On Vertex Matching Polynomial of Graphs

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Abstract

Let G be a graph with n vertices. The vertex matching polynomial $M_v(G,x)$ of the graph G is defined as the sum of $(-1)^r q_v(G,r)x^{n-r}$ in which $q_v(G,r)$ is the number of $r-$ vertex independent sets. In this paper, we extend some important properties of the matching polynomial to the vertex matching polynomial $M_v(G,x)$. The matching and vertex matching polynomials of some important class of graphs and some applications in nanostructures are presented.

Keywords: Matching polynomial, vertex matching polynomial.

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1 Introduction

Throughout this paper G is a simple graph, that is, G does not have loops or multiple edges. Let $\{1, ..., n\}$ be the set of vertices of G and $d_i = deg(i)$ denotes the degree of the vertex i. An edge set A is called independent if there is no vertex in common between any two edges A. Also if this set has r elements we call $r-$edge set to be independent. Matching polynomial of graph G is defined by the sum of $(-1)^r q(G,r)x^{n-2r}$ in which $q(G,r)$ is the number of $r-$edge independent set of G, see [6].

A vertex set B is called independent if there is no edges between any two vertices in B and if this set has r elements we call $r-$vertex independent set. A subset X from $V(G)$ is called the maximal independent set of $V(G)$ if and only if $X \subseteq B \subset V(G)$ and B is independent set then $X = B$. Suppose $q_v(G,r)$ is the number of $r-$vertex independent set of G. In this case, $\sum_{r\geq 0} q_v(G,r)$ is called Fibonacci number $(F(G))$ of G [9]. Then we define vertex matching polynomial of G by the sum of $(-1)^r q_v(G,r)x^{n-r}$. We denote this polynomial by $M_v(G,x)$ which $(-1)^n M_v(G,-1)$ equal $F(G)$. This is an extension of the matching polynomial of graph [8]. In this paper, some basic properties of this new matching polynomial of graphs are investigated. We now introduce some notation which will be kept throughout. An empty graph is a graph without edges and the complement of a graph G is denoted by $\bar{G}$, where $e \in E(\bar{G})$ if and only if $e \notin E(G)$. It is easy to see that $\bar{K}_n$ is an empty graph with exactly n vertices. If G is a graph then $G - \cup_{i=1}^{k} u_i$ is obtained from G by omitting vertices $u_i, 1 \leq i \leq k$.

Suppose G and H are two graphs with disjoint vertex and edge sets. The disjoint union of G and H is a graph T such that $V(T) = V(G) \cup V(H)$ and $E(T) = E(G) \cup E(H)$. The join of G and H is a graph $G + H$ such that $V(T) = V(G) \cup V(H)$ and $E(T) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. The line graph G is denoted by $L(G)$ in which $V(L(G)) = E(G)$ and $E(L(G)) = \{e_i e_j | e_i \in E(G), e_j \in E(G), e_i \cap e_j \neq \emptyset\}$.

The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. Throughout this paper our notation is standard and taken mainly from [1-5, 7, 10, 11].

2 The vertex matching polynomial

This section is concerned with the use of algebraic techniques in the study of vertex matching polynomial of graphs. Some results of this polynomial are the same as matching polynomial.

Theorem 1. Let G be a (n, m)-graph, $n = 2m$ and $L(G)$ is the line graph of G. Then $M_v(L(G), x^2) = M(G, x)$

Proof. Obviously, $q_v(L(G), r) = q(G, r)$, since e_1 and e_2 are independent in $E(G)$ if and only if e_1 and e_2 are independent in $V(L(G))$. Therefore $M(G, x) = \sum_{r \geq 0} (-1)^r q(G, r) x^{n-2r} = \sum_{r \geq 0} (-1)^r q_v(L(G), r) x^{2m-2r} = M_v(L(G), x^2)$. □

