Unit distance representations of the Petersen graph in the plane

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Abstract

The well-known Petersen graph $G(5, 2)$ admits drawings in the ordinary Euclidean plane in such a way that each edge is represented as a line segment of length 1. When two vertices are drawn as the same point in the Euclidean plane, drawings are said to be degenerate. In this paper we investigate all such degenerate drawings of the Petersen graph and various relationships among them. A heavily degenerate unit distance planar representation, where the representation of a vertex lies in the interior of the representation of an edge, it does not belong to, is also shown.

Keywords: Petersen graph; unit distance graphs; graph representation; degenerate representation; planar representation; graph coloring; rigidity;
Math. Subj. Class.: 05C62, 05C12, 05C15, 05C60, 05C10

1 Introduction

Let P be the well known *Petersen graph* $G(5, 2)$, see for instance [13]. For simplicity the vertices of the Petersen graph are labeled by natural numbers as in Figure 1.

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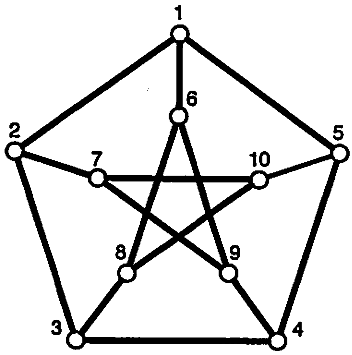


Figure 1: The vertices of the Petersen graph P are labeled as shown.

Define a *representation* ρ of a graph G in a set M as a map ρ_V from $V(G)$ into M and a map ρ_E from $E(G)$ into 2^M, such that if v is an end-vertex of an edge $e = u \sim v$, then $\rho_V(v) \in \rho_E(e)$ (and $\rho_V(u) \in \rho_E(e)$), see [5]. If the converse is true, namely, that for $v \in V(G)$ and $e \in E(G)$, if $\rho_V(v) \in \rho_E(e)$ is true, then v is an end-vertex of $e = u \sim v$, the representation is called a *realization*. If there is no danger of confusion we drop the subscripts and denote both mappings ρ_V and ρ_E by ρ. Observe, that representations of edges of G may cross, which is different than in the usual embedding case.

In this paper we consider only representations in the Euclidean n-space, i.e. (*Euclidean*) representations that have $M = \mathbb{R}^n$ and where edge $e = u \sim v$ is represented by the line segment between the representations of its end-vertices. We will mainly consider representations in the Euclidean plane, which will be called *planar*, see [5, 12]. All graphs in this paper are assumed to be simple, having no loops and no multiple edges.

If the mapping ρ is not injective on $V(H)$, where $H \subseteq G$ is a subgraph of a graph G, the representation is called *degenerate on H.* If the mapping ρ is not injective on $V(G)$, it is called *degenerate.* A representation is called an *immersion* if representations of any two edges share only a finite number of points. In an Euclidean immersion two edges share at most one point and an Euclidean realization of a graph is always an immersion.

An Euclidean representation is called a *unit distance representation* if each edge is represented as a line segment of length 1. A graph is called a *unit distance graph* if it has a unit distance planar realization. Again, edges of unit distance graphs are allowed to cross, thus not every unit distance graph is a planar graph. For example, the well known Petersen graph is a unit distance graph, as can be seen in Figure 6(b).

Since each Euclidean realization of a connected graph on at least three vertices is non-degenerate, non-degenerate representations that are not realizations have the property that a representation of a vertex lies in the interior of a representation of an edge. Such degeneracies are sometimes called *non-simplicial* or *heavily degenerate* representations and are not considered in this paper (e.g., see Figure 8). There are even cases, where non-simplicial representations are not immersions.

If all continuous motions of the points of the representation ρ maintain the distances given by edge constraints, the representation is called *rigid* [3]. A representation that is not rigid, is called *flexible*. The number of independent coordinates required to define the position and orientation of an object $\mathcal{O}$ is called the *degree of freedom* (a.k.a. *dof*, $\mathcal{D}_{\mathcal{O}}$) [8]. For a planar representation $\mathcal{M}$, the degree of freedom is given by Gruebler's Equation, see [7]. Graphs with pre-described edge length contain only *revolute* joints, which are joints that allow only one rotational freedom. Thus, for a graph G with pre-described edge length, the degree of freedom is given by equation $\mathcal{D}_G = 3(n-1) - 2f$, where $f = \sum_{v \in V(G)} d(v) - 1$ and $d(u)$ is a degree of a vertex $u \in V(G)$. Since the Petersen graph has 15 edges and 10 vertices with degree 3, $\mathcal{D}_P = 3(15-1) - 2(10(3-1)) = 2$.

Sometimes a representation can be over-constrained [7]. For example, the Wheel graph W_7, is rigid in the plane and contains one redundant constraint. One can remove any edge that connects the hub vertex with the cycle, and the graph $G^*_{21} := W_7 - r$, that is obtained and can be seen in Figure 3(b), remains rigid in the plane. Application of Gruebler's equation on the graph W_7 would yield a value of $\mathcal{D}_{W_7}$ less than zero.

2 Degenerate representations of a graph

A mapping $f : V(G) \to V(H)$ from a graph G into a graph H is called a *graph morphism* or a *graph homomorphism*, if f maps vertices of G into vertices of H, such that $u \sim v \in E(G)$ implies $f(u) \sim f(v) \in E(H)$ (f preserves adjacencies). Each graph morphism $f : V(G) \to V(H)$ induces a unique mapping (denoted by the same letter) $f : E(G) \to E(H)$, such that for $e = u \sim v$, $f(e) = f(u) \sim f(v)$. A homomorphism that is bijective both on $V(G)$ and $E(G)$, is called a *graph isomorphism* (see e.g. [6]). We say that an isomorphism of a graph G onto itself is a *graph automorphism*.

Each graph morphism $f : G \to H$, surjective on $V(G)$, defines a vertex set partition, where two vertices u and v are equivalent iff $f(u) = f(v)$. Since the image graph H is simple, with no loops, the converse is almost true, see [6].

Theorem 2.1. *A vertex set partition Θ of $V(G)$ arises from a graph morphism, surjective on $V(G)$, if and only if Θ defines a proper vertex coloring.*

Proof. Let Θ be a vertex coloring of a graph G, and let K_ℓ denote the complete graph on vertices $v_1, \ldots, v_\ell$. If Θ is a proper $|\Theta|$-coloring, then we can map each color class to one vertex of $K_{|\Theta|}$, thus there exist the bijective mapping $f : \Theta \to K_{|\Theta|}$, such that $f(V_i) = v_i$, $1 \le i \le |\Theta|$. The mapping $g : G \to K_{|\Theta|}$ defined with $g(u) = j$ iff $u \in f^{-1}(v_j)$ is well defined graph morphism and since every part of Θ is non-empty, g is surjective on $V(G)$. Conversely, if the partition does arise from a graph homomorphism, then vertices in the same partition cannot have an edge between them, since the image graph is loopless. Because the graph homomorphism is surjective on $V(G)$, Θ is a proper $|\Theta|$-coloring. □

For a graph G with a (vertex) coloring Θ, define a *colored graph* as a pair (G, Θ). We say that G is the *underlying graph* of graph (G, Θ).

