## On Steiner Quasigroups of Cardinality 21

By

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**Abstract:** *In* [12] *Quackenbush has expected that there should be subdirectly irreducible Steiner quasigroups (squags), whose proper homomorphic images are entropic (medial). The smallest interesting cardinality for such squags is* 21. *Using the tripling construction given in* [1] *we construct all possible nonsimple subdirectly irreducible squags of cardinality* 21 (SQ(21)s). *Consequently, we may say that there are 4 distinct classes of nonsimple* SQ(21)s, *based on the number n of sub-*SQ(9)s *for n* = 0, 1, 3, 7. *The squags of the first three classes for n* = 0, 1, 3 *are nonsimple subdirectly irreducible having exactly one proper homomorphic image isomorphic to the entropic* SQ(3) (equivalently, *having* 3 *disjoined sub-*SQ(7)*s*). *For n* = 7, *each squag* SQ(21) *of this class has* 3 *disjoint sub-*SQ(7)s and 7 *sub-*SQ(9)s, *we will see that this squag is isomorphic to the direct product* SQ(7) × SQ(3). *For n* = 0, *each squag* SQ(21) *of this class is a nonsimple subdirectly irreducible having three disjoint sub-*SQ(7)*s and no sub-*SQ(9)*s. In section* 5, *we describe an example for each of these classes Finally, we review all well-known classes of simple* SQ(21)s.

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### 1. Introduction

A Steiner quasigroup (briefly squag) is a groupoid $\boldsymbol{Q} = (Q; *)$ satisfying the identities:

$$x * x = x \quad , \quad x * y = y * x \quad , \quad x * (x * y) = y .$$

A Steiner loop (briefly sloop) is a groupoid $\boldsymbol{L} = (L; \cdot, 1)$ with neutral element 1 satisfying the identities:

$$x \cdot x = 1 \quad , \quad x \cdot y = y \cdot x \quad , \quad x \cdot (x \cdot y) = y .$$

Notice that both squags and sloops are quasigroups [6, 8].

We use the abbreviations SL($n$) and SQ($n$) for a sloop and a squag of cardinality $n$, respectively. Squags satisfying the entropic (medial) law :

$$(x * y) * (z * w) = (x * z) * (y * w)$$

will be called entropic (medial) squags. There is a one to one correspondence between squags (sloops) and Steiner triple systems [6, 7, 8, 12].

A Steiner triple system is a pair $(P; T)$, where $P$ is a set of points and $T$ is a set of 3-element subsets of $P$ called blocks such that for each two distinct points $p_1, p_2 \in P$, there is a unique block $b \in T$ with $\{p_1, p_2\} \subseteq b$. If the cardinality of the set of points $P$ is equal to $n$, the Steiner triple system $(P; T)$ will be denoted by STS($n$). It is well-known that a necessary and sufficient condition for the existence of an STS($n$) is $n \equiv 1$ or 3 (mod 6 ) [8, 11, 12].

In [12] Quackenbush proved that the congruences of squags are permutable, regular, and Lagrangian. A subsquag $S$ of a squag $Q$ is called normal iff $S$ is a congruence class of a congruence relation on $Q$. In [12] Quackenbush proved also that if $S_1$ and $S_2$ are two disjoint subsquags of $Q$ and $|Q| = 3|S_1| = 3|S_2|$, then the three subsquags $S_1$, $S_2$ and $S_3 := Q - (S_1 \cup S_2)$ form the congruence classes of a congruence on $Q$.

An STS is planar if it is generated by every triangle (a three distinct points are called a triangle if they do not form a block) and contains a triangle. A planer STS($n$) exists for each $n \geq 7$ and $n \equiv 1$ or 3 (mod 6) [7]. The squag and sloop associated with a planar triple system are also called planar. In [12] Quackenbush has shown as in the next theorem that the only non-simple finite planar squag has 9 elements.

**Theorem 1** [12]. *Let $(P; T)$ be a planar* STS($n$) *and let $Q = (P; *)$ and $L = (L = P \cup \{1\}; \cdot, 1)$ be the corresponding squag and sloop respectively. Then either $Q$ is simple or $n = 9$ and also $L$ is simple or $n = 7$.*

Accordingly, we may say that there is always a simple SQ($n$) and SL($n$+1) for all $n > 9$ and $n \equiv 1$, or 3 (mod 6). Also, all semi-planar squags and semi-planar sloops are simple.

A semi-planar squag (sloop) is a simple squag (sloop) each of whose triangles generates either the whole squag (sloop) or a sub-SQ(9) (a sub-SL(8)) " cf. [2, 12]". The author [4, 2] has given a construction of semi-planar squags (sloops) of cardinality $3\,n\,(2n+2)$ for each possible $n > 3$.

An extensive study of squags can be found in [6, 8, 12]. In this article, we will use some basic concepts of universal algebra [9] and some other concepts of graph theory [10].

We return our attention to squags. It is well-known that all squags of the classes SQ($n$)s for $n$ = 3, 7, 13, 15 and 19 are simple and SQ(9) is the direct power $SQ(3)^2$. The next admissible order for squags are of cardinality 21 (SQ(21)s). Notice that the SL(22)s associated with SQ(21)s (or STS(21)s) are always simple. Using computer programs in [11] the authors proved that there are more than 2000000 STS(21)s consisting of 3 disjoint sub-STS(7)s (so this class is called Wilson-type STS(21)).

In general, we divide the class of all SQ(21)s (STS(21)s) into two essentially classes, nonsimple SQ(21)s and simple SQ(21)s. Other than the direct product SQ(7) × SQ(3), there are two classes of nonsimple SQ(21)s, one of them has a proper congruence with congruence class of cardinality 7 (3 disjoint sub-SQ(7)s) and the another has a proper congruence with congruence class of cardinality 3 (7 sub-SQ(9)s). Based on the number of sub-SQ(9)s we divide the class of all nonsimple SQ(21)s with 3 disjoint sub-SQ(7)s in to 4 classes. Each of these classes has exactly three disjoint sub-SQ(7)s and $n$ sub-SQ(9)s (for $n$ = 0, 1, 3, 7) This classification supplies us with the following classes:

**Nonsimple SQ(21)s:**

1. For $n = 7$, we have an SQ(21) containing three disjoint sub-SQ(7)s and 7 sub-SQ(9)s, we will show that this squag is isomorphic to the direct product SQ(7) × SQ(3).
2. For $n$ = 0, 1, or 3 the SQ(21) contains three disjoint sub-SQ(7)s and $n$ sub-SQ(9)s.
   In [12] Quackenbush has expected that there should be nonsimple subdirectly irreducible squags, all of whose proper homomorphic images are entropic (medial) squags. Indeed, all SQ(21)s of these three classes (for $n$ = 0, 1, 3) are nonsimple subdirectly irreducible each of whose proper homomorphic image is isomorphic to the entropic SQ(3).
3. For $n = 0$, we get SQ(21)s have three disjoint sub-SQ(7)s and no sub-SQ(9)s, so all SQ(21)s of this class are nonsimple subdirectly irreducible. Indeed, SL(22)s associated with such SQ(21)s supply us with examples for semi-planar SL(22)s. Of course, this class of semi-planar SL(22)s are not planar and have exactly three sub-SL(8)s, but no sub-SL(10)s. It will be convenient to note at this point that the two smallest well known cardinalities of non-planar semi-planar sloops are 20 and 28 " cf. [2]".
4. If an SQ(21) has 7 sub-SQ(9)s and at most one sub-SQ(7), we will show that SQ(21) is nonsimple subdirectly irreducible having only one homomorphic image isomorphic to the SQ(7).

