Some formulas for numbers of line segments and lines in a rectangular grid

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Abstract

We present a formula for the number of line segments connecting $q+1$ points of an $n_1 \times \cdots \times n_k$ rectangular grid. As corollaries, we obtain formulas for the number of lines through at least q points and, respectively, through exactly q points of the grid. The well-known case $k = 2$ is so generalized. We also present recursive formulas for these numbers assuming $k = 2$, $n_1 = n_2$. The well-known case $q = 2$ is so generalized.

Keywords: Rectangular grid, lattice points, recursive formulas

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1 Introduction

Let us consider a rectangular grid

$$G(n_1, \ldots, n_k) = \{0, \ldots, n_1 - 1\} \times \cdots \times \{0, \ldots, n_k - 1\},$$

where $k, n_1, \ldots, n_k \geq 2$. Call its points *gridpoints*. Given $q \geq 2$, we say that a line is a *q-gridline* if it goes through exactly q gridpoints. We write $l_q(n_1, \ldots, n_k)$ for the number of q-gridlines, and $l_{\geq q}(n_1, \ldots, n_k)$ for the number of gridlines through at least q gridpoints. In other words, $l_{\geq q}(n_1, \ldots, n_k)$ is the sum of all $l_p(n_1, \ldots, n_k)$'s with $p \geq q$.

We also say that a line segment is a *q-gridsegment* if its endpoints and exactly $q-2$ interior points are gridpoints. Let $c_q(n_1, \ldots, n_k)$ denote the number of all q-gridsegments. (If $q > 2$, some of them may partially overlap.) In other words, $c_q(n_1, \ldots, n_k)$ is the number of line segments between such gridpoints that are visible to each other through $q-2$ gridpoints.

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Our first problem, discussed in Section 2, is to find formulas for these numbers. They are well-known [5, 6, 7] in case of $k = 2$. We will see that Mustonen's [5, 6] ideas work also in the general case.

These "explicit" formulas may have theoretical value but their practical value is small. They are computationally tedious and do not tell much about the behaviour of the functions. This motivates to look for recursive or asymptotic formulas. We do not discuss asymptotic formulas here but refer to [3] and note that we are pursuing this matter further [4].

So our second problem is to find recursive formulas. We consider only the case $k = 2$, $n_1 = n_2$ and denote $l_q(n) = l_q(n,n)$, $l_{\geq q}(n) = l_{\geq q}(n,n)$, $c_q(n) = c_q(n,n)$. Mustonen [5] conjectured and Ernvall-Hytönen et al. [3] proved recursive formulas for $l_{\geq 2}(n)$. We will in Section 3 settle this problem generally. In fact, Mustonen [6] has already done so, but, in our opinion, his results deserve publication also in a journal.

We will complete our paper with remarks in Section 4.

2 Explicit formulas

Theorem 1. *For all* $k, n_1, \ldots, n_k \geq 2$, $q \geq 1$,

$$c_{q+1}(n_1, \ldots, n_k) = \frac{1}{2} f_q(n_1, \ldots, n_k),$$

where

$$f_q(n_1, \ldots, n_k) = \sum_{\substack{-n_1 < i_1 < n_1 \\ \cdots \\ -n_k < i_k < n_k \\ \gcd(i_1, \ldots, i_k) = q}} (n_1 - |i_1|) \cdots (n_k - |i_k|).$$

Proof. Consider a line segment c whose endpoints $A = (x_1, \ldots, x_k)$ and $B = (x_1 + i_1, \ldots, x_k + i_k)$ are gridpoints. Then c is a $(q+1)$-gridsegment if and only if also the points

$$\left(x_1 + \frac{t}{q} i_1, \ldots, x_k + \frac{t}{q} i_k\right), \quad t = 1, \ldots, q-1,$$

are gridpoints and there are no other gridpoints in c. This happens if and only if $\gcd(i_1, \ldots, i_k) = q$. Fix $i_1, \ldots, i_k$ with the properties

$$\gcd(i_1, \ldots, i_k) = q \quad \text{and} \quad -n_j < i_j < n_j, \quad j = 1, \ldots, k. \tag{1}$$