The following notation and theory are used in Section 4. Each vertex set partition Θ, where $k = |\Theta|$, gives rise to the corresponding integer partition $n = n_1 + n_2 + \ldots + n_k$, where $n_1 \ge n_2 \ge \ldots \ge n_k$, $n = |V(G)|$ and the i-th part of partition Θ has cardinality n_i. We call each part of the vertex partition a *vertex class*. Further on, we will denote the integer partition $n = n_1 + n_2 + \ldots + n_k$ with $n_{1,2,\ldots,k}$.

Let G be a graph and $S \subseteq V(G)$. If for every pair of vertices $u, v \in S$ holds that $u \sim v \notin E(G)$, then we call S to be an *independent set*. Let G be a graph with n vertices whose maximal independent set has cardinality m. Then, while looking for all surjective morphisms, only integer partitions of n whose maximal element has size m have to be considered. Let $R[n, m]$ be the number of all integer partitions of integer n with the maximal addend $\le m$. As can be seen in [1], $R[n, m]$ equals to the number of partitions of n with at most m parts.

Let G be a graph. Degenerate (not necessary planar) unit distance representation $\rho : G \to M$ can be decomposed into two mappings $f : G \to H$ and $\rho^* : H \to M$, where f is a graph morphism surjective on both $V(G)$ and $E(G)$, ρ^* is (non-degenerate) unit distance realization and $\rho := \rho^* \circ f$. Thus, to find all degenerate unit distance representations of the graph G, one has to find all unit distance realizations of all non-isomorphic epimorphic (on both $V(G)$ and $E(G)$) images of G. By Theorem 2.1, in order to find all epimorphic images of the graph G, one has to find all proper vertex colorings of G.

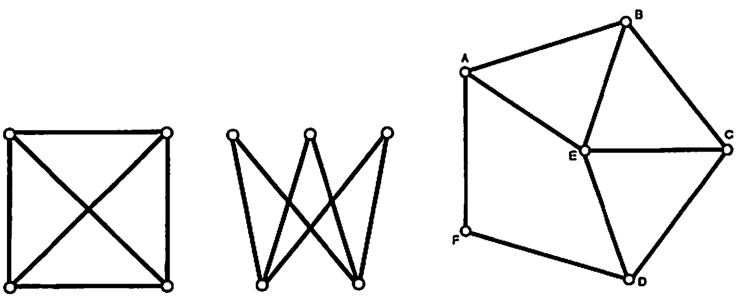
Let $G_{n_{1,2,\ldots,k}}$ denote the set of all non-isomorphic colored (labeled) graphs, that are obtained from G, while coloring it with all proper k colorings that give rise to the integer partition $n_{1,2,\ldots,k}$. Informally speaking, we can describe $G_{n_{1,2,\ldots,k}}$ as the set of all different $n_{1,2,\ldots,k}$ colorings of G. Let $G^*_{n_{1,2,\ldots,k}}$ denote the set of all non-isomorphic (non-labeled) underlying graphs of graphs from the set $G_{n_{1,2,\ldots,k}}$. Informally speaking, we can describe $G^*_{n_{1,2,\dots,k}}$ as the set of all different $n_{1,2,\dots,k}$ drawings of the graph G.

Thus, to obtain all unit distance degenerate planar representations of G, one has to find all unit distance realizations of graphs in set $G^*_{n_{1,2,\dots,k}}$ for every integer partition $n_{1,2,\dots,k}$. When coloring G with colors from the set $\{1,\dots,k\}$ and obtaining corresponding graph epimorphisms, the symmetry of G and $n_{1,2,\dots,k}$ needs to be taken into account.

3 Critical subgraphs of the Petersen graph

The property of being a unit distance graph is a hereditary property of a graph, see [4]. Hence, if a graph G contains a subgraph H which is not a unit distance graph, G too is not a unit distance graph.

Let G be a graph, $F \subseteq E(G)$ a subset of its edges and $e \in E(G)$ an edge. A graph G with edges from F removed will be denoted as $G - F := (V(G), E(G) \backslash F)$ and $G - e := G - \{e\}$. Let K_n denote the complete graph on n vertices and C_n the cycle on n vertices. The complete bipartite graph $K_{m,n}$ is a graph with two sets of vertices, one with n members and one with m members, such that each vertex in one set is adjacent to every vertex in the other set and to no vertex in its own set. Let W_i be the Wheel graph on i vertices (the skeleton of the $(i-1)$-gonal pyramid), let r be one of its spokes (a spoke is an edge that connects the hub vertex with one of the vertices on the cycle) and define $W_i^* := W_i - r$.



K_4 $K_{2,3}$ W_6^*

Figure 2: (a) The complete graph K_4. (b) The complete bipartite graph $K_{2,3}$. (c) A planar realization of the graph W_6^*.

Theorem 3.1. *If a graph G contains K_4, $K_{2,3}$ or W_6^* as a subgraph, then G is not a unit distance graph.*

Proof. The proof when G contains K_4 or $K_{2,3}$ is straight-forward and it will be left out (see e.g. [2]). All we have to prove is that W_6^* is not a unit distance graph. Label the vertices of W_6^* as in Figure 2(c). Define W_6^{**} as the induced subgraph of the graph W_6^* on the vertex set $\{A, B, C, D, E\}$. The graph W_6^{**} has only one unit distance planar realization, say ρ, which is rigid in the plane. Define a representation ρ' of the graph W_6^* in such a way, that $\rho'(v) = \rho(v)$ for each $v \in V(W_6^{**})$. Since $|\rho'(A)\rho'(D)| = 2$, there can be only one path of length 2 between points $\rho'(A)$ and $\rho'(D)$, but in the graph W_6^* there are two, $A \sim E \sim D$ and $A \sim F \sim D$. Thus, ρ' maps vertices F and E into one point $\rho'(F)$. Representation ρ' is degenerate and not an immersion, hence it is not a realization. □

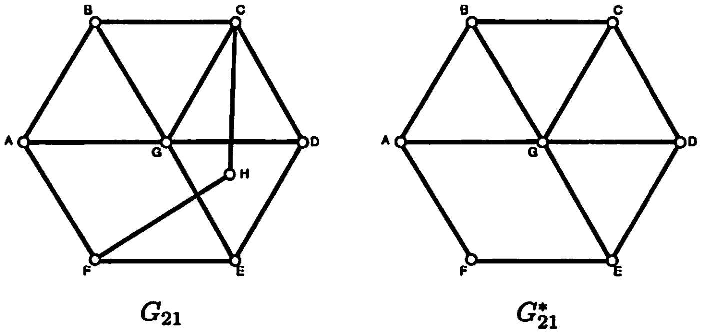


Figure 3: (a) A planar realization of the graph G_{21}. (b) A planar rigid unit distance realization of the graph G_{21}^*, defined as the induced subgraph of G_{21} on the vertex set $V(G_{21}) \setminus \{H\}$.

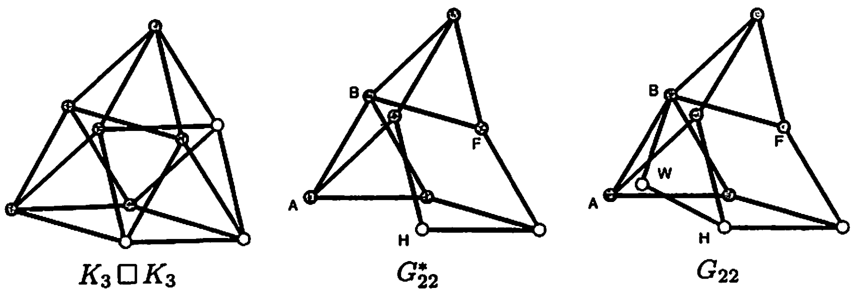
The proofs of Lemma 3.2 and Corollary 3.3 will be omitted, since they can be easily obtained from Figure 3(b).