In [1] a tripling construction for Steiner quasigroups (squags) of cardinality $3n$ was given for all $n > 4$ and $n \equiv 1$ or 3 (mod 6). By applying this construction, we may give in section 5 an example for each class of the nonsimple SQ(21)s with 3 disjoint sub-SQ(7)s.

**Simple SQ(21)s:**

Finally, we review all well known constructions of simple SQ(21)s. By applying these constructions given in [3, 4, 5, 7] we get simple SQ(21)s having:

1. Neither sub-SQ(7)s nor sub-SQ(9)s.
2. 3 sub-SQ(9)s but no sub-SQ(7)s.
3. One sub-SQ(9) and no sub-SQ(7)s.

4. One sub-SQ(7) and 3 sub-SQ(9)s.
5. One sub-SQ(7) and one sub-SQ(9).
6. One sub-SQ(7) and no sub-SQ(9)s.

## 2. Some properties of subsquags of SQ($3n$)s

Consider the 3-element squag $I_3 = (\{0, 1, 2\} ; \oplus)$ with $x \oplus y = 2x + 2y \pmod 3$, where " + " is the binary operation of the 3-element group $Z_3$. Indeed, the triple system STS($3n$) associated with the direct product squag SQ($n$) × $I_3$ contains always three sub-STS($n$)s, since a squag is idempotent.

In this section, we will show that if an SQ($3n$) (STS($3n$)) has three disjoint planar sub-SQ($n$)s (sub-STS($n$)s), then these three subsquags (subsystems) are the only subsquags (subsystems) of cardinality $n$. Moreover, any other subsquag (subsystem) of SQ($3n$) (STS($3n$)) is of cardinality 9. These properties can be generalized in the following two lemmas.

**Lemma 2.** *Let $Q$ be a squag of cardinality $3n$ and $Q$ has a congruence $\theta$ with $Q/\theta = \{A_1, A_2, A_3\}$. Then any subsquag $B$ with $B \not\subset A_i$ for $i$ = 1, 2 or 3 must satisfies:* $|B \cap A_1| = |B \cap A_2| = |B \cap A_3|$.

**Proof.** If $B \cap A_1 = \varnothing$, then there is an element $a \in B \cap A_2$ otherwise $B \subseteq A_3$ contradicting that $B \not\subset A_i$ for $i$ = 1, 2 or 3. But $a \in B \cap A_2$ implies that $a * (B \cap A_3) \subseteq B \cap A_1$, which contradicts the assumption that $B \cap A_1 = \varnothing$. Therefore, $B \cap A_i \neq \varnothing$ for each $i$. Choosing an element $a_1 \in B \cap A_1$, then $a_1 * (B \cap A_2) \subseteq B \cap A_3$. Consider the map $\delta_{a_1}(x) := a_1 * x$ from the set $(B \cap A_2)$ to the set $B \cap A_3$, so the map $\delta_{a_1}$ is bijective. Hence we may directly say that $|B \cap A_2| = |B \cap A_3|$. Similarly, one can show that $|B \cap A_1| = |B \cap A_2|$. This completes the proof. □

The following lemma shows that an SQ($3n$) with three simple planar disjoint sub-SQ($n$) has no more sub-SQ($n$)s.

**Lemma 3.** *Let $Q$ be a squag of cardinality $3n$ and $Q$ has a congruence $\theta$ with $Q/\theta = \{A_1, A_2, A_3\}$, in which $A_i$ for $i$ = 1, 2 and 3 are three simple planar subsquags. Then any other nontrivial subsquags of $Q$ must be of cardinality 9 for all possible number $n > 3$ and $n \equiv 1$ or 3 (mod 6).*

**Proof.** Let $Q = (P; *)$ be a squag SQ($3n$) and let $Q/A_I = \{A_1, A_2, A_3\}$, in which $A_i$ are simple planar squags for $i$ = 1, 2 and 3. Suppose $Q$ contains another normal subsquag $B$, then if $B \subseteq A_i$ for some $i$, $B$ is a normal subsquag of $A_i$. Thus $|B|$ = 1, or $B = A_i$. Assume $B \supset A_i$ for some $i$, hence $B$ is not a normal subsquag of $Q$, where the maximal normal subsquag of $Q$ is of cardinality $n$. According to Lemma 2, thus $B$ must intersects $A_i$ for each $i$, and also $|B \cap A_1| = |B \cap A_2| = |B \cap A_3| = 1$, so $|B| = 3$.

On the other hand, if $\boldsymbol{B}$ is a subsquag satisfying $A_i \subseteq B \subseteq Q$, then $A_i$ is normal in $\boldsymbol{B}$, which means that $B/A_i$ is a squag of cardinality 1 or 3. This means that $B = A_i$ or $B = Q$. If $\boldsymbol{B}$ is a proper subsquag of $A_i$, then $|B| = 1$, or 3. According to Lemma 2, any proper subsquag $\boldsymbol{B}$ with $|B| > 3$ must intersects $A_i$ for each $i$. and also $|B \cap A_1| = |B \cap A_2| = |B \cap A_3| = 1$ or 3. This means that $|B| = 3$ or 9. Therefore, the only possible nontrivial subsquag $\boldsymbol{B}$ of $Q$ is of cardinality 9. This completes the proof of the lemma. □

Notice that the smallest value $n$ verifying the above Lemma is $n = 7$. Moreover, if only one of the subsquags $A_1$, $A_2$ or $A_3$ is simple planar and $B \not\subset A_i$ for $i = 1, 2$ and 3, then we get the same result that $|B \cap A_1| = |B \cap A_2| = |B \cap A_3| = 1$ or 3. This means also that the only possible nontrivial subsquag $\boldsymbol{B}$ of $Q$ is of cardinality 9. The parallel result of lemma 3 for STS is given as follows:

**Corollary 4.** *Let $T$ be an* STS($3n$) *having three disjoint sub-*STS($n$) $T_1$, $T_2$ *and* $T_3$ *one of them planar, then there is no more subsystems of cardinality n. Moreover, any other subsystem* $\boldsymbol{B} \not\subset T_i$ for $i = 1, 2$ and 3 *of the triple system $T$ is of cardinality* 9.

