# Some ternary codes invariant under the group $M^cL{:}2$

Jamshid Moori*
School of Mathematical Sciences
North-West University (Mafikeng)
Mmabatho 2735, South Africa

B. G. Rodrigues †
School of Mathematical Sciences
University of KwaZulu-Natal
Durban 4041, South Africa

**Abstract**

We examine designs $\mathcal{D}_i$ and ternary codes $C_i$, where $i \in \{112, 113, 162, 163, 274\}$, constructed from a primitive permutation representation of degree 275 of the sporadic simple group $M^cL$. We prove that $\dim(C_{113}) = 22$, $\dim(C_{162}) = 21$, $C_{113} \supset C_{162}$ and $M^cL{:}2$ acts irreducibly on $C_{162}$. Furthermore we have $C_{112} = C_{163} = C_{274} = V_{275}(GF(3))$, $\mathrm{Aut}(\mathcal{D}_{112}) = \mathrm{Aut}(\mathcal{D}_{163}) = \mathrm{Aut}(\mathcal{D}_{113}) = \mathrm{Aut}(\mathcal{D}_{162}) = \mathrm{Aut}(C_{113}) = \mathrm{Aut}(C_{162}) = M^cL{:}2$ while $\mathrm{Aut}(\mathcal{D}_{274}) = \mathrm{Aut}(C_{112}) = \mathrm{Aut}(C_{163}) = \mathrm{Aut}(C_{274}) = S_{275}$. We also determine the weight distributions of $C_{113}$ and $C_{162}$ and that of their duals.

## 1 Introduction

In [8, 7] the authors examined binary codes obtained from some primitive permutation representation of the sporadic simple group $M^cL$ [2], namely on 275 and 2025 points respectively. This paper is concerned with ternary codes obtained from the representation of degree 275 and we prove the

*Support of NRF and North-West University (Mafikeng) are acknowledged.
†Support of the University of KwaZulu-Natal Competitive Grant is acknowledged. The authors wish to thank the Department of Mathematical Science at Clemson University for the computing facilities.

following main result (Theorem 1). The proof of Theorem 1 follows from a series of lemmas in Sections 6 and 7.

**Theorem 1** *Let $G$ be the McLaughlin group* M$^c$L *and $\mathcal{D}_i$ and $C_i$ where $i \in \{112, 113, 162, 163, 274\}$ be the designs and ternary codes constructed from the primitive rank-3 permutation action of $G$ on the cosets of $U_4(3)$. Then the following holds:*

(i) $\text{Aut}(\mathcal{D}_{112}) = \text{Aut}(\mathcal{D}_{113}) = \text{Aut}(\mathcal{D}_{162}) = \text{Aut}(\mathcal{D}_{163}) = \text{Aut}(C_{113}) = \text{Aut}(C_{162}) =$ M$^c$L:2.
(ii) $\dim(C_{113}) = 22$, $\dim(C_{162}) = 21$, $C_{113} \supset C_{162}$ *and* M$^c$L:2 *acts irreducibly on $C_{162}$.*
(iii) $C_{112} = C_{163} = C_{274} = V_{275}(GF(3))$.
(iv) $\text{Aut}(\mathcal{D}_{274}) = \text{Aut}(C_{112}) = \text{Aut}(C_{163}) = \text{Aut}(C_{274}) = S_{275}$.

We also show that the code $C_{162}$ is the 21-dimensional irreducible representation of M$^c$L over $GF(3)$ contained in the 22-dimensional representation of M$^c$L:2 over $GF(3)$ (see [10]). We outline our notation in Section 2, and describe the background results and a construction method in Section 3. A brief overview of the simple sporadic group M$^c$L is given in Section 4, and in Section 5 we describe the construction of the designs and the corresponding ternary codes.