Lemma 3.2. *Let the graph G_{21}^* be defined as in Figure 3(b). Then it has only one rigid unit distance realization in the plane, which maps vertices F and G into points that are unit distance apart.*

Corollary 3.3. *Let the graph G_{21} be defined as in Figure 3(a). A unit distance planar realization of G_{21} does not exist.*

The Cartesian product $K \square H$ of graphs K and H is the graph with vertex set $V(K \square H) = V(K) \times V(H)$ and edge set $E(K \square H) = \{(a, b) \sim (c, d) \mid (a = c \wedge b \sim d \in E(H)) \vee (a \sim c \in E(K) \wedge b = d)\}$.

Lemma 3.4. *Let the graph G_{22}^* be defined as in Figure 4(b) and let ρ be its arbitrary unit distance planar realization, if exists. Then, $|\rho(A)\rho(H)| = |\rho(H)\rho(F)| = 1$.*



$K_3 \square K_3$ $\quad$ G_{22}^* $\quad$ G_{22}

Figure 4: (a) A unit distance planar realization of the graph $K_3 \square K_3$ with one degree of freedom. The positions of grey vertices are determined by the angle between the white vertices. (b) A unit distance planar realization of the graph G_{22}^* with one degree of freedom. The positions of grey vertices are determined by the angle between the white vertices and the vertex F. (c) A planar realization of the graph G_{22} defined with the vertex set $V(G_{22}^*) \cup \{W\}$ and the edge set $E(G_{22}^*) \cup \{B \sim W, W \sim H\}$.

Proof. Let K_3 be the equilateral triangle. It is well known that the Cartesian product $K_3 \square K_3$, which can be seen on Figure 4(a), is a unit distance graph. The graph G_{22}^* is obviously a subgraph of the graph $K_3 \square K_3$, thus a unit distance graph. Let ρ be a unit distance planar realization of G_{22}^*. Denote the vertices of the graph G_{22}^* as in Figure 5. Let us to simplify the writing, until the end of this proof, write u instead of $\rho(u)$, where $u \in V(G_{22}^*)$ (e.g. we will write A instead of $\rho(A)$, etc).

Let ω represent the sixth root of unity $\frac{1+i\sqrt{3}}{2}$. Then in the complex plane, place A at the origin, B at ω, and E at 1. Let α denote the unit vector from B to F, so that F is at point $\omega + \alpha$. Since multiplying by ω corresponds to rotation by 60 degrees, C is at point $\omega + \omega\alpha$. Then D is defined to be the 4th point of rhombus $ABCD$, so that, as complex numbers, $C - B = D - A$. Solving for D, we get D $=\omega\alpha$. Similarly, let G be defined as the 4th point of rhombus $EBFG$, so that, as complex numbers, $B - E = F - D$, hence $G = 1 + \alpha$. To find the location of H, we observe that the distance between D and G, being $||D - G|| = ||(1+\alpha) - \omega\alpha|| = ||(1 + \alpha(1-\omega)|| \leq ||1|| + ||\alpha||||1 - \omega|| = 2$, implies that there are exactly two points at unit distance from both D and G, unless we have equality in the previous relation, which occurs when $\alpha = \omega$, in which case there is one point. It is then easy to verify by inspection that α and $1 + \omega\alpha$, are unit distance apart from both D and G. Thus, $H = \alpha$ or $H = 1 + \omega\alpha$. In the first case, inspection shows that H is unit distance from A and F as well, completing the proof. $\square$

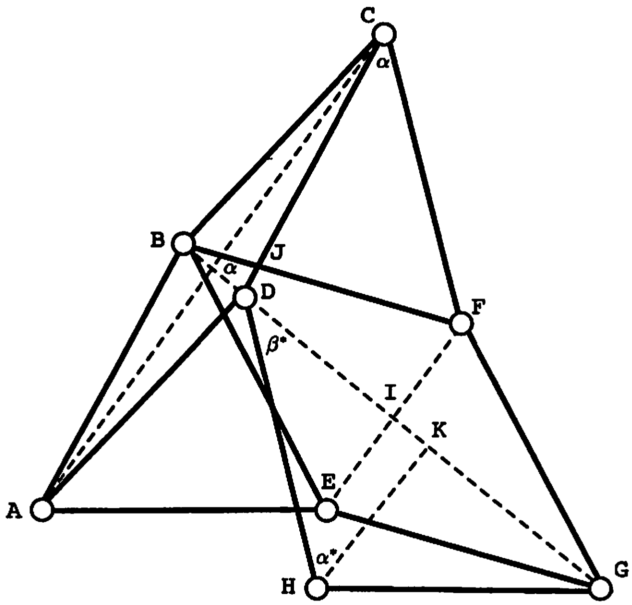


Figure 5: A planar unit distance realization of the graph G_{22}^{*} with detailed notation.

Corollary 3.5. *Let the graph G_{22} be defined as in Figure 4(c). A unit distance planar realization of G_{22} does not exist.*

Proof. To construct a planar unit distance realization of G_{22} one needs to extend a unit distance realization ρ of G_{22}^{*} with a unit distance realization of two missing edges, $\rho(B)\rho(W)$ and $\rho(W)\rho(H)$. As it has been proven in Lemma 3.4, rectangle $\rho(B)\rho(A)\rho(H)\rho(F)$ is a rhombus with all sides of equal length 1. Thus the representation of the path $B \sim W \sim H$ is of length 2 and must contain either $\rho(A)$ or $\rho(F)$. Hence, a representation of a vertex lies inside a representation of an edge it does not belong to. There is more, a representation is not even an immersion. Thus, it is not a unit distance planar realization. □

4 Unit distance planar representations of the Petersen graph

Now we turn to planar unit distance representations of the Petersen graph. For each such (possibly degenerate) planar unit distance representation there is an *underlying graph.* For instance, it is possible to represent the Petersen graph in such a way, that the underlying graph is the unit distance triangle K_3. Such a graph representation is degenerate and not an immersion, as can be seen in Figure 6(a). It is also possible to construct a planar unit distance realization of the Petersen graph, as in Figure 6(b).

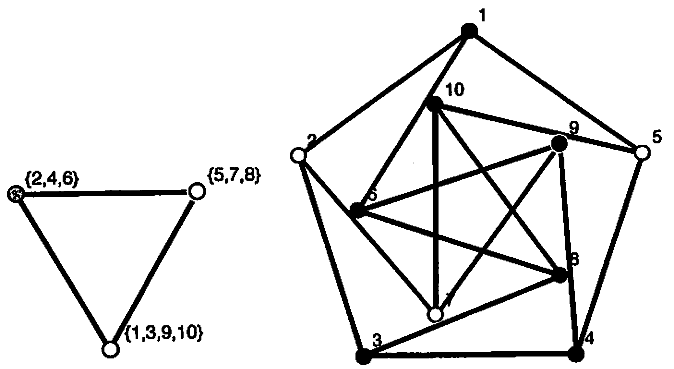


Figure 6: (a) A degenerate planar unit distance representation of the Petersen graph. (b) The Petersen graph can be drawn in the plane in a non-degenerate way with all edges of the same length.