In the following two lemmas we study the relations between the sub-SQ(9)s contained in an SQ(21).

**Lemma 5.** *Let $Q = (P; *)$ be an* SQ(21), *and $A$ be a sub-*SQ(9) (*or a sub-*SQ(7)) *and $\boldsymbol{B}$ be a sub-*SQ(9), *then $A$ and $\boldsymbol{B}$ intersects in a sub-*SQ(3). *Moreover, if $A$ is a sub-*SQ(9), then *there is always a third sub-*SQ(9) *passing through $A \cap B$.*

**Proof.** Assume that $A \cap B = \{x\}$ or $\varnothing$. For $a \in A - \{x\}$ *we have* $a * B \cap B = \varnothing$ and $a * B \cap A = \{a * x\}$ or $\varnothing$, otherwise if $a * b = b`$ or $a * b = a` \neq a * x$, then $a = b * b`$ or $a * a` = b$ contradicting that $A$ and $B$ are subsquags. Moreover, $|a * B| = |B|$, which implies that $Q$ contains at least, $|B \cup A| = 9 + 9 - 1$ (or $9 + 7 - 1$) and $|a * B - (B \cup A)| = 9 - 1$ elements contradicting that $Q$ is of cardinality 21. Therefore, the intersection of a sub-SQ(7) and a sub-SQ(9) or of two sub-SQ(9)s is always a sub-SQ(3).
Let $A = \{x, y, z, a_1, a_2, \ldots, a_6\} = \{x, y, z\} \cup A_1$, $B = \{x, y, z, b_1, b_2, \ldots, b_6\} = \{x, y, z\} \cup B_1$ and $C_1 = \{c_1, c_2, \ldots, c_6\} = P - (A \cup B)$. We want to prove that $C_1$ generates a sub-SQ(9) passing through $\{x, y, z\}$. Now, we have $b_i * A_1 = C_1$ for $i = 1, \ldots, 6$ (or equivalently $B_1 * A_1 = C_1$), otherwise if $b_i * a_j = a_k$, then $a_j * a_k = b_i$ contradicting that $A$ is a subsquag and $B \cap A = \{x, y, z\}$. We get the same contradiction if we assume that $b_i * a_j = b_k$. So we have $A_1 * C_1 = B_1$. To show that $C = \{x, y, z\} \cup C_1$ forms a sub-SQ(9), we have $w * c_i \in C_1$ for each $w \in \{x, y, z\}$ and $c_i \in C_1$. Otherwise, if $w * c_i = a_j$ for $w \in \{x, y, z\}$, then $w * a_j = c_i \notin A$ contradicting that $A$ is subsquag. Also $c_i * c_j \in C$, otherwise if $c_i * c_j = a_k$, then $c_i * a_k = c_j$ contradicting that $A_1 * C_1 = B_1$. We get the same contradiction if

we assume that $w * c_i = b_j$. Therefore, the subset $C = \{x, y, z\} \cup C_1$ forms a sub-SQ(9) passing through the intersection $B \cap A = \{x, y, z\}$. □

According to this Lemma, if a squag SQ(21) contains more than one sub-SQ(9), then it must contain at least three sub-SQ(9)s. Also, each three sub-SQ(9)s are passing through a common sub-SQ(3).

The next lemma shows that if a squag SQ(21) contains more than three sub-SQ(9)s, then it must contains exactly 7 sub-SQ(9)s.

**Lemma 6.** *Let $A$, $B$, and $C$ be three sub-*SQ(9)*s of an* SQ(21) = $Q$ *passing through the sub-*SQ(3) = $\{x, y, z\}$. *Then each other sub-*SQ(9) *intersects each of $A$, $B$ and $C$ in a block (sub-*SQ(3)*) parallel (not intersect) to* $\{x, y, z\}$. *Moreover, any* SQ(21) *contains at most 7 sub-*SQ(9)*s.*

**Proof.** Consider as in the above lemma $A = \{x, y, z, a_1, a_2, \dots, a_6\} = \{x, y, z\} \cup A_1$, $B = \{x, y, z, b_1, b_2, \dots, b_6\} = \{x, y, z\} \cup B_1$, $C = \{x, y, z\} \cup \{c_1, c_2, \dots, c_6\} = \{x, y, z\} \cup C_1$ and $Q = \{x, y, z\} \cup A_1 \cup B_1 \cup C_1$. Assume that $D$ be a sub-SQ(9) and $D \cap A = \{x, a_i, a_j\}$ ; i. e., $D \cap A$ is not parallel to $\{x, y, z\}$. So D - $\{x, a_i, a_j\} \subseteq B \cup C - A$, but $B \cup C - A = B_1 \cup C_1 = \{b_1, b_2, \dots, b_6\} \cup \{c_1, c_2, \dots, c_6\}$. So D - $\{x, a_i, a_j\}$ is a 6-element subset $\subseteq B_1 \cup C_1$ , then each of D - $\{x, a_i, a_j\} \cap B_1$ and D - $\{x, a_i, a_j\} \cap C_1$ is a 3-element subset. These two 3-element subsets form two blocks lying in each of $B_1$ and $C_1$, respectively. Otherwise, $D = B$ or $D = C$. This implies that each of $D \cap B$ and $D \cap C$ contains more than three elements, which is impossible. Therefore, $D \cap A = \{a_i, a_j, a_k\}$ and $\{a_i, a_j, a_k\} \cap \{x, y, z\} = \varnothing$. In general, we may say that $D$ intersects each of $A$ , $B$ and $C$ in three parallel blocks (sub-SQ(3)) to $\{x, y, z\}$.
Accordingly, if an SQ(21) has more than three sub-SQ(9)s $\{A, B, C\}$, then the fourth sub-SQ(9) $D$ intersects $A$, $B$ and $C$ in three parallel distinct blocks (sub-SQ(3)s) $D \cap A$, $D \cap B$ and $D \cap C$. These blocks are different from the intersection block $A \cap B \cap C$. Again, according to Lemma 5 there are three times three sub-SQ(9)s $\{A, D, D_A\}$, $\{B, D, D_B\}$ and $\{C, D, D_C\}$, each passing through a blocks (sub-SQ(3)s) $D \cap A$, $D \cap B$ and $D \cap C$, respectively. Therefore, this SQ(21) contains 7 sub-SQ(9)s $A, B, C, D, D_A, D_B$ and $D_C$. Also, there are 7 parallel sub-SQ(3)s $A \cap B \cap C$, $A \cap D \cap D_A$, $B \cap D \cap D_B$, $C \cap D \cap D_C$, $A \cap D_B \cap D_C$, $B \cap D_A \cap D_C$ and $C \cap D_A \cap D_B$, and three sub-SQ(9)s are passing through each of them. If we assume that an SQ(21) contains more than 7 sub-SQ(9)s; i. e., assume that $E$ is another sub-SQ(9), then $E$ must intersect each sub-SQ(9) of the 7 sub-SQ(9)s in an sub-SQ(3). This needs more than three parallel sub-SQ(3)s in the sub-SQ(9), which is impossible. □

Notice that the direct product SQ(7) × SQ(3) has exactly 7 sub-SQ(9)s.