# 2 Terminology and notation

Our notation will be standard, and it is as in [1] and ATLAS [2]. For the structure of groups and their maximal subgroups we follow the ATLAS notation. The groups $G.H$, $G : H$, and $G^{\cdot}H$ denote a general extension, a split extension and a non-split extension respectively. For a prime $p$, the symbol $p^m$ denotes an elementary abelian group of that order. The notation $p_+^{1+2n}$ and $p_-^{1+2n}$ are used for extraspecial groups of order $p^{1+2n}$. If $p$ is an odd prime, the subscript is $+$ or $-$ according as the group has exponent $p$ or $p^2$. For $p = 2$ it is $+$ or $-$ according as the central product has an even or odd number of quaternionic factors.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a $t$-$(v, k, \lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, and every $t$ distinct points are together incident with precisely $\lambda$ blocks. The **complement** of $\mathcal{D}$ is the structure $\tilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \tilde{\mathcal{I}})$, where $\tilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The **dual** structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, P) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is **symmetric** if it has the same number of points and blocks, and **self dual** if it is isomorphic to its dual.

The code $C_F$ of the design $\mathcal{D}$ over the finite field $F$ is the space spanned by the incidence vectors of the blocks over $F$. We take $F$ to be a prime field $F_p$, in which case we write also $C_p$ for $C_F$, and refer to the dimension of $C_p$ as the **$p$-rank** of $\mathcal{D}$. In the general case of a 2-design, the prime must divide the order of the design, i.e. $r - \lambda$, where $r$ is the replication number for the design, that is, the number of blocks through a point. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to $F$. For any code $C$, the **dual** or **orthogonal** code $C^{\perp}$ is the orthogonal subspace under the standard inner product. The **hull** of a design's code over some field is the intersection $C \cap C^{\perp}$. If a linear code over a field of order $q$ is of length $n$, dimension $k$, and minimum weight $d$, then we write $[n, k, d]_q$ to represent this information. A **constant word** in the code is a codeword all of whose coordinate entries are the same. The all-one vector will be denoted by $\jmath$, and is the constant vector of weight the length of the code. Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of $C$.

# 3 Preliminary results

The designs and codes in this paper come from the following standard construction (Result 1), described in [4, Proposition 1] and in [5]. We would like to point out here that Result 1, Result 2(i) and Remark 1 have been modified from the original versions by replacing "orbit" to "self-paired orbit". The self-paired assumption is only required to guarantee that the designs constructed are self-dual[1].

**Result 1** *[4, Proposition 1] Let $G$ be a finite primitive permutation group acting on the set $\Omega$ of size $n$. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be a self-paired orbit of the stabilizer $G_\alpha$ of $\alpha$. If*

$$\mathcal{B} = \{\Delta^g : g \in G\},$$

*then $\mathcal{B}$ forms a self-dual 1-$(n, |\Delta|, |\Delta|)$ design with $n$ blocks, with $G$ acting as an automorphism group on this structure, primitive on the points and blocks of the design.*

[1] We thank the referee for pointing out this error in the original versions of these results.

**Remark 1** *Note that if we form any union of orbits of the stabilizer of a point, including the orbit consisting of the single point, and take its images under the full group, we will still get a symmetric 1-design with the group operating. Thus the orbits of the stabilizer can be regarded as building blocks. Because of the maximality of the point stabilizer, there is only one orbit of length 1: see [4].*

The following result which can be found in [9], deals with the automorphism groups of the designs and codes constructed from a finite primitive permutation group in a manner described in Result 1.

**Result 2** (i) *Let $\mathcal{D}$ be a self-dual* 1*-design obtained by taking all the images under $G$ of a non-trivial self-paired orbit $\Delta$ of the point stabilizer in any of $G$'s primitive representations, and on which $G$ acts primitively on points and blocks, then the automorphism group of $\mathcal{D}$ contains $G$.*

(ii) *If $C$ is a linear code of length $n$ of a symmetric $1-(v,k,k)$ design $\mathcal{D}$ over a finite field $\mathbb{F}_q$, then the automorphism group of $\mathcal{D}$ is contained in the automorphism group of $C$.*

# 4 The M$^c$L group and its automorphism group

We consider $G$ to be the sporadic simple group M$^c$L of McLaughlin. Note that M$^c$L has an involutary outer automorphism, so its automorphism group is a split extension of M$^c$L by $\mathbb{Z}_2$, denoted by M$^c$L:2.