It can be easily observed that for the Petersen graph $n = 10$, $m = 4$ and $R[10,4] = 23$. Table 1 lists all different integer partitions (that will be further on denoted as *cases*) for the Petersen graph. For the integer partition $n_{1,2,\ldots,k}$ observed in each case, we denote by:

- n_i ... the size of i-th vertex class,
- k ... the number of colors used in any of the proper k colorings that give rise to $n_{1,2,\ldots,k}$,
- a ... the number of all graphs in $G_{n_{1,2,\ldots,k}}$,
- d ... the number of all graphs in $G^*_{n_{1,2,\ldots,k}}$,
- r ... the number of all unit distance graphs in set $G^*_{n_{1,2,\ldots,k}}$.

It can be easily observed that $0 \le r \le d \le a$.

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	k	a	d	r
1	4	4	2								3	0	0	0
2	4	4	1	1							4	0	0	0
3	4	3	3								3	1	1	1
4	4	3	2	1							4	2	1	1
5	4	3	1	1	1						5	2	1	0
6	4	2	2	2							4	1	1	0
7	4	2	2	1	1						5	2	2	2
8	4	2	1	1	1	1					6	1	1	1
9	4	1	1	1	1	1	1				7	1	1	1
10	3	3	3	1							4	3	3	2
11	3	3	2	2							4	4	2	1
12	3	3	2	1	1						5	10	7	2
13	3	3	1	1	1	1					6	5	5	0
14	3	2	2	2	1						5	14	5	1
15	3	2	2	1	1	1					6	19	15	4
16	3	2	1	1	1	1	1				7	7	7	2
17	3	1	1	1	1	1	1	1			8	2	2	1
18	2	2	2	2	2						5	2	2	0
19	2	2	2	2	1	1					6	11	10	1
20	2	2	2	1	1	1	1				7	12	12	1
21	2	2	1	1	1	1	1	1			8	5	5	1
22	2	1	1	1	1	1	1	1	1		9	1	1	0
23	1	1	1	1	1	1	1	1	1	1	10	1	1	1
										Total		106	85	23

Table 1: Vertex set partitions of the Petersen graph.

We have to consider each of 23 cases separately. For each case we state all different k-colorings of the Petersen graph that give rise to integer partition $n_{1,2,\ldots,k}$ observed in that case. We state which of them have a unit distance realization in the plane, which of them are rigid in the plane and if not rigid, with how many degrees of freedom $\mathcal{D}_G$ are they realizable with. For those cases that do not have a unit distance planar realization, an obstruction (a critical subgraph) will be presented. The results will be shown in a table for each case. The totals for each case are shown in Table 1. All different unit distance planar representations of the Petersen graph are shown in Figure 10.

The computer system VEGA [10, 11, 9] was used in order to find good drawings of graphs, calculate the authomorphisms, calculate all epimorphisms and help constructing the posets of the contracted graphs.

In cases 1 through 9, size m of the maximal set of vertices in vertex partition, is 4. Thus, four vertices of the Petersen graph are colored with one color and the corresponding surjective graph morphism contracts them into one, as can be seen on graph shown in Figure 7. Due to the symmetry of the Petersen graph it has essentially only one independent set of size 4. We may take it to be $\{1, 3, 9, 10\}$.

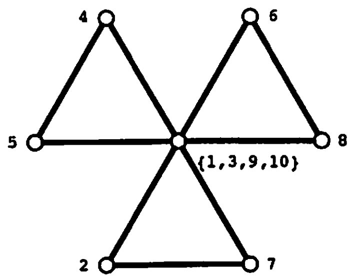


Figure 7: The Petersen graph with the vertex set $\{1, 3, 9, 10\}$ contracted. It is clearly realizable with two degrees of freedom.

Case 1. $(4, 4, 2)$ We may rule out case 1. There is no 3-vertex coloring of P with two color classes having 4 vertices and one color class having two vertices. If there would be one, we would be able to pick four pairwise non-adjacent vertices among vertices $\{2, 4, 5, 6, 7, 8\}$ in the graph in Figure 7 and color them with the same color. But we can only pick three of them.

Case 2. $(4, 4, 1, 1)$ This case is similar to case 1 and can be ruled out too.

Case 3. $(4, 3, 3)$ There is only one distinct vertex coloring of P giving rise to only one rigid equilateral triangle, as can be seen in Figure 6(a).

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
3.1.	$\{1,3,9,10\}$, $\{2,4,6\}$, $\{5,7,8\}$	yes	$G_{3.1}$	yes / 0

Table 2: Case 3. $(4, 3, 3)$

Case 4. $(4, 3, 2, 1)$ There are two distinct vertex colorings of P, both giving rise to the same rigid unit distance planar realization.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
4.1.	$\{1,3,9,10\}$, $\{2,4,6\}$, $\{5,7\}$, $\{8\}$	yes	$G_{4.1}$	yes / 0
4.2.	$\{1,3,9,10\}$, $\{2,4,8\}$, $\{5,6\}$, $\{7\}$	yes	$G_{4.1}$	yes / 0

Table 3: Case 4. $(4, 3, 2, 1)$

Case 5. $(4, 3, 1, 1, 1)$ There are two distinct vertex colorings of P, which, when applied to P, give rise to only one graph. That graph contains $K_{2,3}$ as a subgraph. Hence, a unit distance planar realization of it does not exist, although there exists a unit distance 3-space realization with two degrees of freedom.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
5.1.	{1, 3, 9, 10}, {2, 4, 6}, {5}, {7}, {8}	no	$K_{2,3}$	
5.2.	{1, 3, 9, 10}, {2, 4, 8}, {5}, {6}, {7}	no	$K_{2,3}$	

Table 4: Case 5. (4, 3, 1, 1, 1)

Case 6. (4, 2, 2, 2) There is only one distinct vertex coloring of P, which, when applied to P, gives rise to K_4. As we proved, K_4 does not have a unit distance planar realization, but has a unit distance 3-space realization, which is rigid in 3-space (a regular tetrahedron).

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
6.1.	{1, 3, 9, 10}, {2, 4}, {5, 6}, {7, 8}	no	K_4	

Table 5: Case 6. (4, 2, 2, 2)

Case 7. (4, 2, 2, 1, 1) There are two distinct vertex colorings of P. In this case, both of them have a unit distance planar realization, first is rigid and the other one realizable as a unit distance graph with one degree of freedom.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
7.1.	{1, 3, 9, 10}, {2, 4}, {5, 6}, {7}, {8}	yes	$G_{7.1}$	yes/0
7.2.	{1, 3, 9, 10}, {2, 4}, {5, 7}, {6}, {8}	yes	$G_{7.2}$	no/1

Table 6: Case 7. (4, 2, 2, 1, 1)

Case 8. (4, 2, 1, 1, 1, 1) There is only one distinct vertex coloring of P, which, when applied to P, gives rise to a graph with a unit distance planar realization, realizable with one degree of freedom.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
8.1.	{1, 3, 9, 10}, {2, 4}, {5}, {6}, {7}, {8}	yes	$G_{8.1}$	no/1

Table 7: Case 8. (4, 2, 1, 1, 1, 1)

Case 9. (4, 1, 1, 1, 1, 1, 1) Again, there is only one distinct vertex coloring of P, which, when applied to P, gives rise to a graph with a unit distance planar realization, realizable with two degrees of freedom.