We can choose x_j in $n_j - |i_j|$ ways, and so the number of all possible choices of A is $(n_1 - |i_1|) \cdots (n_k - |i_k|)$. Summing over all $i_1, \ldots, i_k$ satisfying (1), we obtain $f_q(n_1, \ldots, n_k)$. We halve it, because $-i_1, \ldots, -i_k$ lead to same gridsegments as $i_1, \ldots, i_k$. So the theorem follows. ■

Corollary 1. *Let $k, n_1, \ldots, n_k, q \geq 2$. Then*

$$l_{\geq q}(n_1, \ldots, n_k) = \frac{1}{2}(f_{q-1}(n_1, \ldots, n_k) - f_q(n_1, \ldots, n_k)).$$

Proof. Let L be the set of p-gridlines over all $p \geq q$. Take $l \in L$. It contains $p - q + 1$ q-gridsegments. On the other hand, it contains $p - q$ $(q + 1)$-gridsegments. If $N(r, l)$ stands for the number of r-gridsegments on l, we therefore have $N(q, l) - N(q + 1, l) = 1$, and so

$$\begin{aligned} l_{\geq q}(n_1, \ldots, n_k) = \sum_{l \in L} 1 &= \sum_{l \in L}(N(q, l) - N(q + 1, l)) \\ &= \sum_{l \in L} N(q, l) - \sum_{l \in L} N(q + 1, l) \\ &= c_q(n_1, \ldots, n_k) - c_{q+1}(n_1, \ldots, n_k) \\ &= \frac{1}{2} f_{q-1}(n_1, \ldots, n_k) - \frac{1}{2} f_q(n_1, \ldots, n_k). \end{aligned}$$

■

Corollary 2. *Let $k, n_1, \ldots, n_k, q \geq 2$. Then*

$$l_q(n_1, \ldots, n_k) = \frac{1}{2}(f_{q+1}(n_1, \ldots, n_k) - 2f_q(n_1, \ldots, n_k) + f_{q-1}(n_1, \ldots, n_k)).$$

Proof. Simply note that

$$l_q(n_1, \ldots, n_k) = l_{\geq q}(n_1, \ldots, n_k) - l_{\geq q+1}(n_1, \ldots, n_k).$$

■

3 Recursive formulas

Let us consider the function

$$e_q(n) = \begin{cases} \phi(\frac{n-1}{q}) & \text{if } q \mid n - 1, \\ 0 & \text{if } q \nmid n - 1, \end{cases}$$

where ϕ is the Euler totient function. We present first the following elementary lemma. If $q = 1$, then (2) is trivial and (3) is well-known (e.g., [2, Exercise 2.16], [3, Lemma 1]).

Lemma 1. *Let $n \geq 2$, $q \geq 1$. Then*

$$\sum_{\substack{i=1 \\ (i,n)=q}}^{n} 1 = e_q(n + 1). \tag{2}$$

Furthermore,

$$\sum_{\substack{i=1\\(i,n)=q}}^{n} i = \sum_{\substack{i=1\\(i,n)=q}}^{n} (n-i) = \frac{1}{2} n e_q(n+1) \tag{3}$$

if $q \neq n$, *and*

$$\sum_{\substack{i=1\\(i,n)=n}}^{n} i = n, \quad \sum_{\substack{i=1\\(i,n)=n}}^{n} (n-i) = 0. \tag{4}$$

Proof. If $q \nmid n$, then both sides of (2) and all sides of (3) are zero. The case $q \mid n$ remains; let $n = kq$. Then

$$\sum_{\substack{i=1\\(i,n)=q}}^{n} 1 = \sum_{\substack{i=1\\(i,k)=1}}^{k} 1 = \phi(k) = \phi(\frac{n}{q}) = e_q(n+1),$$

and (2) follows. To show (3), assume $k > 1$. Because

$$\sum_{\substack{i=1\\(i,n)=q}}^{n} i = \sum_{\substack{i=1\\(n+1-i,n)=q}}^{n} (n+1-i) = \sum_{\substack{i=0\\(n-i,n)=q}}^{n-1} (n-i) = \sum_{\substack{i=1\\(i,n)=q}}^{n} (n-i),$$