It was shown by McLaughlin [6] that there exists a regular graph $\mathcal{G} = (\Omega, \mathcal{E})$ with 275 vertices possessing a transitive automorphism group $Aut(\mathcal{G}) \cong$ M$^c$L:2, with M$^c$L a simple group of order $2^7 \times 3^6 \times 5^3 \times 7 \times 11$. The McLaughlin graph $\mathcal{G}$ is a rank-3 graph of valency 112 on 275 points. The stabilizer of a point in M$^c$L is a maximal subgroup $U$ isomorphic to $U_4(3)$. The orbits under this action are $\{x\}$, $\Phi$ and $\Psi$ with lengths 1, 112 and 162, respectively. Clearly since these lengths are distinct, the corresponding orbits are self-paired. The action of $U$ on $\Phi$ is equivalent to the representation of $U_4(3)$ on the set of totally singular lines of the 4-dimensional unitary space $V$ over the Galois field $GF(9)$ with the stabilizer of a point having the form $3^4{:}A_6$ and orbits of lengths 1, 30 and 81. The action of $U$ on $\Psi$ is equivalent to the representation of $U_4(3)$ on the left cosets of a subgroup isomorphic to $L_3(4)$ with the stabilizer of a point having orbits of lengths 1, 56 and 105. Thus the two point stabilizers of M$^c$L on $\Omega$ are isomorphic to either $3^4{:}A_6$ or $L_3(4)$. From this we conclude that $U \cap U^g \cong 3^4{:}A_6$ or $L_3(4)$, for any two distinct conjugate subgroups isomorphic to $U_4(3)$.

The group M$^c$L has precisely one conjugacy class of involutions and the centralizer of an involution in $M^cL$ is isomorphic to $2{\cdot}A_8$, the unique perfect central extension of the alternating group $A_8$ by a group of order

2. Finkelstein [3] showed that the proper non-abelian simple subgroups of M$^c$L are isomorphic to $A_5$, $A_6$, $A_7$, $L_2(7)$, $U_4(2)$, $U_3(3)$, $L_3(4)$, $U_3(5)$, $U_4(3)$, $M_{11}$ and $M_{22}$. There are two classes of $M_{22}$ subgroups, interchanged by the outer automorphism.

**Theorem 2** *(Finkelstein [3]) The McLaughlin simple group has precisely twelve conjugacy classes of maximal subgroups. The isomorphism types in these classes are as follows:*

(i) *two groups of classical type, namely,* $U_4(3)$ *and* $U_3(5)$*;*

(ii) *four groups of Mathieu type, namely,* $M_{11}$, $M_{22}$ *(two classes) and* $L_3(4){:}2_2$*, the set stabilizer of two points in the canonical representation of* $M_{23}$*;*

(iii) *six p-local subgroups, namely,* $2^4{:}A_7$ *(two classes),* $2{\cdot}A_8$, $3^4{:}M_{10}$, $3_+^{1+4}{:}2.S_5$ *and* $5_+^{1+2}{:}3{:}8$. ■

# 5 Construction of the designs and codes

Notice from Theorem 2 and the ATLAS (see [2]) that there is just one class of maximal subgroups of M$^c$L of index 275, namely the unitary group $U_4(3)$. The M$^c$L group acts as a rank-3 primitive group of degree 275 on the cosets of $U_4(3)$. The stabilizer of a point in this action is $U_4(3)$, and the orbits of $U_4(3)$ have lengths 1, 112, and 162 respectively.

In this paper, using the construction method outlined in Result 1 and Remark 1, we construct the self-dual symmetric designs $\mathcal{D}_i$ where $i \in \{112, 113, 162, 163, 274\}$, from the orbits of $U_4(3)$ and their respective unions, from the primitive permutation representation of M$^c$L of degree 275. The stabilizer of a point $\alpha$ in this representation is a maximal subgroup isomorphic to $U_4(3)$, producing orbits $\{\alpha\}$, $\Delta_1$, $\Delta_2$ of lengths 1, 112 and 162 respectively. The self-dual symmetric 1-designs $\mathcal{D}_i$ are constructed from the sets $\Delta_1$, $\{\alpha\} \cup \Delta_1$, $\Delta_2$, $\{\alpha\} \cup \Delta_2$, and $\Delta_1 \cup \Delta_2$, respectively. Notice that the orbit containing a single element, has been excluded, as it would produce a trivial design. The ternary codes $C_i$ whose properties we will be examining are the ternary span of the rows of the incidence matrices of the designs $\mathcal{D}_i$. In Sections 6 and 7 we deal with these designs and respective ternary codes.

