A NOTE ON q-BERNOULLI NUMBERS AND q-BERNSTEIN POLYNOMIALS

TAEKYUN KIM, CHEON SEOUNG RYOO, AND HEUNGSU YI

ABSTRACT. The purpose of this paper is to investigate some properties of several q-Bernstein type polynomials to express the bosonic p-adic q-integral of those polynomials on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be regarded as either a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we usually assume that $|1-q|_p < 1$. In this paper we define the *q-number of x* by

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}.$$

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *bosonic p-adic q-integral on $\mathbb{Z}_p$* is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [2-5])}. \tag{1}$$

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In [2], the *Carlitz's q-Bernoulli numbers* are inductively defined by

$$\beta_{0,q} = 1, \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{2}$$

with the usual convention of replacing β^i with $\beta_{i,q}$.

The *Carlitz's q-Bernoulli polynomials* are also defined by

$$\beta_{k,q}(x) = (q^x\beta + [x]_q)^k = \sum_{i=0}^{k} \binom{k}{i} q^{ix}\beta_{i,q}[x]_q^{k-i}. \tag{3}$$

In [2], Kim proved that the Carlitz q-Bernoulli numbers and polynomials are represented by p-adic q-integral as follows: for $n \in \mathbb{Z}_+$,

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x), \quad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y). \tag{4}$$

The *Kim's q-Bernstein polynomials* are defined by

$$\widetilde{B}_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \text{ (see [1-8])}, \tag{5}$$

where $n, k \in \mathbb{Z}_+$, and $x \in [0,1]$.

Let f be continuous functions on $[0,1]$. Then the *Kim's q-Bernstein operator of order n for f* is defined by

$$\widetilde{\mathbb{B}}_{n,q}(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \widetilde{B}_{k,n}(x,q), \text{ (see [5])}.$$

In this paper, we consider the p-adic analogue of the extended Kim's q-Bernstein polynomials on $\mathbb{Z}_p$ and investigate some properties of several extended Kim's q-Bernstein polynomials to express the bosonic p-adic q-integral of those polynomials.

2. Extended q-Bernstein Polynomials

In this section we assume that $q \in \mathbb{R}$ with $0 < q < 1$. Let $C[0,1]$ be the set of continuous function on $[0,1]$.

For $f \in C[0,1]$, we consider the *extended Kim's q-Bernstein operator of order n* as follows:

$$\widetilde{\mathbb{B}}_{n,q}(f \mid x_1, x_2) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}$$
$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\widetilde{B}_{k,n}(x_1, x_2 \mid q). \tag{6}$$

For $n, k \in \mathbb{Z}_+$, and $x_1, x_2 \in [0,1]$, the *extended Kim's q-Bernstein polynomials of degree* n are defined by

$$\widetilde{B}_{k,n}(x_1, x_2 \mid q) = \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}. \tag{7}$$

In the special case $x_1 = x_2 = x$, then $\widetilde{B}_{k,n}(x_1, x_2 \mid q) = \widetilde{B}_{k,n}(x, q)$.

From (6) and (7) we can derive the generating function for $\widetilde{B}_{k,n}(x_1, x_2 \mid q)$ as follows:

$$F_q^{(k)}(x_1, x_2 \mid t) = \frac{(t[x_1]_q)^k \exp(t[1-x_2]_{q^{-1}})}{k!}, \tag{8}$$

where $k \in \mathbb{Z}_+$ and $x_1, x_2 \in [0,1]$.

By (8), we get

$$F_q^{(k)}(x_1, x_2 \mid t) = \sum_{n=0}^{\infty} \frac{[x_1]_q^k[1-x_2]_{q^{-1}}^n}{k!n!} t^{n+k}$$
$$= \sum_{n=k}^{\infty} \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}\frac{t^n}{n!} \tag{9}$$
$$= \sum_{n=k}^{\infty} \widetilde{B}_{k,n}(x_1, x_2 \mid q)\frac{t^n}{n!}.$$

Thus, we have

$$\widetilde{B}_{k,n}(x_1, x_2 \mid q) = \begin{cases} \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k, \end{cases}$$

for $n, k \in \mathbb{Z}_+$.