Clearly, we know that if an SQ(21) has a congruence $\theta$ with a homomorphic image $\cong$ the SQ(7), then the SQ(21) has 7 sub-SQ(9)s. In the next theorem we will show that the inverse is also true.

**Theorem 7.** *Let $Q$ be an* SQ(21) *having* 7 *sub-*SQ(9)s. *Then it has a congruence with a congruence class of cardinality* 3.

**Proof.** According to the proof of Lemma 6, we may consider the 7 sub-SQ(9)s $A$, $B$, $C$, $D$, $D_A$, $D_B$ and $D_C$ in which each 3 sub-SQ(9)s of the 7 sub-SQ(9)s intersect in a sub-SQ(3). These sub-SQ(3)s supplies us with a 7 sub-SQ(3)s forming a partition of the SQ(21). The 7 sub-SQ(3)s are:
$S_1 = A \cap B \cap C$, $S_2 = A \cap D \cap D_A$, $S_3 = B \cap D \cap D_B$, $, S_4 = C \cap D \cap D_C$, $S_5 = A \cap D_B \cap D_C$, $, S_6 = B \cap D_A \cap D_C$ $, S_7 = C \cap D_A \cap D_B$.
Moreover, each sub-SQ(9) consists of 3 of these sub-SQ(3)s and each two sub-SQ(3)s of them $S_i$ and $S_j$ ly exactly in one sub-SQ(9) whose elements are $S_i \cup S_j \cup S_k$. I.e., each two $S_i$ and $S_j$ sub-SQ(3)s there is a sub-SQ(9) and a sub-SQ(3) $S_k$ satisfying that sub-SQ(9) = $S_i \cup S_j \cup S_k$.
Now, the set of 7 sub-SQ(3)s forms an equivalent relation on $Q$. Let $(x_1, x_2) \in S_i^2$ and $(y_1, y_2) \in S_j^2$. Then there is a sub-SQ(9) = $S_i \cup S_j \cup S_k$ satisfying that $(x_1 y_1, x_2 y_2) \in S_k^2$. This implies directly that 7 sub-SQ(3)s forms a congruence relation on $Q$. □

In the following section we will study some properties of subsquags of the tripling construction given in [1].

## 3. Construction of SQ($3n$) = [$Q_1$, $Q_2$, $Q_3$; $F(Q_1, Q_2)$, $\alpha$] [1]

In [1] the authors have given a tripling construction of finite squags denoted by [$Q_1$, $Q_2$, $Q_3$; $F(Q_1, Q_2)$, $\alpha$] of cardinality $3n$ for all $n \equiv 1$ or 3 (mod 6 ), where $Q_1$, $Q_2$ and $Q_3$ are any three disjoint SQ($n$)s and $\alpha$ any permutation on the $n$-element set. We will review this construction in the following. For this propose, we need to review briefly some concepts from graph theory [10].

A complete bipartite graph $K_{n,m}$ is a simple graph, whose set of vertices $V(K_{n,m})$ can be partitioned into two disjoint sets $A$ and B such that $|A| = n$ & $|B| = m$ and the set of edges $E(K_{n,m})$ is exactly the set of all edges connecting each vertex of $A$ to each vertex of $B$. A spanning subgraph $F$ of a graph $G$ is called a 1-factor of $G$, if $\deg_F v = 1$; for all $v \in V(G)$. If a graph $G$ is the union of a set of disjoint 1-factors $\{F_1, F_2, \dots, F_n\}$, then the set $F = \{F_1, F_2, \dots, F_n\}$ is called a 1-factorization of $G$ and $G$ itself is called 1-factorable. In fact, every regular bipartite $K_{n,n}$ is 1-factorable [10].

Let $Q_i$ for $i = 1, 2, 3$ be any three disjoint squags $(P_i; *_i)$ of the same cardinality $n$ and let $(P_i, T_i)$ be the corresponding disjoint STS($n$)s for $i = 1, 2, 3$; i.e, $P_i \cap P_j = \varnothing$, for $i \neq j$. Let $K_{n,n}$ be the complete bipartite graph with a set of vertices $V(K_{n,n}) = P_1 \cup P_2$ and $F = \{F_1, F_2, \dots, F_n\}$ be a 1-factorization of $K_{n,n}$.

By taking any permutation $\alpha$ on $\{1, 2, \dots , n\}$, then $\underline{\alpha}$ can be considered as a bijective map from $\{1, 2, \dots , n\}$ on to $P_3$ by $\underline{\alpha} : i \to c_{\alpha(i)}$.

Consider the following set of triples given by:

$$T_{123} := \{\{x , y , \underline{\alpha}(i)\} : x\,y \in F_i \ \& \ \ F_i \in F\}.$$

For $P := P_1 \cup P_2 \cup P_3$ and $T := T_1 \cup T_2 \cup T_3 \cup T_{123}$, then the system $(P; T)$ is an STS($3n$) [1]. We note that $F$ may be any 1-factorization on $P_1 \cup P_2$ and that $\alpha$ may be any permutation on $\{1, 2, \dots , n\}$. This construction of an STS ($3n$) = $(P; T)$ will be denoted by:

$$[((P_1;\ T_1) \cup (P_2;\ T_2)) \cup (P_3;\ T_3);\ \ F(P_1, P_2), \alpha].$$

Also the corresponding squag $(P; *)$ denoted by:

$$[((P_1;\ *_1) \cup (P_2;\ *_2)) \cup (P_3;\ *_3);\ \ F(P_1, P_2), \alpha],$$

where the binary operation " $*$ " on $P$ is defined by :

$$x * y := \begin{cases} x *_i y & \text{if } x, y \in P_i \ \text{ for } i = 1, 2, 3 \\ \underline{\alpha}(i) & \text{if } \ x \in P_1 \,\&\, y \in P_2 \,\&\, x\,y \in F_i \\ z & \text{if } x \in P_1 \,\&\, y \in P_3 \,\&\, x\,z \in F_{\alpha^{-1}(y)} \\ z & \text{if } \ x \in P_3 \,\&\, y \in P_2 \,\&\, y\,z \in F_{\alpha^{-1}(y)}. \end{cases}$$

In addition the binary operation " $*$ " satisfies the identities $x * x = x$ and $x * y = y * x$ (cf. [1]).