the first equality holds. Hence, by (2),

$$2\sum_{\substack{i=1\\(i,n)=q}}^{n} i = \sum_{\substack{i=1\\(i,n)=q}}^{n} i + \sum_{\substack{i=1\\(i,n)=q}}^{n} (n-i) = \sum_{\substack{i=1\\(i,n)=q}}^{n} n = n \sum_{\substack{i=1\\(i,n)=q}}^{n} 1 = n e_q(n+1),$$

and so also the second holds. The claim (4) is trivial. ■

Theorem 2. *Let* $n \geq 2$, $q \geq 1$. *Then*

$$f_q(n) = 2f_q(n-1,n) - f_q(n-1) + r_q(n), \tag{5}$$
$$f_q(n-1,n) = 2f_q(n-1) - f_q(n-2,n-1) + s_q(n). \tag{6}$$

Here $f_q(n) = f_q(n,n)$,

$$r_q(n) = 8(e_q(2) + e_q(3) + \cdots + e_q(n))$$

and so

$$r_q(n) - r_q(n-1) = 8e_q(n),$$

and

$$s_q(n) = 2(n-1)e_q(n).$$

The initial values are $f_q(n) = f_q(n-1,n) = 0$ *for* $n \leq q$.

Proof. If $q > n-1$, then everything is zero. So we assume $q \leq n-1$.

Case 1. $q < n-1$. Since

$$\begin{aligned} f_q(n) &= 4\Big[n(n-q) + \sum_{\substack{0<i,j<n\\(i,j)=q}} (n-i)(n-j)\Big] \\ &= 4\Big[n^2 - qn + \sum_{\substack{0<i,j<n-1\\(i,j)=q}} (n-i)(n-j) + 2\sum_{\substack{i=1\\(i,n-1)=q}}^{n-2} (n-i)\Big], \end{aligned} \tag{7}$$

$$\begin{aligned} &f_q(n-1,n) \\ &\quad= 2\Big[(n-1)(n-q) + (n-1-q)n + 2\sum_{\substack{0<i<n-1\\0<j<n\\(i,j)=q}} (n-1-i)(n-j)\Big] \\ &\quad= 2\Big[2n^2 - 2(q+1)n + q + 2\sum_{\substack{0<i,j<n-1\\(i,j)=q}} (n-1-i)(n-j) \\ &\qquad\qquad + 2\sum_{\substack{i=1\\(i,n-1)=q}}^{n-2} (n-1-i)\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} &f_q(n-1) \\ &\quad= 4\Big[(n-1)(n-1-q) + \sum_{\substack{0<i,j<n-1\\(i,j)=q}} (n-1-i)(n-1-j)\Big] \\ &\quad= 4\Big[n^2 - (q+2)n + q + 1 + \sum_{\substack{0<i,j<n-1\\(i,j)=q}} (n-1-i)(n-1-j)\Big], \end{aligned} \tag{9}$$

and

$$n^2 - qn - [2n^2 - 2(q+1)n + q] + n^2 - (q+2)n + q + 1 = 1,$$
$$(n-i)(n-j) - 2(n-1-i)(n-j) + (n-1-i)(n-1-j) = 1 + i - j,$$
$$n - i - (n-1-i) = 1,$$
$$\sum_{\substack{0<i,j<n-1\\(i,j)=q}} (1+i-j) = \sum_{\substack{0<i,j<n-1\\(i,j)=q}} 1 = 2\sum_{i=1}^{n-2}\sum_{\substack{j=1\\(i,j)=q}}^{i} 1 - 1,$$