Theorem 2. Let G be a graph with exactly n vertices. If

$M_1 = M_v(G,x)$ and $M_2 = M_v(G-e,x)$ then we have:

$$\begin{aligned}(a)M_1 &= xM_v(G-u,x) - x^{d(u)}M_v(G-\{u,\cup_{i=1}^{d(u)}u_i\},x)\\ (b)M_2 &= x^2M_v(G-u-v,x) + x^{d(u)+1}M_v(G-\{u,\cup_{i=1}^{d(u)}u_i\},x)\\ &+ x^{d(v)+1}M_v(G-\{v,\cup_{i=1}^{d(v)}v_i\},x) + x^{d(v)+d(u)-1}M_v(G',x)\end{aligned}$$

where $u,v \in V(G)$, $e = uv$, $G' = G - \{\cup_{i=1}^{d(v)}v_i, \cup_{i=1}^{d(u)}u_i\}$ and $uu_i, vv_i \in E(G)$.

Proof. a) To prove the theorem, we first show that

$$q_v(G,r) = q_v(G-u,r) + q_v(G-\{u,\cup_{i=1}^{d(u)}u_i\},r-1). \qquad (1)$$

Suppose that X is a r-vertex independent set. Then $q_v(G-u,r)$ is the number of r-vertex independent set that are not contain u. Also $q_v(G-\{u,\cup_{i=1}^{d(u)}u_i\},r-1)$ is the number of r-vertex independent set containing u. Combine these relations to prove (1). By $T = q(G-\{u,\cup_{i=1}^{d(u)}u_i\},r-1)$ and we have

$$\begin{aligned}M_1 &= \sum_{r\geq 0}(-1)^r q(G,r)x^{n-r}\\ &= \sum_{r\geq 0}(-1)^r x^{n-r}q(G-u,r) + \sum_{r\geq 0}(-1)^r x^{n-r}T\\ &= x\sum_{r\geq 0}(-1)^r x^{n-1-r}q(G-u,r) - x^{d(u)}\sum_{r\geq 0}(-1)^r x^{n-d(u)-r}T,\end{aligned}$$

which completes the prove.

b) Consider three cases for u and v. If A contains all of $r-$ vertex independent set containing u and v then we have $|A| = q_v(G-\{\cup_{i=1}^{d(u)}u_i, \cup_{i=1}^{d(v)}v_i\},r-2)$. Suppose B is the set of an $r-$ vertex independent set which is not contain u and v. Then $|B| = q_v(G-\{u,v\},r)$. We now assume C contains all of $r-$ vertex independent set which is containing one of u or v. Then

$$|C| = q_v(G - \{u, \cup_{i=1}^{d(u)} u_i\}, r-1) + q_v(G - \{v, \cup_{i=1}^{d(v)} v_i\}, r-1).$$

Therefore $q_v(G-e, r) = |A| + |B| + |C|$ and we have

$$\begin{aligned} M_2 &= \sum_{r\geq 0}(-1)^r q_v(G-e,r)x^{n-r} = x^{d(u)+d(v)-1}M_v(G',x) \\ &+ x^2 M_v(G-\{u,v\},x) + x^{d(u)+1}M_v(G-\{\cup_{i=1}^{d(u)}u_i, u\},x) \\ &+ x^{d(v)+1}M_v(G-\{\cup_{i=1}^{d(v)}v_i, v\},x). \end{aligned}$$

This completes the proof. □

Lemma 3. The quantity $q_v(G,r)$ is the number of complete subgraphs G with r vertices.

Proof. A subset A of G is independent if and only if $[A]$ is a complete subgraph of $\bar{G}$.

Corollary 4. The vertex matching polynomial of P_n, $n \geq 3$, satisfies the following equation $M_v(P_1, x) = x - 1, M_v(P_2, x) = x^2 - 2x$ and $M_v(P_n, x) = x[M_v(P_{n-1}, x) - M_v(P_{n-2}, x)]$.