It is easy to check that

$$\widetilde{B}_{n-k,n}(1-x_2, 1-x_1 \mid q^{-1}) = \widetilde{B}_{k,n}(x_1, x_2 \mid q) \tag{10}$$

and that for $0 \le k \le n$,

$$\begin{aligned}
[1-x_2]_{q^{-1}}\widetilde{B}_{k,n-1}(x_1,x_2 \mid q) &+ [x_1]_q\widetilde{B}_{k-1,n-1}(x_1,x_2 \mid q) \\
&= [1-x_2]_{q^{-1}}\binom{n-1}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k-1} \\
&\quad + [x_1]_q\binom{n-1}{k-1}[x_1]_q^{k-1}[1-x_2]_{q^{-1}}^{n-k} \\
&= \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k} \\
&= \widetilde{B}_{k,n}(x_1,x_2 \mid q).
\end{aligned} \tag{11}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $x_1, x_2 \in [0,1]$ and $n, k \in \mathbb{Z}_+$,*

$$[1-x_2]_{q^{-1}}\widetilde{B}_{k,n}(x_1,x_2 \mid q) + [x_1]_q\widetilde{B}_{k-1,n}(x_1,x_2 \mid q) = \widetilde{B}_{k,n+1}(x_1,x_2 \mid q).$$

It is also easy to see that for $k \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x_1, x_2 \in [0,1]$,

$$\frac{\partial}{\partial x_1}\widetilde{B}_{k,n}(x_1,x_2 \mid q) = \frac{\log q}{q-1}q^{x_1}n\widetilde{B}_{k-1,n-1}(x_1,x_2 \mid q)$$

and

$$\frac{\partial}{\partial x_2}\widetilde{B}_{k,n}(x_1,x_2 \mid q) = \frac{\log q}{1-q}q^{x_2}n\widetilde{B}_{k,n-1}(x_1,x_2 \mid q).$$

These show that the partial derivatives of $\widetilde{B}_{k,n}(x_1,x_2 \mid q)$ are also q-polynomials of degree $n-1$. Therefore, we obtain the following lemma.

Lemma 2. *For $k \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x_1, x_2 \in [0,1]$,*

$$\frac{\partial}{\partial x_1}\widetilde{B}_{k,n}(x_1,x_2 \mid q) = \frac{\log q}{q-1}n\{(q-1)[x_1]_q\widetilde{B}_{k-1,n-1}(x_1,x_2 \mid q)+\widetilde{B}_{k-1,n-1}(x_1,x_2 \mid q)\}$$

and

$$\frac{\partial}{\partial x_2}\widetilde{B}_{k,n}(x_1,x_2 \mid q) = \frac{\log q}{1-q}n\{(q-1)[x_2]_q\widetilde{B}_{k,n-1}(x_1,x_2 \mid q)+\widetilde{B}_{k,n-1}(x_1,x_2 \mid q)\}.$$

If $f = 1$, then we get from (6)

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(1 \mid x_1, x_2) &= \sum_{k=0}^{n} \widetilde{B}_{k,n}(x_1, x_2 \mid q) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k} \\ &= (1 + [x_1]_q - [x_2]_q)^n,\end{aligned} \tag{12}$$

where $n \in \mathbb{N}$ and $x_1, x_2 \in [0, 1]$. Therefore we have

$$\frac{1}{(1 + [x_1]_q - [x_2]_q)^n} \widetilde{\mathbb{B}}_{n,q}(1 \mid x_1, x_2) = 1.$$

If $f(t) = t$, we also get from (6) that for $n \in \mathbb{N}$ and $x_1, x_2 \in [0, 1]$,

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(t \mid x_1, x_2) &= \sum_{k=0}^{n} \left(\frac{k}{n}\right) [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k} \binom{n}{k} \\ &= \sum_{k=1}^{n} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k} \binom{n-1}{k-1} \\ &= [x_1]_q \sum_{k=0}^{n-1} \binom{n-1}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k-1}.\end{aligned}$$