### 3. 1 Construction $3 \otimes_\alpha Q_l$

By choosing SQ($n$) = $Q_l$ = $(P_1;\ *_1) \cong (P_2;\ *_2) \cong (P_3;\ *_3)$ we will denote the construction $[((P_1;\ *_1) \cup (P_2;\ *_2)) \cup (P_3;\ *_3\ ),\ F(P_1, P_2),\ \alpha]$ by $3 \otimes_\alpha Q_l$. Based on the number of subsquags of an SQ(21) and using the construction $3 \otimes_\alpha Q_l$ , we will construct in the next section all classes of nonsimple subdirectly irreducible SQ(21)s in which all of whose proper homomorphic images are entropic (medial).

Let $P_0 = \{1, 2, 3, 4, 5, 6, 7\}$ and $T_0 := (P_0;\ T_0)$ be an STS(7). Let $P_1 = \{a_1, \dots , a_7\}$, $P_2 = \{b_1, \dots , b_7\}$ and $P_3 = \{c_1, \dots , c_7\}$ such that $P_1 \cap P_2 \cap P_3 = \emptyset$. Consider three isomorphic STS(7)s $T_i = (P_i\,;\ T_i)$ to $T_0$ for $i = 1, 2, 3$ by meaning $\{i, j, k\} \in T_0 \Leftrightarrow \{\ a_i, a_j, a_k\} \in T_1 \Leftrightarrow \{b_i, b_j, b_k\} \in T_2 \Leftrightarrow \{c_i, c_j, c_k\} \in T_3$. Also, let $Q_i = (P_i;\ \ *_i\ )$ be the corresponding squags of $T_i = (P_i;\ T_i)$ for $i = 1, 2, 3$. Consider the set of 1-factors defined by $F_i = \{a_l\, b_k : a_l * a_k = a_i\ ,\ \ a_i, a_l, a_k \in P_1$ and $b_i, b_l, b_k \in P_2\}$, then the class $F = \{F_1, F_2, \dots , F_7\}$ forms a 1-factorization of the complete bipartite graph $\mathbf{K}_{n,\,n}$ on the sets of vertices $P_1 \cup P_2$ . The constructed STS(21) = $(P\,;\ T)$ and the associated squag SQ(21) = $(P;\ *\ )$ will be denoted by

$3 \otimes_\alpha T_1$ and $3 \otimes_\alpha Q_1$, respectively. We take here $P = P_1 \cup P_2 \cup P_3$, the set of triples $T = T_1 \cup T_2 \cup T_3 \cup T_{123}$ and $T_{123} = \{\{a_i, b_j, c_{\alpha(k)}\} : a_i b_j \in F_k\}$.

If we choose the permutation $\alpha$ equal to the identity map on the set $\{1, 2, \ldots, 7\}$, then the constructed squag $Q = 3 \otimes_\alpha Q_1$ is isomorphic to the direct product of $SQ(7) = Q_1$ and the 3-element squag SQ(3).

Let $3 \otimes_\alpha Q_1 = (P = P_1 \cup P_2 \cup P_3; *)$ be the squag associated with the STS(21) $= 3 \otimes_\alpha T_1$. Notice that the squag $3 \otimes_\alpha Q_1 = (P; *)$ contains the disjoint normal subsquags $Q_1 = (P_1; *_1)$, $Q_2 = (P_2; *_2)$ and $Q_3 = (P_3; *_3)$.

For each block $\{i, j, k\} \in T_0$ there is a sub-1-factorization $f = \{f_i = \{a_i b_i, a_j b_k, a_k b_j\} \subseteq F_i, f_j = \{a_j b_j, a_i b_k, a_k b_i\} \subseteq F_j, f_k = \{a_k b_k, a_i b_j, a_j b_i\} \subseteq F_k\}$ of $F$. Conversely, if there is a sub-1-factorization of $K_{3,3}$ on the two 3-element subsets $\{a_i, a_j, a_k\} \subseteq P_1$ and $\{b_i, b_j, b_k\} \subseteq P_2$, then $\{i, j, k\}$ is a block $\in T_0$.

In the next lemma we show that these sub-1-factorizations of $K_{3,3}$ are the only sub-1-factorizations of $F$. This means that there is a one-one correspondence between the set of blocks of $T_0$ and the sub-1-factorizations of $K_{3,3}$ in $F$.

**Lemma 8.** *The 1-factorization $F$ of the complete bipartite graph $K_{7,7}$ on the set $P_1 \cup P_2$ has exactly 7 sub-1-factorizations of $K_{3,3}$.*

**Proof.** Let $f$ be a sub-1-factorization on $K_{r,r}$ of $F$. Then the order $r$ of the complete bipartite graph $K_{r,r}$ is less than or equal 7/2, hence $r = 3$ is the only interesting nontrivial case. Indeed, there is always a sub-1-factorization on the two blocks $\{a_i, a_j, a_k\} \in T_1$ and $\{b_i, b_j, b_k\} \in T_2$ that is the sub-1-factorization:
$f = \{f_i = \{a_i b_i, a_j b_k, a_k b_j\} \subseteq F_i, f_j = \{a_j b_j, a_i b_k, a_k b_i\} \subseteq F_j, f_k = \{a_k b_k, a_i b_j, a_j b_i\} \subseteq F_k\}$.