Apply Theorem 1(a), to compute $q_v(P_n, r)$. One easily can see that $q_v(P_n, 0) = 1, q_v(P_n, 1) = n$ and $q_v(P_n, 2) = |E(\bar{G}| = \frac{(n-1)(n-2)}{2} = C(n-1, 2)$. Suppose $r = 3$. Then from eq.(1),

$$\begin{aligned} q_v(P_n, 3) &- q_v(P_{n-1}, 3) = q_v(P_{n-2}, 2) = \frac{(n-3)(n-4)}{2}, \\ q_v(P_m, 3) &= \sum_{n=5}^{m} q_v(P_n, 3) - q_v(P_{n-1}, 3) = C(n-2, 3), \\ q_v(P_m, 4) &= C(m-3, 4), \quad q_v(P_m, 5) = C(m-4, 5). \end{aligned}$$

Therefore by induction $q_v(P_n, r) = C(n-r+1, r), 0 \leq r \leq [\frac{n+1}{2}]$. Now we compute $M_v(P_n, x)$. With above computation $M_v(P_n, x) = \sum_{r=0}^{[\frac{n+1}{2}]}(-1)^r C(n-r+1, r)x^{n-r}$.

3 Polynomial of Join and Union

In this section, some results related to vertex matching polynomial of join and union of two graphs are proved. We also compute this polynomial for some well-known graphs.

Theorem 5. Suppose $G_i, 1 \leq i \leq k$ are graphs. Then
a) $\frac{M_v(\sum_{i=1}^k G_i, x)}{x^{\sum_{i=1}^k |G_i|}} = \sum_{i=1}^k \frac{M_v(G_i,x)}{x^{|G_i|}} - (n-1)$. In particular, $M_v(kG, x) = kx^{(k-1)|G|}M_v(G, x) - (n-1)x^{k|G|}$
b) $M(G+H, x) = x^{|H|}M(G, x) + x^{|G|}M(H, x) + M(K_{|G|,|H|}, x) - 2x^{|G|+|H|}$.

Proof. a) Suppose G and H are two graphs and A is a $r-$vertex independent set of $V(G+H)$. It is easy to see that $A \subseteq V(G)$ or $A \subseteq V(H)$. Thus $q_v(G+H, r) = q_v(G, r) + q_v(H, r), r = 1, 2, ..., |G| + |H|$ and $q_v(G+H, 0) = 1$. Therefore

$$\begin{aligned} M_v(G+H, x) &= \sum_{r=0}^{|G|+|H|}(-1)^r q_v(G+H, r)x^{|G|+|H|-r} \\ &= x^{|G|+|H|} + x^{|H|}\sum_{r=1}^{|G|}(-1)^r q_v(G, r)x^{|G|-r} \\ &+ x^{|G|}\sum_{r=1}^{|H|}(-1)^r q_v(H, r)x^{|H|-r} \\ &= x^{|H|}\sum_{r=0}^{|G|}(-1)^r q_v(G, r)x^{|G|-r} + x^{|G|}\sum_{r=0}^{|H|}(-1)^r q_v(H, r)x^{|H|-r} \\ &- x^{|G|+|H|} = x^{|H|}M_v(G, x) + x^{|G|}M_v(H, x) - x^{|G|+|H|}. \end{aligned}$$

Thus $\frac{M_v(G+H,x)}{x^{|G|+|H|}} = \frac{M_v(G,x)}{x^{|G|}} + \frac{M_v(H,x)}{x^{|H|}} - 1$ and by induction the

assertion is proved .
b) The same argument as part (a) shows that $q(G+H,r) = q(G,r)+q(H,r)+q(K_{|G|,|H|},r), r = 1,2,...,|G|+|H|$ and $q_v(G+H,0) = 1$. Thus $M(G+H,x) = x^{|H|}M(G,x) + x^{|G|}M(H,x) + M(K_{|G|,|H|},x) - 2x^{|G|+|H|}$. This completes the proof. □

Corollary 6. The following results on proper graphs are hold $M_v(K_n,x) = nx^{(n-1)}(x-1)-(n-1)x^n$ and $M_v(\bar{K}_n,x) = (x-1)^n$.

Proof. We notice that $q_v(\bar{K}_n,r) = C(n,r), K_n = n\bar{K}_1$. Then the previous theorem, completes the proof. □

Theorem 7. Suppose $G_i, 1 \leq i \leq k$ are graphs. Then we have $M_v(\cup_{i=1}^{k} G_i,x) = \prod_{i=1}^{k} M_v(G_i,x)$.

