the following result.

Theorem 1.2. Let G and H both be connected graphs, where $\gamma(G) = m \geq 2$ and $\gamma(H) = n \geq 2$, every vertex of $V(G)$ is either a leaf or is attached by precisely k_1 leaves and H is a $[k_2, n]$−packable graph. If $k_1 \leq k_2$, then $k_1\gamma(G)\gamma(H) \leq \rho(G \times H)$ and $\gamma(G \times H) \leq k_2\gamma(G)\gamma(H)$.

The *generalized Petersen graph* $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 0 \leq i \leq n-1,$ subscripts modulo $n\}$.

In this paper, we consider the packing number of generalized Petersen graphs $P(n,2)$ and prove that $\rho(P(n,2)) = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \geq 5)$.

2 The packing number of $P(n, 2)$

Let $m = \lfloor \frac{n}{7} \rfloor$, $t = n \bmod 7$, then $n = 7m + t$. Let S be a packing of $P(n,2)$ with $|S| = \rho(P(n,2))$. And let

$$
\begin{aligned}
V_i &= \{v_{7i+j} : 0 \leq j \leq 6\}, \quad 0 \leq i \leq m-1, \\
U_i &= \{u_{7i+j} : 0 \leq j \leq 6\}, \quad 0 \leq i \leq m-1, \\
V_m &= \{v_{7m+j} : 0 \leq j \leq t-1\}, \text{ if } t = 0 \text{ then } V_m = \emptyset, \\
U_m &= \{u_{7m+j} : 0 \leq j \leq t-1\}, \text{ if } t = 0 \text{ then } U_m = \emptyset, \\
S_i &= S \cap (U_i \cup V_i), \quad 0 \leq i \leq m, \\
s_i &= |S_i|, \quad 0 \leq i \leq m,
\end{aligned}
$$

then,

$$
\begin{aligned}
\textstyle\bigcup_{i=0}^{m}(U_i \cup V_i) &= V(P(n,2)), \\
(U_i \cup V_i) \cap (U_j \cup V_j) &= \emptyset, \quad 0 \leq i < j \leq m.
\end{aligned}
$$

Let $V'(k,x) = \{v_{k+j}, u_{k+j} \ : \ 0 \le j \le x-1, \ (k+j) \bmod n\}$, we have

Lemma 2.1. Let S be an arbitrary packing of $P(n,2)(n \ge 7)$, then $|S \cap V'(i,7)| \le 3(0 \le i \le n-1)$.

Proof. By contradiction. Suppose that there exists a set $V'(i,7)(0 \le i \le n-1)$, say $V'(7,7)$, with $|S \cap V'(7,7)| \ge 4$.

Since $d(x,y) \ge 3$ for any two vertices of x and y in S, there are at most two vertices of $\{u_7, u_9, u_{11}, u_{13}\}$ belong to S and there is at most one vertex of $\{u_8, u_{10}, u_{12}\}$ belong to S. Hence, there is at least one vertex of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ belong to S. By symmetry, we only need to consider the cases $S \cap \{v_7, v_8, v_9, v_{10}\} \ne \emptyset$.

Let $X_i = V'(7,7) - N[N[v_i]](7 \le i \le 10)$, then $|S \cap X_i| \ge |S \cap V'(7,7) - \{v_i\}| = 4 - 1 = 3$. Notice that S is a packing, for any vertex $v \in V(P(n,2))$, $|N[v] \cap S| \le 1$.

Case 1. Suppose $v_7 \in S$, then $N[N[v_7]] \cap V'(7,7) = V'(7,3)$, $|X_7| = |V'(10,4)| = 8$. Since every vertex $x \in X_7$ is adjacent to at least two vertices of X_7, we have $|X_7| \ge |S \cap X_7| \times 3 \ge 3 \times 3 = 9 > 8 = |X_7|$, a contradiction(see Figure 2.1 (1)).

Case 2. Suppose $v_8 \in S$, then $N[N[v_8]] \cap V'(7,7) = V'(7,4)$, $|X_8| = |V'(11,3)| = 6$. Since u_{12} is adjacent to just one vertex of X_8, and every vertex in $X_8 - \{u_{12}\}$ is adjacent to at least two vertices of X_8, we have $|X_8| \ge 1 \times 2 + 2 \times 3 = 8 > 6 = |X_8|$, a contradiction(see Figure 2.1 (2)).

Case 3. Suppose $v_9 \in S$, then $N[N[v_9]] \cap V'(7,7) = V'(7,5)$, $X_9 = V'(12,2)$. Since $d(x,y) \ge 3$ for any two vertices of x and y in S, there are at most two vertices of X_9 belong to S, a contradiction with $|S \cap X_9| \ge 3$(see Figure 2.1 (3)).