Thus, we have

$$\frac{1}{(1 + [x_1]_q - [x_2]_q)^{n-1}} \widetilde{\mathbb{B}}_{n,q}(t \mid x_1, x_2) = [x_1]_q.$$

Note also that if $f(t) = t^2$, then we get from (6)

$$\begin{aligned}&\widetilde{\mathbb{B}}_{n,q}(t^2 \mid x_1, x_2) \\ &= \frac{n-1}{n} [x_1]_q^2 (1 + [x_1]_q - [x_2]_q)^{n-2} + \frac{[x_1]_q}{n} (1 + [x_1]_q - [x_2]_q)^{n-1},\end{aligned}$$

where $n \in \mathbb{N}$ and $x_1, x_2 \in [0, 1]$.

In the special case of $x_1 = x_2 = x$,

$$\widetilde{\mathbb{B}}_{n,q}(t^2 \mid x_1, x_2) = \widetilde{\mathbb{B}}_{n,q}(t^2 \mid x, x) = \frac{n-1}{n} [x]_q^2 + \frac{[x]_q}{n}. \tag{13}$$

Notice from (13) that

$$\lim_{n \to \infty} \widetilde{\mathbb{B}}_{n,q}(t^2 \mid x, x) = [x]_q^2.$$

We see from (6) that for $n \in \mathbb{N}$ and $x_1, x_2 \in [0,1]$,

$$\begin{aligned}
\widetilde{\mathbb{B}}_{n,q}(f \mid x_1, x_2) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \widetilde{B}_{k,n}(x_1, x_2 \mid q) \\
&= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x_1]_q^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j [x_2]_q^j \\
&= \sum_{l=0}^{n} \binom{n}{l} [x_2]_q^l \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} f\left(\frac{k}{n}\right) \left(\frac{[x_1]_q}{[x_2]_q}\right)^k .
\end{aligned}$$

Also, we have from the definition of $\widetilde{B}_{k,n}(x_1, x_2 \mid q)$ that for $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $x_1, x_2 \in [0,1]$,

$$\begin{aligned}
&\frac{n-k}{n} \widetilde{B}_{k,n}(x_1, x_2 \mid q) + \frac{k+1}{n} \widetilde{B}_{k+1,n}(x_1, x_2 \mid q) \\
&= \frac{(n-1)!}{k!(n-k-1)!} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x_1]_q^{k+1} [1-x_2]_{q^{-1}}^{n-k-1} \\
&= ([x_1]_q + [1-x_2]_{q^{-1}}) \widetilde{B}_{k,n-1}(x_1, x_2 \mid q) \\
&= ([x_1]_q + 1 - [x_2]_q) \widetilde{B}_{k,n-1}(x_1, x_2 \mid q).
\end{aligned} \tag{14}$$

We note from the binomial theorem that

$$\widetilde{B}_{k,n}(x_1, x_2 \mid q) = \left(\frac{[x_1]_q}{[x_2]_q}\right)^k \sum_{l=k}^{n} \binom{l}{k} \binom{n}{l} (-1)^{l-k} [x_2]_q^l .$$

It is possible to write $[x_1]_q^k$ as a linear combination of $\widetilde{B}_{k,n}(x_1, x_2 \mid q)$ by using the degree evaluation formulae and mathematical induction:

$$\frac{1}{(1+[x_1]_q - [x_2]_q)^{n-1}} \sum_{k=1}^{n} \frac{\binom{k}{1}}{\binom{n}{1}} \widetilde{B}_{k,n}(x_1, x_2 \mid q) = [x_1]_q .$$

By the same method, we get

$$\frac{1}{(1+[x_1]_q - [x_2]_q)^{n-2}} \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} \widetilde{B}_{k,n}(x_1, x_2 \mid q) = [x_1]_q^2 .$$

Continuing this process, we obtain the following theorem.