Proof. By $q_v(G \cup H,r) = \sum_{k=0}^{r} q_v(G,k)q_v(H,r-k)$ and definition the proof is straightforward. □

Theorem 8. The vertex matching polynomial of the Hamming graph $H_{m,n}$ are computed as follows: $\sum_{r\geq 0}(-1)^r C(m,r)P(n,r)x^{mn-r}$.

Proof. it is well-known that $H_{m,n} \cong K_m \times K_n$ and

$$\begin{aligned} q_v(K_m \times K_n, r) &= \frac{mn(m-1)(n-1)...(m-r+1)(n-r+1)}{r!} \\ &= \frac{P(m,r)P(n,r)}{r!}, r \geq 1. \end{aligned}$$

Thus $M_V(K_m \times K_n, x) == \sum_{r\geq 0}(-1)^r C(m,r)P(n,r)x^{mn-r}$. □

4 Applications to Nanostructures

In this section, we obtain the maximal and maximum vertex independent set of some well-known graphs. For example, the

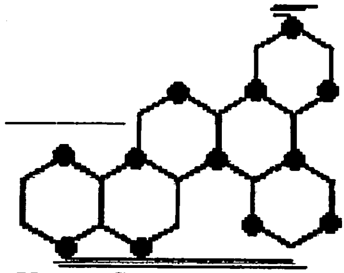

Figure 1: The Independent Vertex Set with Maximum Number of Vertices.

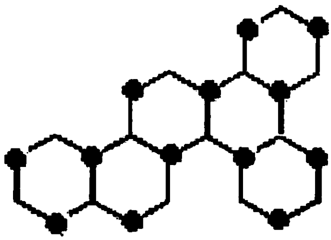

Figure 2: The Independent Vertex Set without Maximum Number of Vertices.

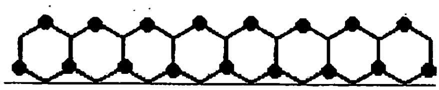

Figure 3:

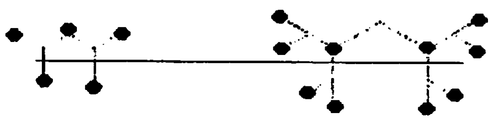

Figure 4: The Maximum Independent Vertex Set of Dendrimers.

path P_n, the hexagonal graph and some denderimers.

Example 9. Suppose $M_v(P_n)$ is the number of maximal vertex independent set of P_n. Then $M_v(P_1) = 1, M_v(P_2) = 2, M_v(P_3) = 2, M_v(P_n) = M_v(P_{n-2}) + M_v(P_{n-3})$.

Theorem 10. Let G be a catacondensed benzenoid graph[10]. Then the size of the maximum vertex independent set of G is $2n + 1$, where n is the number of hexagons. In particular if G

is a catacondensed benzenoid chain then

$$\begin{aligned} M_v(G,x) &= x^{|V(G)|} - |V(G)|x^{|V(G)|-1} \\ &+ |E(\bar{G})|x^{|V(G)|-2} + ... + 2x^{|V(G)|-2n-1}. \end{aligned}$$

Proof. consider Figure 1 and assume that the independent vertices of G are the form of Figure 1. Then the number of these independent vertices are $2n$. But if these vertices are the form of Figur2. In other words, one of two vertices of common edge is choosed . Then the number of these vertices are $2n+1$. Therefore maximum is $2n+1$. Now we compute $q_v(G, 2n+1)$. If G is a catacondensd chain by n Hexagoe. Thus $q_v(G, 2n+1) = 2$ and $q_v(G,x) = x^{|V(G)|} - |V(G)|x^{|V(G)|-1} + |E(\bar{G})|x^{|V(G)|-2} + ... - 2x^{|V(G)|-2n-1}$. □

Example 11. If $n = 2^p - 1$. Then Figure 4 shows size of maximum vertex independent set in this denderimers are $\frac{2^{p+1}-1}{3}$ or $2\frac{2^p-1}{3}$ for $p = 2k-1$ or $p = 2k$, respectively, see Figure 4 for details.

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