Now, we want to prove that these are all sub-1-factorizations of $F$. Assume that there is another sub-1-factorization on the two blocks $\{a_i, a_j, a_k\} \in T_1$ and $\{b_{i'}, b_{j'}, b_{k'}\} \in T_2$. If $i = i'$, then $a_i b_i \in F_i$, hence $f_i = \{a_i b_i, a_j b_k, a_k b_j\} \subseteq F_i$ is the unique sub-1-factor in $F_i$ containing $\{a_i, a_j, a_k\}$, therefore $j = j'$ and $k = k'$. The same result if we begin with $j = j'$ or $k = k'$. Notice that if $a_i b_{j'}$ and $a_j b_i$ are two edges with $i \neq j'$ and $i \neq j$ in one factor $F_h$, then $\{h, i, j'\}$ and $\{h, i, j\}$ are two blocks in $T_0$, hence $j' = j$ and $h = k = k'$. This means that $\{i', j', k'\} = \{i, j, k\}$ if $i = i'$.
Now, assume that there is a sub-1-factorization on the two blocks $\{a, a_j, a_k\} \in T_1$ and $\{b_{i'}, b_{j'}, b_{k'}\} \in T_2$ with $\{i, j, k\} \cap \{i', j', k'\} = \varnothing$, then there is a sub-1-factor $f = \{a_i b_{i'}, a_j b_{j'}, a_k b_{k'}\} \subseteq F_h$ with $h \notin \{i, j, k, i', j', k'\}$. Otherwise, $a_h b_h$ is an edge in $f$ contradicting that $\{i, j, k\} \cap \{i', j', k'\} = \varnothing$. This means that $F_h = \{a_h b_h, a_i b_{i'}, a_j b_{j'}, a_k b_{k'}, a_{i'} b_i, a_{j'} b_j, a_{k'} b_k\}$. Also, we may assume that there is another sub-1-factor $f' = \{a_i b_{j'}, a_j b_{k'}, a_k b_{i'}\} \subseteq F_{h'}$ with $h' \notin \{i, j, k, i', j', k'\}$ and $h \neq h'$ contradicting that $\{i, j, k, i', j', k', h, h'\}$ must be 7-element set. Since the number of blocks of $T_0$ is 7, the 1-factorizations $F$ on the set $P_1 \cup P_2$ has exactly 7 sub-1-

facorizations. So each two blocks $\{a_i, a_j, a_k\} \in T_1$ and $\{b_i, b_j, b_k\} \in T_2$ for each $\{i, j, k\} \in T_0$ there is a sub-1-facorization $f = \{f_i \subseteq F_i, f_j \subseteq F_j, f_k \subseteq F_k\}$ of $K_{3,3}$. □

Accordingly, we may easily verify the following lemma.

**Lemma 9.** *Let* $C_1 = \{a_i, a_j, a_k\}$ *and* $C_2 = \{b_i, b_j, b_k\}$ *be two subsquags of* $Q_1$ *and* $Q_2$, *respectively. Also, let* $\alpha_1$ *be equal to* $\alpha$ *restricted to the line* $\{i, j, k\}$. *Then* $3 \otimes_{\alpha_1} C_I = (C_I \cup C_2 \cup C_3; *)$, *where* $C_3 = \{c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$, *is a subsquag of* $3 \otimes_\alpha Q_1$, *if and only if* $\{\alpha(i), \alpha(j), \alpha(k)\}$ *is a line in* $T_0$.

**Proof.** Since $C_1 = \{a_i, a_j, a_k\}$ and $C_2 = \{b_i, b_j, b_k\}$ are two subsquags of $Q_1$ and $Q_2$, respectively. Then there is a sub-1-factorization $f = \{f_i \subseteq F_i, f_j \subseteq F_j, f_k \subseteq F_k\}$ on $C_1 \cup C_2$.

Let $3 \otimes_{\alpha_1} C_1 = (C_I \cup C_2 \cup C_3; *)$ be a subsquag of $3 \otimes_\alpha Q_1$. According to the definition of $3 \otimes_{\alpha_1} C_1$, then $f_i$ related to $c_{\alpha(i)}$, $f_j$ related to $c_{\alpha(j)}$ and $f_k$ related to $c_{\alpha(k)}$, so the set $C_3 = \{c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ must be a block in $T_3$, hence $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in $T_0$.

On the other direction, if $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in $T_0$, then $C_3 = \{c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ is a block in $T_3$. The sub-1-factorization $f = \{f_i = \{a_i b_i, a_j b_k, a_k b_j\} \subseteq F_i, f_j = \{a_j b_j, a_i b_k, a_k b_i\} \subseteq F_j, f_k = \{a_k b_k, a_i b_j, a_j b_i\} \subseteq F_k\}$ forms with the block $C_3 = \{c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ the subsquag $3 \otimes_{\alpha_1} C_1 = (C_I \cup C_2 \cup C_3; *)$. □

The next lemma shows that the converse of the above lemma is also true.

**Lemma 10.** *Let* $S$ *be a sub-*SQ(9) *of* $3 \otimes_\alpha Q_1$, *then there is a sub-1-factorization of* $F$ *on two sub-*SQ(3)s $C_1 = \{a_i, a_j, a_k\} \in T_1$ *and* $C_2 = \{b_i, b_j, b_k\} \in T_2$ *and a sub-*SQ(3) $C_3 = \{c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\} \in T_2$ *satisfying* $S = 3 \otimes_{\alpha_1} C_1 = (C_I \cup C_2 \cup C_3; *)$ *such that* $\{\alpha(i), \alpha(j), \alpha(k)\}$ *is a line of* $T_0$. *Here* $\alpha_1$ *is equal to the permutation* $\alpha$ *restricted to the line* $\{i, j, k\}$ *and the binary operation " * " is the same binary operation of* $3 \otimes_\alpha Q_1$.

**Proof .** According to Lemma 2, we may say that $S \cap L_1 = C_1 = \{a_i, a_j, a_k\}$, $S \cap L_2 = C_2 = \{b_i, b_j, b_k\}$ and $S \cap L_3 = C_3 = \{c_{i'}, c_{j'}, c_{k'}\}$ are three sub-SQ(3)s. Also, there is a sub-1-factorization $f = \{f_i = \{a_i b_i, a_j b_k, a_k b_j\} \subseteq F_i, f_j = \{a_j b_j, a_i b_k, a_k b_i\} \subseteq F_j, f_k = \{a_k b_k, a_i b_j, a_j b_i\} \subseteq F_k\}$ on $C_1 \cup C_2$ forms with the block $C_3 = = \{c_{i'}, c_{j'}, c_{k'}\}$ a sub-SQ(9). According to the construction $3 \otimes_\alpha Q_1$ we have: $f_i$ related with $c_{\alpha(i)}$, $f_j$ related with $c_{\alpha(j)}$ and $f_k$ related with $c_{\alpha(k)}$, so $i' = \alpha(i)$, $j' = \alpha(j)$ and $k' = \alpha(k)$. Since $S$ is a sub-SQ(9), then $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines in $T_0$. According to the definition of the set of blocks $T_{123}$ and by using the sub-1-factorization $f$, then the subsquag $S$ can be represented by the construction $S = 3 \otimes_{\alpha_1} C_1 = (C_1 \cup C_2 \cup C_3; *)$. The proof is complete. □

This means that the only possible sub-SQ(9)s of the squag $3 \otimes_\alpha Q_1$ are determined by the set of elements $S = \{a_i, a_j, a_k, b_i, b_j, b_k, c_{(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ such that $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\} \in T_0$ and the intersection $Q_i \cap S$ is always a sub-SQ(3) for $i = 1, 2$ and 3.
In the following section we will construct all possible subdirectly irreducible squags of cardinality 21 each of whose proper homomorphic images are entropic.