Theorem 3. *For $j \in \mathbb{Z}_+$ and $x_1, x_2 \in [0,1]$,*

$$\frac{1}{(1+[x_1]_q - [x_2]_q)^{n-j}} \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} \widetilde{B}_{k,n}(x_1, x_2 \mid q) = [x_1]_q^j .$$

We get from Theorem 3 that

$$\frac{1}{(1+[x_1]_q-[x_2]_q)^{n-j}}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}\widetilde{B}_{k,n}(x_1,x_2\mid q)=\sum_{k=0}^{j}q^{\binom{k}{2}}\binom{x_1}{k}_q[k]_q!S_q(k,j-k),$$

where $[k]_q!=[k]_q[k-1]_q\cdots[2]_q[1]_q$ and $S_q(k,j-k)$ is the *q-Stirling numbers of the second kind.*

3. q-Bernstein Polynomials associated with the bosonic p-adic q-integral on $\mathbb{Z}_p$.

In this section we assume that $q\in\mathbb{C}_p$ with $|1-q|_p<1$. We easily get from (1) that for $n\in\mathbb{Z}_+$,

$$\int_{\mathbb{Z}_p}[1-x+x_1]_{q^{-1}}^n d\mu_{q^{-1}}(x_1)=(-1)^n q^n\int_{\mathbb{Z}_p}[x+x_1]_q^n d\mu_q(x_1). \tag{15}$$

We also get from (4) and (15) that for $n\in\mathbb{Z}_+$,

$$\beta_{n,q^{-1}}(1-x)=(-1)^n q^n\beta_{n,q}(x). \tag{16}$$

Hence we get from (2), (3) and (16) that if $n>1$, then

$$q^2\beta_{n,q}(2)-(n+1)q^2+q=q(q\beta+1)^n=\beta_{n,q}.$$

Thus, we have

$$\beta_{n,q}(2)=(n+1)-\frac{1}{q}+\frac{1}{q^2}\beta_{n,q}. \tag{17}$$

Also, it is easy to see that

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n d\mu_q(x)=(-1)^n q^n\beta_{n,q}(-1)=\beta_{n,q^{-1}}(2).$$

Therefore we get the following equation (18) from (15), (16) and (17): if $n>1$, then

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n d\mu_q(x)=q^2\beta_{n,q^{-1}}+(n+1)-q. \tag{18}$$

Taking double bosonic p-adic q-integral on $\mathbb{Z}_p$, we get from (18) that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\widetilde{B}_{k,n}(x_1,x_2\mid q)d\mu_q(x_1)d\mu_q(x_2)\\&=\binom{n}{k}\int_{\mathbb{Z}_p}[x_1]_q^k d\mu_q(x_1)\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_q(x_2).\end{aligned} \tag{19}$$

Thus, we obtain the following theorem.

Theorem 4. *For $x_1, x_2 \in [0,1]$ and $n, k \in \mathbb{Z}_+$,*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2)$$
$$= \begin{cases} \binom{n}{k}\beta_{k,q}(q^2\beta_{n-k,q^{-1}} + (n-k+1) - q), & \text{if } n > k+1 \\ (k+1)\beta_{k+1,q}\beta_{1,q^{-1}}(2), & \text{if } n = k+1 \\ 0, & \text{if } n < k \\ \beta_{k,q}, & \text{if } n = k \\ 1, & \text{if } n = k = 0 \end{cases}$$

We get from the q-symmetric properties of the q-Bernstein polynomials, (10) that for $n, k \in \mathbb{Z}_+$,

$$\begin{aligned} &\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2) \\ &= \sum_{l=0}^{k}\binom{n}{n-k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{k-l}[1-x_2]_{q^{-1}}^{n-k}d\mu_q(x_1)d\mu_q(x_2) \\ &= \binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_q(x_2)\Big\{1 - k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_q(x_1) \\ &\qquad + \sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{k-l}d\mu_q(x_1)\Big\}. \end{aligned} \tag{20}$$

We also get from (20) that for $n, k \in \mathbb{Z}_+$,

$$\begin{aligned} &\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2) \\ &= \binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_q(x_2)\Big\{\Big(1 - k - \frac{k}{[2]_q}\Big) \\ &\qquad + \sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k+l}(q^2\beta_{k-l,q^{-1}} + k - l + 1 - q)\Big\}. \end{aligned} \tag{21}$$

Therefore we obtain the following theorem by (19) and (21).