Case 10. (3, 3, 3, 1) There are three distinct vertex colorings of P, which, when applied to P, give rise to three graphs. First and second have a unit distance planar realization, first is rigid and the other one realizable

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
9.1.	{1,3,9,10}, {2}, {4}, {5}, {6}, {7}, {8}	yes	$G_{9.1}$	no/2

Table 8: Case 9. (4, 1, 1, 1, 1, 1, 1)

with one degree of freedom. The third coloring, which, when applied to P, gives rise to K_4, does not have a unit distance planar realization.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
10.1.	{1,3,7}, {2,4,6}, {5,8,9}, {10}	yes	$G_{4.1}$	yes/0
10.2.	{1,3,9}, {2,4,6}, {5,7,8}, {10}	yes	$G_{10.2}$	no/1
10.3.	{1,3,10}, {2,4,6}, {5,8,9}, {7}	no	K_4	

Table 9: Case 10. (3, 3, 3, 1)

Case 11. (3, 3, 2, 2) There are four distinct vertex colorings of P, which, when applied to P, give rise to four graphs. First three are isomorphic to K_4, which does not have a unit distance planar realization. The last one has a rigid unit distance planar realization.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
11.1.	{1,3,7}, {2,4,6}, {5,8}, {9,10}	no	K_4	
11.2.	{1,3,7}, {2,4,8}, {5,6}, {9,10}	no	K_4	
11.3.	{1,3,10}, {2,4,6}, {5,7}, {8,9}	no	K_4	
11.4.	{1,4,8}, {2,9,10}, {3,5}, {6,7}	yes	$G_{4.1}$	yes/0

Table 10: Case 11. (3, 3, 2, 2)

Case 12. (3, 3, 2, 1, 1) There are ten distinct vertex colorings of P, which, when applied to P, give rise to seven non-isomorphic graphs. Only four of them are unit distance graphs, first three are rigid and the other one realizable with one degree of freedom. All other six graphs that are not realizable, contain $K_{2,3}$ as a subgraph.

Case 13. (3, 3, 1, 1, 1, 1) There are five distinct vertex colorings of P, which, when applied to P, give rise to five different graphs. All of them contain $K_{2,3}$ as a subgraph, thus are not realizable with unit distances in the plane.

Case 14. (3, 2, 2, 2, 1) There are 14 distinct vertex colorings of P, which, when applied to P, give rise to five different graphs. Only one of them has a rigid unit distance planar realization. Among the other four, three contain $K_{2,3}$ and one K_4.

Case 15. (3, 2, 2, 1, 1, 1) There are 19 distinct vertex colorings of P, which, when applied to P, give rise to 15 different graphs. Only four of

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
12.1.	{1,3,7}, {2,4,6}, {5,8}, {9}, {10}	yes	$G_{7.1}$	yes/0
12.2.	{1,3,7}, {2,4,6}, {5,9}, {8}, {10}	no	$K_{2,3}$	
12.3.	{1,3,7}, {2,4,8}, {5,6}, {9}, {10}	no	$K_{2,3}$	
12.4.	{1,3,7}, {2,4,10}, {5,6}, {8}, {9}	no	$K_{2,3}$	
12.5.	{1,3,9}, {2,4,6}, {5,7}, {8}, {10}	no	$K_{2,3}$	
12.6.	{1,3,10}, {2,4,6}, {5,7}, {8}, {9}	yes	$G_{7.1}$	yes/0
12.7.	{1,3,10}, {2,4,6}, {5,8}, {7}, {9}	yes	$G_{7.1}$	yes/0
12.8.	{1,3,10}, {2,4,6}, {5,9}, {7}, {8}	no	$K_{2,3}$	
12.9.	{1,4,8}, {2,9,10}, {3,5}, {6}, {7}	yes	$G_{12.9}$	no/1
12.10.	{1,4,10}, {2,8,9}, {3,5}, {6}, {7}	no	$K_{2,3}$	

Table 11: Case 12. (3, 3, 2, 1, 1)

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
13.1.	{1,3,7}, {2,4,6}, {5}, {8}, {9}, {10}	no	$K_{2,3}$	
13.2.	{1,3,7}, {2,4,8}, {5}, {6}, {9}, {10}	no	$K_{2,3}$	
13.3.	{1,3,9}, {2,4,6}, {5}, {7}, {8}, {10}	no	$K_{2,3}$	
13.4.	{1,3,10}, {2,4,6}, {5}, {7}, {8}, {9}	no	$K_{2,3}$	
13.5.	{1,4,8}, {2,9,10}, {3}, {5}, {6}, {7}	no	$K_{2,3}$	

Table 12: Case 13. (3, 3, 1, 1, 1, 1)

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
14.1.	{1,3,7}, {2,4}, {5,6}, {8,9}, {10}	no	$K_{2,3}$	
14.2.	{1,3,7}, {2,4}, {5,6}, {9,10}, {8}	no	$K_{2,3}$	
14.3.	{1,3,7}, {2,4}, {5,8}, {6,10}, {9}	no	$K_{2,3}$	
14.4.	{1,3,7}, {2,4}, {5,9}, {6,10}, {8}	yes	$G_{7.1}$	yes/0
14.5.	{1,3,9}, {2,4}, {5,6}, {7,8}, {10}	no	$K_{2,3}$	
14.6.	{1,3,9}, {2,4}, {5,7}, {6,10}, {8}	yes	$G_{7.1}$	yes/0
14.7.	{1,3,9}, {2,4}, {5,8}, {6,10}, {7}	no	$K_{2,3}$	
14.8.	{1,3,10}, {2,4}, {5,6}, {7,8}, {9}	no	$K_{2,3}$	
14.9.	{1,3,10}, {2,4}, {5,6}, {8,9}, {7}	no	$K_{2,3}$	
14.10.	{1,3,10}, {2,4}, {5,7}, {8,9}, {6}	no	K_4	
14.11.	{1,3,10}, {2,4}, {5,9}, {6,7}, {8}	no	$K_{2,3}$	
14.12.	{1,3,10}, {2,4}, {6,7}, {8,9}, {5}	no	$K_{2,3}$	
14.13.	{1,4,10}, {2,6}, {3,7}, {8,9}, {5}	no	K_4	
14.14.	{1,4,10}, {2,6}, {3,9}, {7,8}, {5}	no	K_4	

Table 13: Case 14. (3, 2, 2, 2, 1)

them have a unit distance planar realization, three out of four are rigid and one is realizable in the plane with one degree of freedom. Ten of eleven graphs, that are not realizable, contain $K_{2,3}$ as a subgraph and one contain K_4 as a subgraph.

Case 16. (3, 2, 1, 1, 1, 1, 1) There are seven distinct vertex colorings of P, which, when applied to P, give rise to seven different graphs. Only two