# 6 $\mathcal{D}_{113}$, $C_{113}$, $\mathcal{D}_{162}$ and $C_{162}$

We start by taking the union of the orbit consisting of a single point and the orbit of length 112, namely $\{\alpha\} \cup \Delta_1$. By taking its images under M$^c$L we get the blocks of a self-dual symmetric 1-$(275, 113, 113)$ design which we denote by $\mathcal{D}_{113}$. Similarly $\Delta_2$ produces a self-dual symmetric 1-(275, 162, 162) design $\mathcal{D}_{162}$. Lemma 3 below deals with these designs and in Lemma 4 we show that M$^c$L:2 is the automorphism group of their associated ternary codes $C_{113}$ and $C_{162}$, and we also examine some of the properties of these codes.

**Lemma 3** Aut$(\mathcal{D}_{113})$ *and* Aut$(\mathcal{D}_{162})$ *are isomorphic to M$^c$L:2.*

**Proof:** Let $\overline{G} = \text{Aut}(\mathcal{D}_{113})$. Then M$^c$L is contained in $\overline{G}$ by Result 2(i). By Magma computations (see Subsection 8.1), $|\overline{G}| = 2 \times |\text{M}^c\text{L}|$. Also Magma computations shows that $\overline{G}$ is generated by the permutations which we denote $x, y$ and there is a permutation $\alpha$ of order 2 in $\overline{G} - \text{M}^c\text{L}$ of cycle type $1^{11}2^{132}$ (see Subsection 8.2). Hence $\overline{G} = \text{M}^c\text{L} : \langle\alpha\rangle$. Furthermore since $\mathcal{D}_{162} = \tilde{\mathcal{D}}_{113}$, we have $\text{Aut}(\mathcal{D}_{162}) = \text{Aut}(\tilde{\mathcal{D}}_{113}) = \text{Aut}(\mathcal{D}_{113})$. ■

**Lemma 4** *$C_{113}$ is a $[275, 22, 113]_3$ code, ${C_{113}}^{\perp}$ is a $[275, 253, 6]_3$ code, $C_{162}$ is a $[275, 21, 114]_3$ code and ${C_{162}}^{\perp}$ is a $[275, 254, 5]_3$ code. Moreover $\jmath$ is in $C_{113}$ and ${C_{162}}^{\perp}$, $C_{162}$ is self-orthogonal and is contained in $C_{113}$. Also* Aut$(C_{113})$ *and* Aut$(C_{162})$ *are isomorphic to M$^c$L:2 and M$^c$L:2 acts irreducibly on $C_{162}$.*

**Proof:** The statements given in the first sentence are verified entirely by using the Magma programme in 8.1. Let $\text{Aut}(C_{113}) = \Gamma$. Then by Result 2(ii) and Lemma 3, we have that $\text{M}^c\text{L:2} \subseteq \Gamma$. Our computations show that $|\Gamma| = 1796256000 = |\text{M}^c\text{L:2}|$ and hence $\Gamma = \text{M}^c\text{L:2}$.

Since $\mathcal{D}_{162}$ is the complement of $\mathcal{D}_{113}$, the difference of any two codewords in $C_{113}$ is in $C_{162}$. As these differences span a subcode of dimension 21 in $C_{113}$, this subcode must be $C_{162}$. The weight distribution of $C_{113}$ and $C_{162}$ are listed in Table I and Table II, respectively. In these tables, $i$ represents the weight of a codeword and $A_i$ denotes the number of codewords of weight $i$.

Self orthogonality of $C_{162}$ follows, since all its weights are divisible by 3. Also $\jmath$ is orthogonal to the codewords corresponding to the blocks of $\mathcal{D}_{162}$, since these codes have weights 162 (divisible by 3). So we have that $\jmath \in {C_{162}}^{\perp}$. Since $\jmath$ can be written, in many ways, as the sum of a codeword in $C_{113}$ and a codeword in $C_{162}$, it is also in $C_{113}$.