Theorem 5. *For $k \in \mathbb{Z}_+$ with $k \geq 2$,*

$$\beta_{k,q} = \Big(1 - k - \frac{k}{[2]_q}\Big) + \sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k+l}(q^2\beta_{k-l,q^{-1}} + k - l + 1 - q).$$

Note that for $m, n, k \in \mathbb{Z}_+$,

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}(x_1, x_2 \mid q)\widetilde{B}_{k,m}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2)$$
$$= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x_1]_q^{2k}d\mu_q(x_1)\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_q(x_2) \tag{22}$$
$$= \binom{n}{k}\binom{m}{k}\beta_{2k,q}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_q(x_2).$$

Note also from (10) that for $m, n, k \in \mathbb{Z}_+$,

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}(x_1, x_2 \mid q)\widetilde{B}_{k,m}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2)$$
$$= \sum_{l=0}^{2k}\binom{n}{k}\binom{m}{k}\binom{2k}{l}(-1)^{2k+l}\Big\{\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}d\mu_q(x_1)$$
$$\times\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_q(x_2)\Big\}$$
$$= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_q(x_2)\Big\{1-2k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_q(x_1)$$
$$+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k+l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}d\mu_q(x_1)\Big\}. \tag{23}$$

Therefore we see from (22) and (23) that the following theorem holds.

Theorem 6. *For $k \in \mathbb{N}$,*

$$\beta_{k,q} = \left(1-2k-\frac{2k}{[2]_q}\right)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k+l}(q^2\beta_{2k-l,q^{-1}}+2k-l+1-q).$$

Note that for $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ and $s \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\prod_{i=1}^{s}\widetilde{B}_{k,n_i}(x_1, x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2)$$
$$= \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\int_{\mathbb{Z}_p}[x_1]_q^{sk}d\mu_q(x_1)\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk}d\mu_q(x_2) \tag{24}$$
$$= \prod_{i=1}^{s}\binom{n_i}{k}\beta_{sk,q}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk}d\mu_q(x_2).$$

Note also from the binomial theorem that $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ and $s \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\prod_{i=1}^{s}\widetilde{B}_{k,n_i}(x_1,x_2 \mid q)d\mu_q(x_1)d\mu_q(x_2)$$
$$=\sum_{l=0}^{sk}\left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\binom{sk}{l}(-1)^{sk+l}\Big\{\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{sk-l}d\mu_q(x_1) \tag{25}$$
$$\times\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+\cdots+n_s-sk}d\mu_q(x_2)\Big\}.$$

Therefore we get from (24) and (25) that

$$\beta_{sk,q}=\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{sk+l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{sk-l}d\mu_q(x_1)$$
$$=1-sk\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_q(x_1) \tag{26}$$
$$+\sum_{l=0}^{sk-2}\binom{sk}{l}(-1)^{sk+l}q\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{sk-l}d\mu_q(x_1).$$

Thus we see from (26) that the following theorem holds.

Theorem 7. *For $s \in \mathbb{N}$ and for $k \in \mathbb{N}$ satisfying $sk \geq 2$,*

$$\beta_{sk,q}=\left(1-sk-\frac{sk}{[2]_q}\right)+\sum_{l=0}^{sk-2}\binom{sk}{l}(-1)^{sk+l}(q^2\beta_{sk-l,q^{-1}}+sk-l+1-q).$$

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DIVISION OF GENERAL EDUCATION, KWANGWOON UNIVERSITY, SEOUL 139-701, KOREA
E-mail address: tkkim@kw.ac.kr

DEPARTMENT OF MATHEMATICS, HANNAM UNIVERSITY, DAEJEON 306-791, KOREA
E-mail address: ryoocs@hnu.kr

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, KOREA
E-mail address: hsyi@kw.ac.kr