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
15.1.	{1,3,7}, {2,4}, {5,6}, {8}, {9}, {10}	no	$K_{2,3}$	
15.2.	{1,3,7}, {2,4}, {5,8}, {6}, {9}, {10}	no	$K_{2,3}$	
15.3.	{1,3,7}, {2,4}, {5,9}, {6}, {8}, {10}	no	$K_{2,3}$	
15.4.	{1,3,7}, {2,4}, {6,10}, {5}, {8}, {9}	no	$K_{2,3}$	
15.5.	{1,3,9}, {2,4}, {5,6}, {7}, {8}, {10}	no	$K_{2,3}$	
15.6.	{1,3,9}, {2,4}, {5,7}, {6}, {8}, {10}	yes	$G_{15.6}$	no/1
15.7.	{1,3,9}, {2,4}, {5,8}, {6}, {7}, {10}	no	$K_{2,3}$	
15.8.	{1,3,9}, {2,4}, {6,10}, {5}, {7}, {8}	no	$K_{2,3}$	
15.9.	{1,3,10}, {2,4}, {5,6}, {7}, {8}, {9}	no	$K_{2,3}$	
15.10.	{1,3,10}, {2,4}, {5,7}, {6}, {8}, {9}	no	$K_{2,3}$	
15.11.	{1,3,10}, {2,4}, {5,9}, {6}, {7}, {8}	no	$K_{2,3}$	
15.12.	{1,3,10}, {2,4}, {6,7}, {5}, {8}, {9}	yes	$G_{15.12}$	yes/0
15.13.	{1,3,10}, {2,4}, {8,9}, {5}, {6}, {7}	no	$K_{2,3}$	
15.14.	{1,3,10}, {2,9}, {4,6}, {5}, {7}, {8}	no	$K_{2,3}$	
15.15.	{1,3,10}, {2,9}, {4,7}, {5}, {6}, {8}	yes	$G_{15.15}$	yes/0
15.16.	{1,4,10}, {2,6}, {3,7}, {5}, {8}, {9}	no	$K_{2,3}$	
15.17.	{1,4,10}, {2,6}, {3,9}, {5}, {7}, {8}	no	K_4	
15.18.	{1,4,10}, {2,8}, {3,6}, {5}, {7}, {9}	no	$K_{2,3}$	
15.19.	{1,4,10}, {2,9}, {3,6}, {5}, {7}, {8}	yes	$G_{15.19}$	no/1

Table 14: Case 15. $(3,2,2,1,1,1)$

of them have a unit distance planar realization, first is rigid and the other realizable in the plane with one degree of freedom. All other five graphs that are not realizable, contain $K_{2,3}$ as a subgraph.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
16.1.	{1,3,7}, {2,4}, {5}, {6}, {8}, {9}, {10}	yes	$G_{16.1}$	yes/0
16.2.	{1,3,9}, {2,4}, {5}, {6}, {7}, {8}, {10}	no	$K_{2,3}$	
16.3.	{1,3,9}, {2,10}, {4}, {5}, {6}, {7}, {8}	no	$K_{2,3}$	
16.4.	{1,3,10}, {2,4}, {5}, {6}, {7}, {8}, {9}	no	$K_{2,3}$	
16.5.	{1,4,8}, {2,9}, {3}, {5}, {6}, {7}, {10}	yes	$G_{16.5}$	no/1
16.6.	{1,4,10}, {2,6}, {3}, {5}, {7}, {8}, {9}	no	$K_{2,3}$	
16.7.	{1,4,10}, {2,8}, {3}, {5}, {6}, {7}, {9}	no	$K_{2,3}$	

Table 15: Case 16. $(3,2,1,1,1,1,1)$

Case 17. $(3,1,1,1,1,1,1,1)$ There are two distinct vertex colorings of P, which, when applied to P, give rise to two non-isomorphic graphs. First one has a unit distance planar realization, which is realizable with one degree of freedom. The second one contains $K_{2,3}$ as a subgraph, thus it is not realizable in the plane. Note that the application of the Gruebler's equation for case 17.1 would yield $\mathcal{D}_G = 0$, because of a redundant spoke.

Case 18. $(2,2,2,2,2)$ Again, another case where there are two distinct vertex colorings of P, which, when applied to P, give rise to two non-

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
17.1.	{1,3,7}, {2}, {4}, {5}, {6}, {8}, {9}, {10}	yes	$G_{17.1}$	no/1
17.2.	{1,3,9}, {2}, {4}, {5}, {6}, {7}, {8}, {10}	no	$K_{2,3}$	

Table 16: Case 17. $(3,1,1,1,1,1,1,1)$

isomorphic graphs. Both contain $K_{2,3}$ as a subgraph and are not realizable in the plane.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
18.1.	{1,3}, {2,4}, {5,6}, {7,8}, {9,10}	no	$K_{2,3}$	
18.2.	{1,3}, {2,10}, {4,7}, {5,6}, {8,9}	no	$K_{2,3}$	

Table 17: Case 18. $(2,2,2,2,2)$

Case 19. $(2,2,2,2,1,1)$ There are eleven distinct vertex colorings of P, which, when applied to P, give rise to ten non-isomorphic graphs. Only one among them is realizable in the plane with one degree of freedom. Other nine contain $K_{2,3}$ as a subgraph, thus they are not realizable in the plane.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
19.1.	{1,3}, {2,4}, {5,6}, {7,8}, {9}, {10}	no	$K_{2,3}$	
19.2.	{1,3}, {2,4}, {5,6}, {8,9}, {7}, {10}	no	$K_{2,3}$	
19.3.	{1,3}, {2,4}, {5,6}, {9,10}, {7}, {8}	no	$K_{2,3}$	
19.4.	{1,3}, {2,4}, {5,7}, {6,10}, {8}, {9}	no	$K_{2,3}$	
19.5.	{1,3}, {2,4}, {5,8}, {6,7}, {9}, {10}	no	$K_{2,3}$	
19.6.	{1,3}, {2,4}, {5,8}, {6,10}, {7}, {9}	no	$K_{2,3}$	
19.7.	{1,3}, {2,4}, {6,7}, {8,9}, {5}, {10}	no	$K_{2,3}$	
19.8.	{1,3}, {2,4}, {6,7}, {9,10}, {5}, {8}	no	$K_{2,3}$	
19.9.	{1,3}, {2,9}, {4,6}, {5,7}, {8}, {10}	no	$K_{2,3}$	
19.10.	{1,3}, {2,9}, {4,6}, {5,8}, {7}, {10}	yes	$G_{19.10}$	no/1
19.11.	{1,3}, {2,10}, {4,7}, {5,6}, {8}, {9}	no	$K_{2,3}$	

Table 18: Case 19. $(2,2,2,2,1,1)$

Case 20. $(2,2,2,1,1,1,1)$ There are twelve distinct vertex colorings of P, which, when applied to P, give rise to twelve non-isomorphic graphs. Only one among them is realizable in the plane with two degrees of freedom. Ten of other eleven graphs, that are not realizable in the plane, contain $K_{2,3}$ as a subgraph, and the other one contains K_4 as a subgraph.

The graph $G_{20.11}$ has two degrees of freedom. It is not so obvious, but a non-simplicial representation of $G_{20.11}$ exists and can be seen in Figure 8.

Case 21. $(2,2,1,1,1,1,1,1)$ There are five distinct vertex colorings of P, which, when applied to P, give rise to five non-isomorphic graphs.

case	proper coloring	realizable	graph	rig./$\mathcal{D}_G$
20.1.	{1, 3}, {2, 4}, {5, 6}, {7}, {8}, {9}, {10}	no	$K_{2,3}$	
20.2.	{1, 3}, {2, 4}, {5, 7}, {6}, {8}, {9}, {10}	no	$K_{2,3}$	
20.3.	{1, 3}, {2, 4}, {6, 7}, {5}, {8}, {9}, {10}	no	$K_{2,3}$	
20.4.	{1, 3}, {2, 4}, {6, 10}, {5}, {7}, {8}, {9}	no	$K_{2,3}$	
20.5.	{1, 3}, {2, 4}, {7, 8}, {5}, {6}, {9}, {10}	no	$K_{2,3}$	
20.6.	{1, 3}, {2, 9}, {4, 6}, {5}, {7}, {8}, {10}	no	$K_{2,3}$	
20.7.	{1, 3}, {2, 9}, {4, 10}, {5}, {6}, {7}, {8}	no	$K_{2,3}$	
20.8.	{1, 3}, {2, 10}, {4, 6}, {5}, {7}, {8}, {9}	no	$K_{2,3}$	
20.9.	{1, 3}, {2, 10}, {4, 7}, {5}, {6}, {8}, {9}	no	K_4	
20.10.	{1, 4}, {2, 8}, {9, 10}, {3}, {5}, {6}, {7}	no	$K_{2,3}$	
20.11.	{1, 4}, {2, 9}, {3, 6}, {5}, {7}, {8}, {10}	yes	$G_{20.11}$	no/2
20.12.	{1, 4}, {2, 10}, {3, 6}, {5}, {7}, {8}, {9}	no	$K_{2,3}$	

Table 19: Case 20. (2, 2, 2, 1, 1, 1, 1)

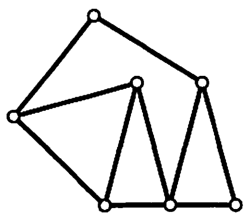


Figure 8: A heavily degenerate unit distance representation of the graph $G_{20.11}$, which is obtained from the Petersen graph.