we have, by (7), (8), (9) and (2),

$$
\begin{aligned}
&f_q(n) - 2f_q(n-1,n) + f_q(n-1) \\
&\qquad = 4 + 4\Big(2\sum_{i=1}^{n-2}\sum_{\substack{j=1\\(i,j)=q}}^{i} 1 - 1\Big) + 8\sum_{\substack{i=1\\(i,n-1)=q}}^{n-2} 1 \\
&\qquad = 8\sum_{i=1}^{n-2}\sum_{\substack{j=1\\(i,j)=q}}^{i} 1 + 8\sum_{\substack{i=1\\(i,n-1)=q}}^{n-2} 1 = 8\sum_{i=1}^{n-2}\sum_{\substack{j=1\\(i,j)=q}}^{i} 1 + 8\sum_{\substack{i=1\\(i,n-1)=q}}^{n-1} 1 \\
&\qquad = 8\sum_{i=1}^{n-2} e_q(i+1) + 8e_q(n) = 8\sum_{i=2}^{n} e_q(i) = r_q(n),
\end{aligned}
$$

and (5) follows.

To show (6), we start from (8), (9), and the fact that

$$
\begin{aligned}
f_q(n-2,n-1) &= 2\Big[(n-2)(n-1-q) + (n-2-q)(n-1) \\
&\qquad\qquad + 2\sum_{\substack{0<i<n-2\\0<j<n-1\\(i,j)=q}} (n-2-i)(n-1-j)\Big] \\
&= 2\Big[2n^2 - 2(q+3)n + 3q + 4 + 2\sum_{\substack{0<i,j<n-1\\(i,j)=q}} (n-2-i)(n-1-j)\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
&2n^2 - 2(q+1)n + q - 4[n^2 - (q+2)n + q + 1] \\
&\qquad\qquad + 2n^2 - 2(q+3)n + 3q + 4 = 0, \\
&(n-1-i)(n-j) - 2(n-1-i)(n-1-j) + (n-2-i)(n-1-j) = j - i,
\end{aligned}
$$

we have

$$
\begin{aligned}
&f_q(n-1,n) - 2f_q(n-1) + f_q(n-2,n-1) \\
&\qquad = 4\sum_{\substack{0<i,j<n-1\\(i,j)=q}} (j-i) + 4\sum_{\substack{i=1\\(i,n-1)=q}}^{n-1} (n-1-i) \\
&\qquad = 0 + 4\sum_{\substack{i=1\\(i,n-1)=q}}^{n-1} (n-1-i) = 2(n-1)e_q(n) = s_q(n).
\end{aligned}
$$

Case 2. $q = n-1$. Simply note that

$$\begin{aligned} f_{n-1}(n) - 2f_{n-1}(n-1,n) + f_{n-1}(n-1) \\ &= 4(n+1) - 2\cdot 2(n-1) + 0 \\ &= 8 = 8e_{n-1}(n) = 8\sum_{i=2}^{n} e_{n-1}(i) = r_{n-1}(n), \end{aligned}$$

$$\begin{aligned} f_{n-1}(n-1,n) - 2f_{n-1}(n-1) + f_{n-1}(n-2,n-1) \\ &= 2(n-1) - 0 + 0 = 2(n-1)e_{n-1}(n) = s_{n-1}(n). \end{aligned}$$

■

Corollary 3. *Let $n, q \geq 2$. Then*

$$l_{\geq q}(n) = 2l_{\geq q}(n-1,n) - l_{\geq q}(n-1) + \rho_{\geq q}(n), \tag{10}$$

$$l_{\geq q}(n-1,n) = 2l_{\geq q}(n-1) - l_{\geq q}(n-2,n-1) + \sigma_{\geq q}(n). \tag{11}$$

Here

$$\rho_{\geq q}(n) = 4\sum_{i=2}^{n}(e_{q-1}(i) - e_q(i))$$

and so

$$\rho_{\geq q}(n) - \rho_{\geq q}(n-1) = 4(e_{q-1}(n) - e_q(n)),$$

and

$$\sigma_{\geq q}(n) = (n-1)(e_{q-1}(n) - e_q(n)).$$

The initial values are $l_{\geq q}(n) = l_{\geq q}(n-1,n) = 0$ for $n < q$, and $l_{\geq q}(q-1,q) = q-1$.