Case 4. Suppose $v_{10} \in S$, then $N[N[v_{10}]] \cap V'(7,7) = V'(8,5)$, $X_{10} = V'(7,1) \cup V'(13,1)$. Since $d(x,y) \geq 3$ for any two vertices of x and y in S, there is at most one vertex of $V'(7,1)$ belong to S, and there is at most one vertex of $V'(13,1)$ belong to S, a contradiction with $|S \cap X_{10}| \geq 3$(see Figure 2.1 (4)). □

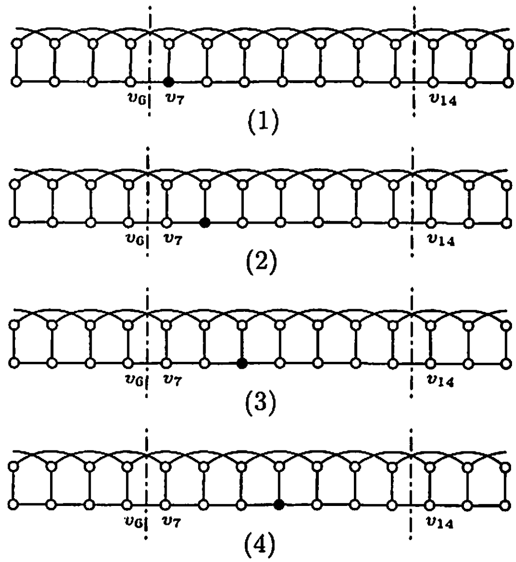

Figure 2.1.

Lemma 2.2. Let S be a packing of $P(n,2)$, if $S \cap \{v_j : 0 \leq j \leq n-1\} \neq \emptyset$, then there exists a vertex set $V'(k_t, t)$ of $V(P(n,2))$ with $|V'(k_t,t) \cap S| \leq \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor$ for $1 \leq t \leq 6$.

Proof. Let $v_i \in S \cap \{v_j : 0 \leq j \leq n-1\}$ and $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = i-2$, then

$$|V'(k_t,t) \cap S| \leq \begin{cases} 0, & t = 1,2, \\ 1, & t = 3,4,5, \\ 2, & t = 6. \end{cases}$$

(see Figure 2.2). □

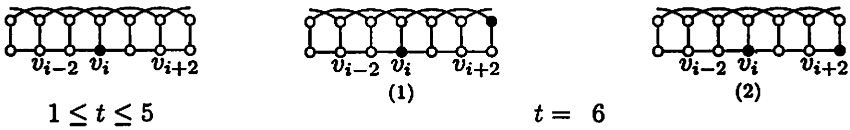

Figure 2.2.

Lemma 2.3. $\rho(P(n,2)) = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$ for $5 \leq n \leq 13$.

Proof. In Figure 2.3, we show packings of $P(n,2)$ for $5 \leq n \leq 13$, where the vertices of S are in dark. Hence we have

$$\rho(P(n,2)) \geq \begin{cases} 1, & n = 5, \\ 2, & n = 6, \\ 3, & n = 7, 8, 9, \\ 4, & n = 10, 11, 12, \\ 5, & n = 13. \end{cases}$$

i.e. $\rho(P(n,2)) \geq \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$ for $5 \leq n \leq 13$.

Let S be an arbitrary packing of $P(n,2)$. If $S \cap \{v_j : 0 \leq j \leq n-1\} = \emptyset$, then $|S| \leq \lfloor \frac{n}{3} \rfloor \leq \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$. If $S \cap \{v_j : 0 \leq j \leq n-1\} \neq \emptyset$, then by Lemma 2.2, there exists a vertex set $V'(k,t)$ of $V(P(n,2))$ with $|V'(k,t) \cap S| \leq \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor$ for $1 \leq t \leq 6$. We relabel the vertex v_i with v_{i+7m-k}, u_i with $u_{i+7m-k} (0 \leq i \leq n-1)$. Then $s_m \leq \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor$. By Lemma 2.1, we have

$$|S| = \begin{cases} s_m \leq 1, & n = 5, \\ s_m \leq 2, & n = 6 \\ s_0 + s_m \leq 3 + 0 = 3, & n = 7, 8, 9, \\ s_0 + s_m \leq 3 + 1 = 4, & n = 10, 11, 12, \\ s_0 + s_m \leq 3 + 2 = 5, & n = 13, \end{cases}$$

i.e. $|S| \leq \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$ for $5 \leq n \leq 13$. Hence $\rho(P(n,2)) = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$ for $5 \leq n \leq 13$. □