In case 21.2, the graph G_{21}, which can be seen in Figure 3(a), is obtained. As it was stated in Corollary 3.3, a planar unit distance realization of G_{21} does not exist.

Only one among five non-isomorphic graphs is realizable in the plane with one degree of freedom. Three out of four other graphs, that are not realizable in the plane, contain $K_{2,3}$ as a subgraph, and the fourth one is the graph G_{21}. Note that the application of the Gruebler's equation for case 21.1 would yield $d = 0$, because of a redundant constraint (the lower right link of the top most triangle in the representation of the graph $G_{21.1}$ in Figure 10 is, following Lemma 3.4, redundant).

Case 22. (2, 1, 1, 1, 1, 1, 1, 1, 1) There is only one vertex coloring of P, which, when applied to P, gives rise to the graph G_{22}, which can be seen in Figure 4(c). As it was stated in Corollary 3.5, a planar unit distance realization of G_{22} does not exist.

The unit distance planar realization of the graph G_{22}^* with one degree of freedom (the angle α, that is labeled in Figure 5) exists and can be seen in Figure 4(b).

case	proper coloring	realizable	graph	rig./$\mathcal{D}_{\mathcal{G}}$
21.1.	{1,3}, {2,4}, {5}, {6}, {7}, {8}, {9}, {10}	yes	$G_{21.1}$	no/1
21.2.	{1,3}, {2,9}, {4}, {5}, {6}, {7}, {8}, {10}	no	G_{21}	
21.3.	{1,3}, {9,10}, {2}, {4}, {5}, {6}, {7}, {8}	no	$K_{2,3}$	
21.4.	{1,4}, {2,8}, {3}, {5}, {6}, {7}, {9}, {10}	no	$K_{2,3}$	
21.5.	{1,4}, {2,9}, {3}, {5}, {6}, {7}, {8}, {10}	no	$K_{2,3}$	

Table 20: Case 21. $(2,2,1,1,1,1,1,1)$

case	proper coloring	real.	graph	rig./$\mathcal{D}_{\mathcal{G}}$
22.1.	{1,3}, {2}, {4}, {5}, {6}, {7}, {8}, {9}, {10}	no	G_{22}	

Table 21: Case 22. $(2,1,1,1,1,1,1,1,1)$

Case 23. $(1,1,1,1,1,1,1,1,1,1)$ Every vertex is colored with different color, thus, the graph morphism is the identity. One of the most symmetrical unit distance realizations of the Petersen graph can be seen in Figure 6(b).

The regular pentagon 12345 in Figure 6 has two degrees of freedom, $\angle 234$ and $\angle 345$. There are at most two representations of vertex 1. Another parameter, the angle $\angle 516$, determines the representation of vertex 6. There are at most two possible representations of vertex 9, determined by the pair of vertices $\{4,6\}$. Similarly, there are at most two possible representations of vertices 7, 10, 8, determined by the pairs of vertices $\{2,9\}$ and $\{5,7\}$ and $\{3,10\}$, respectively. The graph $G^*_{23} := rank - \{6,8\}$ is flexible with 3 degrees of freedom. The constraint that is determined by the unit length of the edge $\{6,8\}$ fixes one degree of freedom. Thus, the Petersen graph has a unit distance planar realization, that is flexible with 2 degrees of freedom.

case	proper coloring	real.	graph	rig./$\mathcal{D}_{\mathcal{G}}$
23.1.	{1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}	yes	$G_{23.1}$	no/2

Table 22: Case 23. $(1,1,1,1,1,1,1,1,1,1)$

Hence, there are 23 (degenerate) unit distance representations of the Petersen graph, if the numbers of identified vertices are taken into account. Graphs $G_{4.1}$ and $G_{7.1}$ are obtained from integer partitions $\{4,3,2,1\}$, $\{3,3,3,1\}$, $\{3,3,2,2\}$ and $\{4,2,2,1,1\}$, $\{3,3,2,1,1\}$, $\{3,2,2,2,1\}$, respectively. If we distinguish the representations by the images of the vertices, then the underlaying graph $G_{4.1}$ occurs in three cases, which can be seen in Figure 9.

Each of other seventeen unit distance representations is defined by the integer partition that raises it. Thus, there are exactly eighteen different

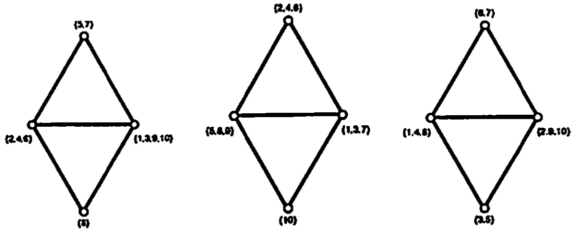


Figure 9: Three different degenerate unit distance planar representations of the Petersen graph, regarded to representations of the vertices.

degenerate, and one non-degenerate unit distance drawings of the Petersen graph, see Figure 10. Thirteen unit distance representations are flexible, three with $\mathcal{D}_G = 2$.

5 Relationships among the representations of the Petersen graph

When we consider the epimorphic images of the Petersen graph, they form a *ranked poset* (rank being the number of vertices) in which the cover relation corresponds to elementary vertex identification (when we identify two non-adjacent vertices).

Identifying two non-adjacent vertices of the Petersen graph $G_{23.1}$ with rank 10 does not provide a unit distance graph, since the only possibility would be the graph G_{22} with rank 9, which is not a unit distance graph. Thus at least two steps of identifying non-adjacent vertices are necessary to obtain a unit distance realization from $G_{23.1}$, either $G_{21.1}$ or $G_{17.1}$. Similarly, the graph $G_{20.11}$ with rank 7 is obtained from $G_{23.1}$ (with rank 10). Hence, the interesting question arises: For a given unit distance representation, which unit distance representations are obtained in one or more steps, if only vertex identification is allowed? The answer for the Petersen graph can be seen in Figure 11(b), which is drawn in such a way, that all arcs point downward eliminating arrowheads. Note, that the poset of the isomorphism classes of epimorphic images of the Petersen graph restricted to the cases that are realizable as unit distance graphs, is not a ranked poset, since the unit distance representation $G_{8.1}$ can be obtained from the unit distance realization $G_{23.1}$ in two distinct ways, one following a path that goes through the unit distance representation $G_{9.1}$ or the other, following a path that visits three graphs that do not have unit distance representations. Both restricted paths can be seen in Figure 11(a).