**4. Subdirectly irreducible squags SQ(21)s with all possible subsquags**

The STS(21) associated with the squag SQ(7) × SQ(3) has exactly three sub-STS(7) and 7 sub-STS(9)s. On the other hand, the planar STS(21) has no nontrivial subsystems. Any squag of cardinality 21 has at most three disjoint subsquags of cardinality 7 and $n$ subsquags of cardinality 9 ($n = 0, 1, 3, 7$). For $n = 7$, we will show that, if an SQ(21) contains three disjoint sub-SQ(7)s and 7 sub-SQ(9)s, then this squag is isomorphic to the direct product SQ(7) × SQ(3).

A nonsimple subdirectly irreducible SQ(21) has exactly one proper congruence. Except for the subdirectly irreducible SQ(21)s having a normal sub-SQ(3) all the other nonsimple subdirectly irreducible SQ(21)s have exactly three disjoint normal sub-SQ(7)s (equivalently, exactly one proper congruence) and $n$ sub-SQ(9)s for $n = 0, 1, 3$. The proper homomorphic images of these SQ(21)s are entropic (more precisely ≅ SQ(3)). This means that these SQ(21)s satisfy the expectation of Quackenbush given in [12]. Example for each case will be given in section 5.

**Theorem 11.** *The constructed sloop* $3 \otimes_\alpha Q_1 = (P = P_1 \cup P_2 \cup P_3;\ *\ )$ *is isomorphic to the direct product of the subsquag* SQ(7) = $Q_1$ *and the* 3*-element squag* SQ(3) *if and only if* $3 \otimes_\alpha Q_1$ *has* 7 *sub-*SQ(9)s, *otherwise* $3 \otimes_\alpha Q_1$ *is nonsimple subdirectly irreducible. Moreover, the constructed sloop* $3 \otimes_\alpha Q_1$ *has exactly n subsquags of cardinality* 9 *if and only if the permutation* $\alpha$ *transfers n lines into n lines of* $T_0$ *for* $n = 0, 1, 3, 7$, *where* $T_0$ *is the set of lines of the projective plane over* GF(2).

**Proof.** Let $3 \otimes_\alpha Q_1$ have 7 sub-SQ(9)s, according to Lemmas 8 and 9 each sub-SQ(9) is determined by the 9-element subset $=\{a_i, a_j, a_k, b_i, b_j, b_k, c_{(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ such that $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\} \in T_0$. This means that $\alpha(T_0) := \{\{\alpha(i), \alpha(j), \alpha(k)\}$: for all $\{i, j, k\} \in T_0\} = T_0$. Consider the map $\varphi$ from $3 \otimes_\alpha Q_1$ to the direct product $Q_1 \times I_3$ by $\varphi(a_i) = (a_i, 0)$, $\varphi(b_i) = (a_i, 1)$ and $\varphi(c_i) = (a_{\alpha^{-1}(i)}, 2)$. It is routine matter to proof that $\varphi(x_1 * x_2) = \varphi(x_1) * \varphi(x_2)$ if $\{x_1, x_2\} \subseteq P_i$ for $i = 1, 2$ or 3. Also, if $\{a_i, b_j, c_k\}$ is a block in $T_{123}$, then the edge $a_i\ b_j \in F_h$ for some $h$, hence $\{i, j, h\} \in T_0$, so $\{a_i, b_j, c_{\alpha(h)}\}$ is a block in $T_{123}$, so $\alpha(h) = k$, hence $\varphi\{a_i, b_j, c_k\} = \{(a_i, 0), (a_j, 1), (a_{\alpha^{-1}(k)}, 2)\}$ is a block in $Q_1 \times I_3$; i. e. $\varphi$ is an isomorphism.

Since $3 \otimes_\alpha Q_1$ has three disjoint subsquags of cardinality 7, then $Q_i$ is a normal subsquag for $i = 1, 2$ and 3, so $3 \otimes_\alpha Q_1$ is not simple. Another possible normal subsquag is the 3-element subsquag $SQ(3) = I_3$ with $|I_3 \cap Q_i| = 1$ for each $i$. But if $3 \otimes_\alpha Q_1$ contains a normal sub-SQ(3), then $3 \otimes_\alpha Q_1$ is isomorphic to the direct product $SQ(7) \times SQ(3)$. Therefore, if $3 \otimes_\alpha Q_1$ has $n$ subsquags with $n < 7$, then the congruence lattice of $3 \otimes_\alpha Q_1$ has exactly one proper atom that is the congruence determined by the three disjoint SQ(7)s $Q_1 = (P_1; *)$, $Q_2 = (P_2; *)$ and $Q_3 = (P_3; *)$. Hence $3 \otimes_\alpha Q_1$ is always subdirectly irreducible for each possible $n < 7$.
Let $\alpha$ transfer the line $\{i, j, k\} \in T_0$ into the line $\{\alpha(i), \alpha(j), \alpha(k)\} \in T_0$. Now, according to Lemmas 8 and 9, we may directly say that the set of elements $S = \{a_i, a_j, a_k, b_i, b_j, b_k, c_{(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ forms a subsquag. Since $\alpha$ is permutation on the set of points $P_0 = \{1, 2, \dots, 7\}$ of the projective plane over GF(2), the possible values for the number of lines transferred into lines are $n = 0, 1, 3$, or 7. □

## 5. Examples of subdirectly irreducible SQ(21)s

In this section we describe how one can construct an example for each class of SQ(21). Let $P_0 = \{1, 2, 3, 4, 5, 6, 7\}$ and $T_0 := (P_0; T_0)$ be an STS(7) where $T_0 = \{124, 156, 137, 235, 267, 346, 457\}$. Similarly as in section 3.1, let $T_i = (P_i; T_i)$ be there STS(7)s isomorphic to $T_0$ for $i = 1, 2$ and 3, and $Q_i = (P_i; *_i)$ be the corresponding squags, where $P_1 = \{a_1, \dots, a_7\}$, $P_2 = \{b_1, \dots, b_7\}$ and $P_3 = \{c_1, \dots, c_7\}$ such that $P_1 \cap P_2 \cap P_3 = \varnothing$. Consider the set of 1-factors defined by $F_i = \{a_l\, b_k : a_l *_1 a_k = a_i,\ \{a_i, a_l, a_k\} \subseteq P_1$ and $\{b_i, b_l, b_k\} \subseteq P_2\}$, then the class $F = \{F_1, F_2, \dots, F_7\}$ forms a 1-factorization of the complete bipartite graph $K_{7,7}$ on $P_1 \cup P_2$ the two disjoint sets of vertices $P_1$ and $P_2$ [1, 12]. The constructed $STS(21) = (P; T)$ and the associated squag $SQ(21) = (P; *)$ will be denoted by $3 \otimes_\alpha T_1$ and $3 \otimes_\alpha Q_1$, respectively. Note that $P = P_1 \cup P_2 \cup P_3$ and the set of triples $T = T_1 \cup T_2 \cup T_3 \cup T_{123}$, where $T_{123} = \{\{a_i, b_j, c_{\alpha(k)}\} : a_i\, b_j \in F_k\}$.