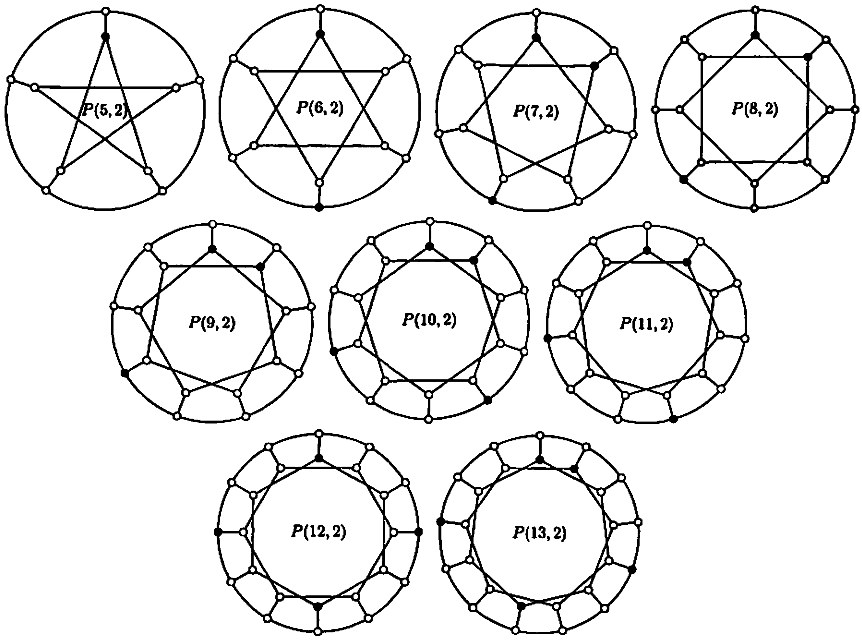

Figure 2.3. The packing of $P(n,2)$ for $5 \le n \le 13$

Lemma 2.4. $\rho(P(n,2)) \ge \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \ge 14)$.

Proof. Let

$$S = \begin{cases} \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-1\}, & t=0, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-2\} & \\ \cup\{u_{7m-7}, v_{7m-4}, u_{7m}\}, & t=1, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-1\}, & t=2, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-2\} & \\ \cup\{u_{7m-7}, v_{7m-4}, v_{7m-1}, u_{7m+2}\}, & t=3, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-2\} & \\ \cup\{u_{7m-7}, v_{7m-4}, v_{7m-1}, u_{7m+3}\}, & t=4, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-1\} & \\ \cup\{v_{7m+2}\}, & t=5, \\ \{u_{7i}, v_{7i+3}, u_{7i+6} \ : \ 0 \le i \le m-1\} & \\ \cup\{v_{7m+2}, u_{7m+5}\}, & t=6, \end{cases}$$

then, for any two vertices of x and y in S, we have $d(x,y) \geq 3$(see Figure 2.4 (1)-(7)). Hence S is a packing of $P(n,2)$ for $0 \leq t \leq 6$ with $|S| = 3m + \lfloor\frac{t+1}{7}\rfloor + \lfloor\frac{t+4}{7}\rfloor = \lfloor\frac{n}{7}\rfloor + \lfloor\frac{n+1}{7}\rfloor + \lfloor\frac{n+4}{7}\rfloor$. Hence $\rho(P(n,2)) \geq \lfloor\frac{n}{7}\rfloor + \lfloor\frac{n+1}{7}\rfloor + \lfloor\frac{n+4}{7}\rfloor$. □

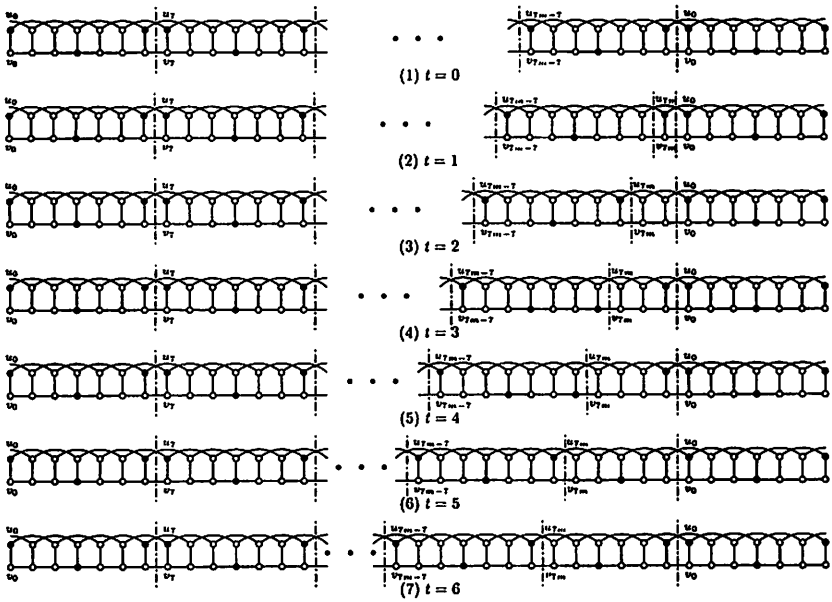

Figure 2.4.