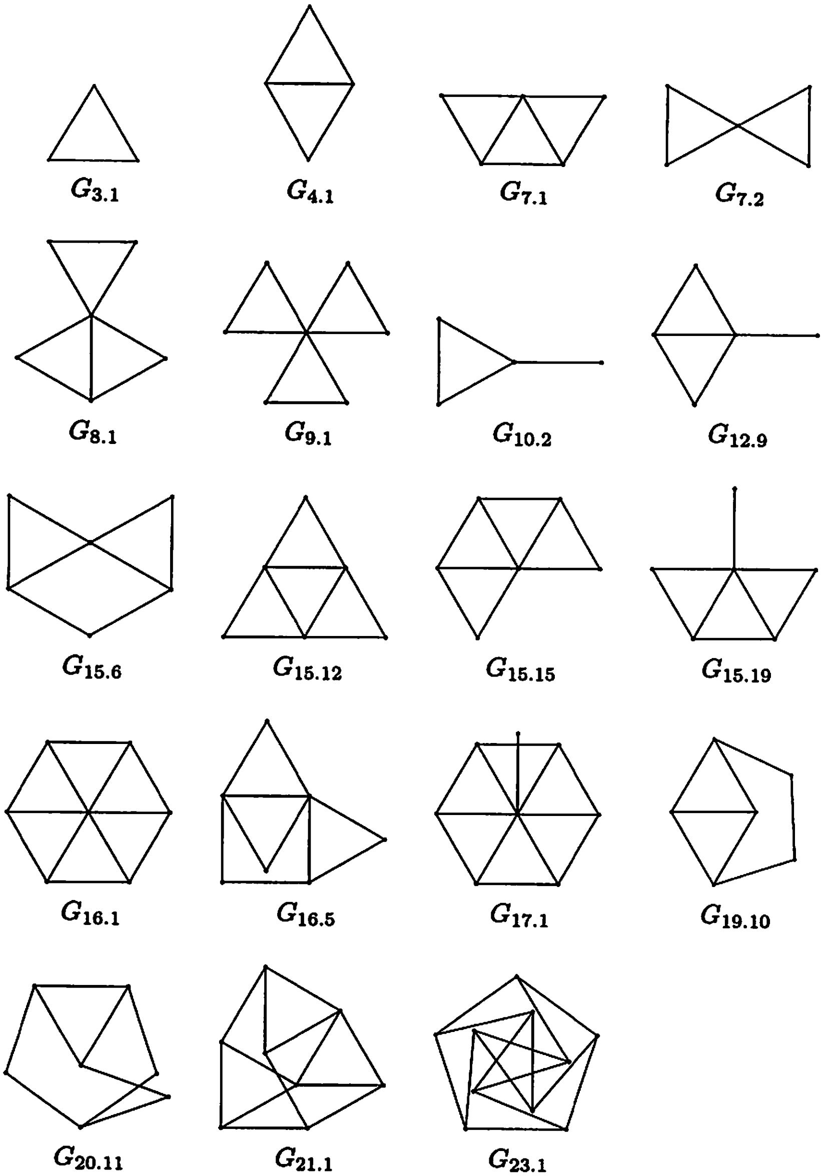


Figure 10: All distinct (degenerate) planar unit distance representations of the Petersen graph P.

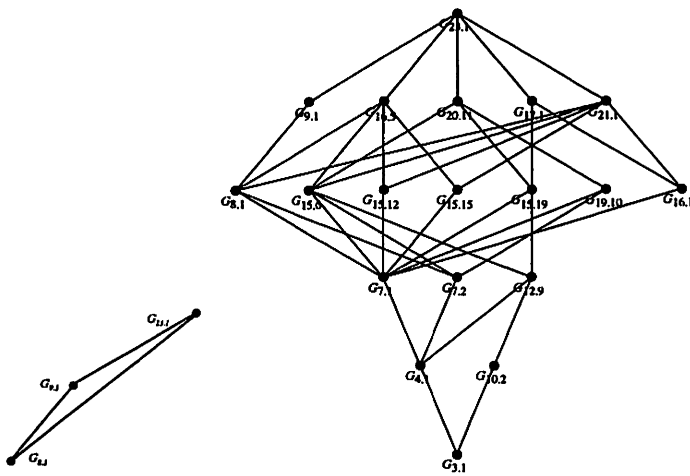


Figure 11: (a) Two distinct paths between the unit distance realization $G_{23.1}$ and the unit distance representation $G_{8.1}$, one contains only graphs without unit distance representation, where the other one visits the unit distance representation $G_{9.1}$. (b) Hasse diagram of the poset of the Petersen graph restricted to the cases that are realizable as unit distance graphs.

We can formulate several interesting problems: Given a graph G, what are all possible unit distance representations of G (including degenerate and highly degenerate ones) and what are their symmetries? What are all possible unit distance realizations of G? Find all subgraphs that are not unit distance embedable in $\mathbb{R}^2$. More generalized problem states: For an arbitrary graph G find a non-singular representation in $\mathbb{R}^2$ minimizing the number of vertex orbits or edge orbits. Another interesting problem is to find an algorithm that embeds a graph with unit distances into the Euclidean plane (space) or answers that a graph can not be embedded with unit distances into given space.

6 Conclusions

In the paper we found all degenerated unit distance representations of the Petersen graph and for each observed its rigidity / flexibility. We considered labeled epimorphic images that are unit distance realizable in the Euclidean plane. We observed the unit distance drawings of the Petersen graph regardless to representations of the vertices. We presented a heavily degenerated unit distance representation of the Petersen graph. We presented the interesting portion of the poset of the isomorphism classes of unit distance planar representations of the Petersen graph.

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References

[1] G. E. Andrews. *The theory of partitions.* Cambridge University Press, 1984.

[2] P. Erdös, F. Harary, W. T. Tutte. On the dimension of a graph. *Mathematika* **12** (1965), 118–122.

[3] J. E. Graver, B. Servatius, H. Servatius. Combinatorial rigidity. *Bull. of the Amer. Math. Soc. 33(1) (1996), 152–164.*

[4] F. Harary. Graph Theory. *Addison-Wesley Publishing Company, 1969.*

[5] N. Hartsfield, G. Ringel. Pearls in Graph Theory: A Comprehensive Introduction. *Revised and augmented. Academic Press, San Diego, 1994.*

[6] P. Hell, J. Nešetřil. Graphs and Homomorphisms. *Oxford University Press, 2004.*

[7] D. H. Myszka. Machines and Mechanisms, Applied Kinematics Analysis. *Prentice Hall; 3 edition (July 2, 2004).*

[8] B. Servatius. The Geometry of Frameworks: Rigidity, Mechanisms and CAD. Geometry at Work *53 , 81–88.*

[9] T. Pisanski, et al. Project Vega home page, http://vega.ijp.si/.

[10] T. Pisanski, M. Boben, A. Žitnik. Interactive conjecturing with Vega. Proceedings of the DIMACS International Workshop, *to appear.*

[11] T. Pisanski, M. Boben, A. Žitnik. Visualization of graphs and related data structures in mathematica. Zbornik radova PrimMath, Mathematica u znanosti, tehnologiji i obrazovanju *2001 (2001), 27–39.*

[12] *T. Pisanski, A. Žitnik. Representations of Graphs and Maps. Work in progress.*

[13] *M. Watkins. A theorem on Tait colorings with an application to the generalized Petersen graphs.* J. Combin. Theory *6 (1969), 152–164.*