If $\gamma \in \text{Aut}(C_{162})$, then since $\gamma(\jmath) = \jmath$ and $C_{113} = \langle C_{162}, \jmath\rangle$, we have $\gamma \in \text{Aut}(C_{113})$. So that $\text{Aut}(C_{162}) \subseteq \text{Aut}(C_{113})$. Now

$$\text{M}^c\text{L:2} = \text{Aut}(\mathcal{D}_{162}) \leq \text{Aut}(C_{162}) \leq \text{Aut}(C_{113}) = \text{M}^c\text{L:2},$$

implies that $\text{Aut}(C_{162}) = \text{M}^c\text{L:2}$.

Finally notice that the group M$^c$L has a unique 21-dimensional irreducible representation over $GF(3)$ contained in the 22-dimensional representation of M$^c$L:2 over $GF(3)$ (see [10]). Using Table II we can easily see that $C_{162}$ does not contain an invariant subspace of dimension 1. Also [10] establishes that M^c^L has no irreducible modules over $GF(3)$ with degrees between 2 and 20. Hence $C_{162}$ is the 21-dimensional $GF(3)$ module on which M^c^L:2 acts irreducibly. ∎

TABLE I
The weight distribution of $C_{113}$

| $i$ | $A_i$ | $i$ | $A_i$ |
|---|---|---|---|
| 0 | 1 | 180 | 1437004800 |
| 113 | 31350 | 182 | 2739290400 |
| 114 | 44550 | 183 | 1392098400 |
| 143 | 2721600 | 185 | 3480246000 |
| 144 | 2494800 | 186 | 1683990000 |
| 149 | 712800 | 188 | 2567149200 |
| 150 | 598752 | 189 | 1181703600 |
| 155 | 26061750 | 191 | 2292364800 |
| 156 | 20047500 | 192 | 1002909600 |
| 161 | 12518550 | 194 | 1087086000 |
| 162 | 8809350 | 195 | 451558800 |
| 164 | 37422000 | 197 | 840925800 |
| 165 | 25174800 | 198 | 331273800 |
| 167 | 485654400 | 200 | 240698304 |
| 168 | 312206400 | 201 | 89812800 |
| 170 | 855760950 | 203 | 171918450 |
| 171 | 525467250 | 204 | 60677100 |
| 173 | 1627857000 | 218 | 9756450 |
| 174 | 954261000 | 219 | 2539350 |
| 176 | 1747919250 | 242 | 226800 |
| 177 | 977649750 | 243 | 30800 |
| 179 | 2694384000 | 275 | 552 |

TABLE II
The weight distribution of $C_{162}$

| $i$ | $A_i$ | $i$ | $A_i$ |
|---|---|---|---|
| 0 | 1 | 180 | 1437004800 |
| 114 | 44550 | 183 | 1392098400 |
| 144 | 2494800 | 186 | 1683990000 |
| 150 | 598752 | 189 | 1181703600 |
| 156 | 20047500 | 192 | 1002909600 |
| 162 | 8809350 | 195 | 451558800 |
| 165 | 25174800 | 198 | 331273800 |
| 168 | 312206400 | 201 | 89812800 |
| 171 | 525467250 | 204 | 60677100 |
| 174 | 954261000 | 219 | 2539350 |
| 177 | 977649750 | 243 | 30800 |

**Remark 2** *In the following we give some observations on the codewords of minimum-weight and of maximum weight from the ternary code* $C_{113}$.

It follows from the weight distribution of $C_{113}$ that the minimum weight of $C_{113}$ is 113 and its maximum weight is 275. Let $W_{113}$ and $W_{275}$ be respectively the sets of minimum and maximum weight codewords of $C_{113}$. Then the following observations give a description of the codewords of each set:

1. Let $W_{113} = \{w \in C_{113} \mid \mathrm{wt}(w) = 113\}$, where $\mathrm{wt}(w)$ denotes the weight of $w$. Then $|W_{113}| = 31350$ and $M^cL$ produces four orbits of lengths 275, 275, 15400 and 15400 respectively on $W_{113}$. Let $(275)_1$, $(275)_2$, $(15400)_1$ and $(15400)_2$ denote the sets containing such codewords. We say that $(275)_1$ contains words of type I, $(275)_2$ contains words of type II, $(15400)_1$ contains words of type III and $(15400)_2$ contains words of type IV. A word $w \in W_{113}$ is of type I (respectively type II) if all its non-zero coordinates positions are 1 (respectively -1). Moreover, a word $w \in W_{113}$ is of type III if 81 non-zero coordinates positions are 1 and 32 non-zero coordinate positions are -1. Finally a word $w \in W_{113}$ is of type IV if $-w$ is of type III.