The $SQ(21) = 3 \otimes_\alpha Q_1$ has three disjoint sub-SQ(7)s for each permutation $\alpha$, so $Q_1$, $Q_2$ and $Q_3$ are always normal in $3 \otimes_\alpha Q_1$. Namely, $Q_1$, $Q_2$ and $Q_3$ form the unique proper congruence of $3 \otimes_\alpha Q_1$ for each $\alpha$.

For each block $\{i, j, k\} \in T_0$, we have the sub-1-factorizations:
$f = \{f_i = \{a_i\, b_i, a_j\, b_k, a_k\, b_j\} \subseteq F_i, f_j = \{a_j\, b_j, a_i\, b_k, a_k\, b_i\} \subseteq F_j, f_k = \{a_k\, b_k, a_i\, b_j, a_j\, b_i\} \subseteq F_k\}$ on $C_1 \cup C_2$ for all $\{i, j, k\} \in T_0$.

We may consider $T_0 = \{124, 156, 137, 235, 267, 346, 457\}$ the set of lines of the projective planar over GF(2) with the set of points $P_0 = \{1, 2, \dots, 7\}$.

**nonsimple SQ(21)s:**

The following 4 examples supplies us with an example for each class of nonsimple SQ(21)s given in section 4. Notice that $\{1, 2, 4\}$ is a line in $T_0$, we choose a permutation $\alpha$ satisfying that $\alpha\{1, 2, 4\} = \{1, 2, 4\}$ in the first three cases as follows:

1. $\alpha_1 = id_{P_0}$; i. e., $\alpha_1$ transfers each line into the same line in $T_0$. The constructed SQ(21) = $3 \otimes_{\alpha_1} Q_1$ has 7 sub-SQ(9)s and three disjoint sub-SQ(7). According to Theorem 10, we may say that $3 \otimes_{\alpha_1} Q_1$ is isomorphic to SQ(7) × SQ(3).
2. $\alpha_2 = (12)$; i. e., $\alpha_2$ transfers only 3 lines into lines, namely, the set of lines {124, 346, 457} is transferred to itself. The constructed SQ(21) = $3 \otimes_{\alpha_2} Q_1$ is subdirectly irreducible having three disjoint sub-SQ(7)s and 3 sub-SQ(9)s.
3. $\alpha_3 = (124)$; i. e., $\alpha_3$ transfers only one line into itself that is the line 124. The constructed SQ(21) = $3 \otimes_{\alpha_3} Q_1$ is subdirectly irreducible having 3 disjoint sub-SQ(7)s and one sub-SQ(9).
4. $\alpha_4 = (1234)$; i. e., $\alpha_4$ transfers no line into a line. The constructed SQ(21) = $3 \otimes_{\alpha_4} Q_1$ is subdirectly irreducible having three disjoint sub-SQ(7)s, but no sub-SQ(9)s. In fact, the STS(21) associated with $3 \otimes_{\alpha_4} Q_1$ has exactly three disjoint sub-STS(7)s, but no sub-STS(9)s. This means that each triangle in the corresponding sloop SL(22) either generates the whole SL(22) or a subsloop of cardinality 8. Accordingly, the sloops SL(22)s associated with these squags supplies us with semi-planar sloops of cardinality 22. We note that the two smallest well-known cardinalities of non-planar semi-planar sloops are 20 and 28 (cf. [2]).

The three sub-SQ(7)s mentioned in the above examples are $Q_1$, $Q_2$ and $Q_3$ and the sub-SQ(9)s are determined by the sets $\{a_i, a_j, a_k, b_i, b_j, b_k, c_{\alpha(i)}, c_{\alpha(j)}, c_{\alpha(k)}\}$ such that $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines in $T_0$.

In fact, each nonsimple subdirectly irreducible SQ(21) has exactly one proper homomorphic image. The constructed SQ(21)s are subdirectly irreducible having exactly one proper homomorphic image isomorphic to the SQ(3). According to Theorem 7, a natural question at this point, to find a construction for subdirectly irreducible SQ(21)s having exactly one proper homomorphic image isomorphic to the SQ(7) (SQ(21)s having 7 sub-SQ(9)s, but no 3 disjoint sub-SQ(7)s).

**Simple SQ(21)s:**

An SQ(21) has a proper congruence if it has 3 disjoint sub-SQ(7)s or 7 sub-SQ(9)s, otherwise it is simple. We note that there is a planar SQ(21) [7] (having neither sub-SQ(7)s nor sub-SQ(9)s). Each planar SQ(21) is simple [12].

The construction given in [4] supplies us with a simple SQ($3n$) having sub-SQ(9)s as the only proper subsquags. By applying this construction for $n = 7$ we get a simple SQ(21) having 3 sub-SQ(9)s but no sub-SQ(7)s [5]. Using the interchange property on a sub-SQ(9) we will get a simple SQ(21) having one sub-SQ(9) and no sub-SQ(7)s.

The construction given in [3] gives an SQ(3*n*) having only one subsquag of cardinality *n*.. By applying this construction for $n = 7$ we get a simple SQ(21) having one sub-SQ(7) and 3 sub-SQ(9)s. Using the interchange property on the sub-SQ(7) we will get two different classes of simple SQ(21)s, the first is a simple SQ(21) with one sub-SQ(7) and one sub-SQ(9) and the second is a simple SQ(21) with one sub-SQ(7) and no sub-SQ(9)s.

Lemma 5 tells us that each two sub-SQ(9)s are intersected in a sub-SQ(3), also, if the SQ(21) contains a sub-SQ(9) and a sub-SQ(7), then the intersection between them is a sub-SQ(3). We may say that there is an SQ(21) with one sub-SQ(7) [1] and an SQ(21) with three disjoint sub-SQ(7)s [1, 11], i.e., there is no SQ(21)s with exactly two disjoint sub-SQ(7)s. Consequently, we are faced with the question: Is there an SQ(21) having two (or more) intersected sub-SQ(7)s?

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