Proof. Recall Corollary 1. ■

Corollary 4. *Let $n, q \geq 2$. Then*

$$l_q(n) = 2l_q(n-1,n) - l_q(n-1) + \rho_q(n), \tag{12}$$

$$l_q(n-1,n) = 2l_q(n-1) - l_q(n-2,n-1) + \sigma_q(n). \tag{13}$$

Here

$$\rho_q(n) = 4\sum_{i=2}^{n}(e_{q-1}(i) - 2e_q(i) + e_{q+1}(i))$$

and so

$$\rho_q(n) - \rho_q(n-1) = 4(e_{q-1}(n) - 2e_q(n) + e_{q+1}(n)),$$

and

$$\sigma_q(n) = (n-1)(e_{q-1}(n) - 2e_q(n) + e_{q+1}(n)).$$

The initial values are $l_q(n) = l_q(n-1,n) = 0$ for $n < q$, and $l_q(q-1,q) = q-1$.

Proof. Recall Corollary 2. ∎

Finally, we tie (5) and (6) together to obtain a single recursive formula for $f_q(n)$ only, and join similarly (10) with (11), and (12) with (13).

Theorem 3. *Let $n, q \geq 2$. Then*

$$f_q(n) = f_q(n-1) + 2\sum_{i=1}^{n} s_q(i) + 2\sum_{i=1}^{n-1} r_q(i) + r_q(n), \tag{14}$$

$$l_{\geq q}(n) = l_{\geq q}(n-1) + 2\sum_{i=1}^{n} \sigma_{\geq q}(i) + 2\sum_{i=1}^{n-1} \rho_{\geq q}(i) + \rho_{\geq q}(n),$$

$$l_q(n) = l_q(n-1) + 2\sum_{i=1}^{n} \sigma_q(i) + 2\sum_{i=1}^{n-1} \rho_q(i) + \rho_q(n).$$

Proof. The formula for $l_{\geq 2}(n)$ has already been proved [3, Theorem 2]. The same proof applies to all the above claims. ∎

Because $c_{q+1}(n) = \frac{1}{2} f_q(n)$ by Theorem 1, a trivial modification of (5), (6) and (14) gives the corresponding formulas also for $c_q(n)$.

4 Remarks

Remark 1. A function d, defined on $G(n_1, n_2)$, is a (two-dimensional) threshold function if it takes two values 0 and 1 and if there is a line $a_1x_1 + a_2x_2 + b = 0$ separating $d^{-1}(\{0\})$ and $d^{-1}(\{1\})$ (i.e., $d(x_1, x_2) = 0 \Leftrightarrow a_1x_1 + a_2x_2 + b \leq 0$). Let $t(n_1, n_2)$ denote the number of such functions. Alekseyev [1, Theorem 3] proved (with different notation) that $t(n_1, n_2) = f_1(n_1, n_2) + 2$. So Theorems 2 and 3 also imply recursive formulas for $t(n) = t(n, n)$.

Remark 2. If $k = 2$ but $n_1 \neq n_2$, the question about recursive formulas is more difficult. Mustonen [5] conjectured a recursive formula for $l_{\geq 2}(n_1, n_2)$. If $k \geq 3$, this question remains open even in case of $n_1 = \cdots = n_k$.

References

[1] M. A. Alekseyev, On the number of two-dimensional threshold functions. *SIAM J. Discrete Math.* *24* (2010), 1617–1631.

[2] T. M. Apostol, *Introduction to Analytic Number Theory.* Third printing. Springer, 1986.

[3] A. M. Ernvall-Hytönen, K. Matomäki, P. Haukkanen, J. K. Merikoski, Formulas for the number of gridlines. *Monatsh. Math. 164* (2011), 157–170.

[4] P. Haukkanen and J. K. Merikoski, unfinished manuscript.

[5] S. Mustonen, On lines and their intersection points in a rectangular grid of points. `http://www.survo.fi/papers/PointsInGrid.pdf`

[6] S. Mustonen, On lines through a given number of points in a rectangular grid of points. `http://www.survo.fi/papers/LinesInGrid2.pdf`

[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. `http://www.research.att.com/~njas/sequences/`