2. Let $W_{275} = \{w \in C_{113} \mid \mathrm{wt}(w) = 275\}$. Then $|W_{275}| = 552$ and $M^cL$ produces four orbits of lengths 1, 1, 275 and 275 respectively on $W_{275}$. Let $(1)_1$, $(1)_2$, $(275)_1$ and $(275)_2$ denote the sets containing such codewords, then $(1)_1 = \{\jmath\}$, $(1)_2 = \{-\jmath\}$ and we say that $(275)_1$ contains words of type III and $(275)_2$ contains words of type IV. A word $w \in W_{275}$ is of type III if 113 coordinates positions are 1 and the remaining 162 coordinate positions are -1. Finally a word $w \in W_{275}$ is of type IV if $-w$ is of type III.

**Remark 3** *In the following we give some observations on the codewords of minimum-weight and of maximum weight from the ternary code $C_{162}$.*

It follows from the weight distribution of $C_{162}$ that the minimum weight of $C_{162}$ is 114 and its maximum weight is 243. Let $W_{114}$ and $W_{243}$ be respectively the sets of minimum and maximum weight codewords of $C_{162}$. Then the following observations give a description of the codewords of each set:

1. Let $W_{114} = \{w \in C_{114} \mid \mathrm{wt}(w) = 114\}$. Then $|W_{114}| = 44550$ and $M^cL$ produces one orbit of length 44550 on $W_{114}$. A word $w \in W_{114}$ is such that the 114 non-zero coordinate positions are split into exactly 57 $+1$ and 57 $-1$.

2. Let $W_{243} = \{w \in C_{114} \mid \mathrm{wt}(w) = 243\}$. Then $|W_{243}| = 30800$ and $M^cL$ produces two orbits of lengths 15400 and 15400 respectively on $W_{243}$. Let $(15400)_1$, $(15400)_2$ denote the sets containing such codewords, then $(15400)_1$ contains words of type I, $(15400)_2$ contains words of type II. A word $w \in W_{243}$ is of type I if 81 coordinates positions are 1 and the remaining 162 coordinate are -1. A word $w \in W_{243}$ is of type II if $-w$ is of type I.

## 7 $\mathcal{D}_{112}$, $C_{112}$, $\mathcal{D}_{163}$, $C_{163}$, $\mathcal{D}_{274}$ and $C_{274}$

In this section we examine the designs $\mathcal{D}_i$ and codes $C_i$, where $i \in \{112, 163, 274\}$.

**Lemma 5** (i) $\text{Aut}(\mathcal{D}_{112}) = \text{Aut}(\mathcal{D}_{163}) =$ M$^c$L:2.
(ii) $C_{112} = C_{163} = C_{274} = V_{275}(GF(3))$.
(iii)$\text{Aut}(\mathcal{D}_{274}) = \text{Aut}(C_{112}) = \text{Aut}(C_{163}) = \text{Aut}(C_{274}) = S_{275}$.

**Proof:** (i) Computations with Magma (similar to Lemma 3) show that $\text{Aut}(\mathcal{D}_{112})$ = M$^c$L:2. Since $\mathcal{D}_{163} = \tilde{\mathcal{D}}_{112}$, we deduce that $\text{Aut}(\mathcal{D}_{163}) = \text{Aut}(\mathcal{D}_{112})$ = M$^c$L:2.

(ii)Using Magma, the row span of the adjacency matrices of $\mathcal{D}_{112}$, $\mathcal{D}_{163}$ and $\mathcal{D}_{274}$, respectively, yield the full space $V_{275}(GF(3))$. That is

$$C_{112} = C_{163} = C_{274} = V_{275}(GF(3)).$$

(iii) Since $\text{Aut}(V_{275}(GF(3))) = S_{275}$, we have

$$\text{Aut}(C_{112}) = \text{Aut}(C_{163}) = \text{Aut}(C_{274}) = S_{275}.$$

$\mathcal{D}_{274}$ is the complement of the trivial design $\mathcal{D}_1$ in which the blocks are single points and all permutations are design automorphisms. Hence $\text{Aut}(\mathcal{D}_{274}) = S_{275}$. ■

*Acknowledgment.* The authors express their gratitude to the referee whose suggestions and comments led to significant improvement of the content and the presentation of this paper.