Lemma 2.5. Let S be a packing of $P(n,2)$ with $|S| = \rho(P(n,2))$, then $S \cap \{v_j : 0 \leq j \leq n-1\} \neq \emptyset$ for $n \geq 14$.

Proof. By contradiction. Assume $S \subseteq \{u_j : 0 \leq j \leq n-1\}$, then $|S| \leq \lfloor\frac{n}{3}\rfloor = \lfloor\frac{7m+t}{3}\rfloor = 2m + \lfloor\frac{m+t}{3}\rfloor$. By Lemma 2.1, $|S| \geq \lfloor\frac{n}{7}\rfloor + \lfloor\frac{n+1}{7}\rfloor + \lfloor\frac{n+4}{7}\rfloor = \lfloor\frac{7m+t}{7}\rfloor + \lfloor\frac{7m+t+1}{7}\rfloor + \lfloor\frac{7m+t+4}{7}\rfloor = 3m + \lfloor\frac{t+1}{7}\rfloor + \lfloor\frac{t+4}{7}\rfloor$.

Since $n \geq 14$, $m = \lfloor\frac{n}{7}\rfloor$, we have $m \geq 2$. If $m \geq t$, then $|S| \leq 2m + \lfloor\frac{m+t}{3}\rfloor \leq 2m + \lfloor\frac{2m}{3}\rfloor < 3m \leq 3m + \lfloor\frac{t+1}{7}\rfloor + \lfloor\frac{t+4}{7}\rfloor \leq |S|$, a contradiction. Since $t = n \bmod 7$, we have $2 \leq m < t \leq 6$.

If $t = 3$, then $|S| \leq 2m + \lfloor \frac{m+t}{3} \rfloor = 2m + 1 + \lfloor \frac{m}{3} \rfloor < 3m + 1 = 3m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor \leq |S|$, a contradiction.

If $t = 4$, then $|S| \leq 2m + \lfloor \frac{m+t}{3} \rfloor \leq 2m + 2 < 3m + 1 = 3m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor \leq |S|$, a contradiction.

If $t = 5$ and $m = 2$, then $|S| \leq 2m + \lfloor \frac{m+t}{3} \rfloor = 6 < 7 = 3m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor \leq |S|$, a contradiction.

If $t = 5$ and $m = 3, 4$, then $|S| \leq 2m + \lfloor \frac{m+t}{3} \rfloor \leq 2m + 3 < 3m + 1 = 3m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor \leq |S|$, a contradiction.

If $t = 6$, then $|S| \leq 2m + \lfloor \frac{m+t}{3} \rfloor = 2m + 2 + \lfloor \frac{m}{3} \rfloor < 3m + 2 = 3m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor \leq |S|$, a contradiction.

Hence $S \cap \{v_j : 0 \leq j \leq n-1\} \neq \emptyset$ for $n \geq 14$. □

By Lemma 2.2 and Lemma 2.5, there exists a vertex set $V'(k,t)$ of $V(P(n,2))$ with $|V'(k,t) \cap S| \leq \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor$ for $1 \leq t \leq 6$, $n \geq 14$. We relabel the vertex v_i with v_{i+7m-k}, u_i with $u_{i+7m-k}(0 \leq i \leq n-1)$. Then $s_m \leq \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor$.

Lemma 2.6. $\rho(P(n,2)) \leq \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \geq 14)$.

Proof. Let S be an arbitrary packing of $P(n,2)$, then by Lemma 2.4, $|S| = \sum_{i=0}^{m-1} s_i + s_m \leq 3 \times m + \lfloor \frac{t+1}{7} \rfloor + \lfloor \frac{t+4}{7} \rfloor = \lfloor \frac{7m+t}{7} \rfloor + \lfloor \frac{7m+t+1}{7} \rfloor + \lfloor \frac{7m+t+4}{7} \rfloor = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor$. Hence $\rho(P(n,2)) \leq \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \geq 14)$. □

From Lemmas 2.3-2.4,2.6, we have

Theorem 2.7. $\rho(P(n,2)) = \lfloor \frac{n}{7} \rfloor + \lfloor \frac{n+1}{7} \rfloor + \lfloor \frac{n+4}{7} \rfloor (n \geq 5)$. □

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