# 8 Appendix

## 8.1 Designs and codes from $M^cL$ group of degree 275

```
''The program, where mcl is a permgroup
on 275 pts given by two permutations''
g:=mcl;
''determine a maximal subgroup isomorphic to U_4(3)''
m1:=Stabilizer(mcl,1);
a1,a2,a3:=CosetAction(mcl,m1);
st:=Stabilizer(a2,1);
orbs:=Orbits(st);#orbs;
v:=Index(a2,st);
v; "degree=",v;
lo:=[#orbs[i]: i in [1..#orbs]];
"seq. of orbit lengths=",lo;
for j:=2 to #lo do
"orbs no",j,"of length",#orbs[j];
blox:=Setseq(orbs[j]^a2);
des:=Design<1,v|blox>;des;
autdes:=AutomorphismGroup(des);
"autgp of order",Order(autdes);
p:=3;
dc:=LinearCode(des,GF(p));
dl:=Dual(dc);
d1:=Dimension(dc);
d2:=Dimension(dl);
d3:=Dimension(dc meet dl);
"p=",p,"dim=",d1,"dimdual=",
d2,"hull=",d3;
end for;
------------
''omiting the trivial designs and
the natural representations''
3
275
degree= 275
seq. of orbit lengths= [ 1, 112, 162 ]
orbs no 2 of length 112
1-(275, 112, 112) Design with 275 blocks
autgp of order 1796256000
p= 3 dim= 275 dimdual= 0 hull= 0
orbs no 3 of length 162
```

```
1-(275, 162, 162) Design with 275 blocks
autgp of order 1796256000
p= 3 dim= 21 dimdual= 254 hull= 21
''constructing the designs D_i from the union of orbits of the
rank-3 action of McL on 275 points''
''orbit 1 joined with orbit 2
b:=Setseq(orbs[1]);
bb:=Seqset(b);
c:=Setseq(orbs[2]);
cc:=Seqset(c);
dd:=bb join cc;
bll:=Setseq(dd^a2);
dss:=Design<1,v|bll>;
dss;
1-(275, 113, 113) Design with 275 blocks
IsSelfDual(dss);
true
 at:=AutomorphismGroup(dss);
 #at;
1796256000
gg:=Sym(275);
aa:=gg!x; ''x is given as a permutation on 275 pts below''
bb:=gg!y; ''y is given as a permutation on 275 pts below''
sb:=sub<gg|aa,bb>;
''mcl:2 as a perm group generated by two permutations"
#sb;
1796256000
''determining an element of order 2 in at - a2''
for a in at do
if not a in a2 then
if #sub<at|a > eq 2 then print a; ''a is given as alpha below''
break;
end if;
end if;
end for;
>dcc:=LinearCode(dss,GF(3));
> Dimension(dcc);
22
> pdd:=PermutationGroup(dcc);
>#pdd;
1796256000
''orbit 1 joined with orbit 3
```

```
> dds:=Design<1,v|blo>;dds;
1-(275, 163, 163) Design with 275 blocks
> IsSelfDual(dds);
true
> ldc:=LinearCode(dds,GF(3));Dimension(ldc);
275
> pdc:=PermutationGroup(ldc);
> pdc eq Sym(275);
true
> att:=AutomorphismGroup(dds); #att;
1796256000
''orbit 2 joined with orbit 3
> dk:=Design<1,v|ff>;dk;
1-(275, 274, 274) Design with 2300 blocks
> IsSelfDual(dk);
true
> ls:=LinearCode(dk,GF(3));Dimension(ls);
275
> pda:=PermutationGroup(ls);
> pda eq Sym(275);
true
```

## 8.2 Generators of $\overline{G} = \mathrm{M^cL}{:}2$ and $\alpha$

$x =$

(1,2)(3,4)(5,7)(6,8)(9,12)(10,13)(11,15)(14,19)
(16,21)(17,22)(18,24)(20,27)(23,31)(25,33)
(26,34)(28,37)(29,38)(30,40)(32,43)(35,47)
(36,48)(39,51)(41,53)(42,54)(44,57)(45,58)
(46,60)(49,63)(50,64)(52,67)(55,71)(56,72)
(59,76)(61,78)(62,79)(66,82)(68,85)(69,86)
(70,88)(73,92)(75,94)(77,97)(80,101)(81,102)
(83,105)(84,106)(87,110)(89,112)(90,113)(91,115)
(93,118)(95,120)(96,121)(98,123)(99,109)
(100,125)(103,129)(104,130)(107,133)(108,134)
(111,137)(114,140)(116,142)(117,128)(119,144)
(122,147)(124,149)(126,139)(131,153)(132,154)
(135,158)(136,159)(138,161)(141,164)(143,167)
(145,170)(146,172)(148,175)(150,177)(151,178)
(152,179)(157,183)(160,187)(162,189)(163,190)
(165,168)(166,193)(169,195)(171,196)(174,197)
(176,192)(180,204)(181,201)(182,206)(188,209)
(191,213)(194,216)(198,221)(199,222)(200,208)
(202,225)(203,227)(205,230)(207,232)(210,218)
(211,233)(212,235)(215,236)(217,238)(219,240)
(220,241)(223,244)(224,245)(226,247)(228,249)
(229,250)(231,252)(234,255)(237,257)(239,242)
(243,259)(246,261)(248,264)(253,267)(254,268)
(256,269)(258,266)(260,270)(262,271)(263,265) (272,274)(273,275);

$y =$

(1,3,5)(4,6,9)(7,10,14)(8,11,16)(12,17,23)(13,18,25)
(15,20,28)(19,26,35)(21,29,39)(22,30,41)(24,32,44)
(27,36,49)(31,42,55)(33,45,59)(34,46,61)(38,50,65)
(40,52,68)(43,56,73)(48,62,80)(51,66,83)(53,69,87)
(54,70,89)(57,74,93)(58,75,95)(60,77,98)(63,81,103)
(64,76,96)(67,84,107)(71,90,114)(72,91,116)(78,99,124)
(79,100,126)(82,104,131)(85,108,112)(86,109,135)
(88,111,138)(92,117,123)(94,119,125)(97,122,148)
(101,127,150)(102,128,151)(106,132,155)(110,136,160)
(113,139,162)(115,141,165)(118,143,168)(120,145,171)
(121,146,173)(129,144,169)(130,152,180)(133,156,182)
(134,157,184)(140,163,191)(142,166,172)(147,174,198)
(149,176,200)(153,181,205)(158,185,207)(159,186,170)
(161,188,210)(164,192,214)(167,194,217)(175,199,223)
(177,201,224)(178,202,226)(179,203,228)(183,193,215)
(187,208,209)(189,211,234)(190,212,221)(195,218,239)
(196,219,233)(197,220,242)(204,229,251)(206,231,253)
(216,237,255)(222,243,260)(225,246,262)(227,248,257)
(230,238,258)(232,254,269)(235,256,267)(240,249,265)
(247,263,261)(250,264,268)(252,266,272)(270,271,273);

$\alpha =$

(1,195)(2,19)(3,200)(4,163)(5,129)(6,152)(7,103)(8,105)(9,96)(10,192)(11,159)
(12,222)(13,75)(14,160)(15,221)(16,140)(17,30)(18,45)(20,63)(21,246)(22,220)
(23,270)(24,272)(25,181)(26,269)(27,34)(28,144)(29,109)(31,121)(32,268)(33,90)
(35,50)(36,256)(38,177)(39,238)(40,178)(41,143)(42,217)(43,85)(44,125)(46,145)
(47,83)(48,58)(49,151)(52,124)(53,126)(54,86)(55,99)(56,112)(57,113)(59,228)
(60,84)(61,260)(65,239)(66,249)(67,225)(68,131)(69,158)(70,170)(71,205)(72,169)
(73,108)(74,190)(76,212)(77,116)(78,213)(79,137)(80,98)(81,119)(82,180)(87,198)
(88,248)(89,231)(91,257)(92,243)(93,168)(94,120)(95,136)(97,240)(100,165)(101,173)
(102,271)(104,186)(106,135)(107,166)(111,251)(114,266)(115,208)(117,171)(118,149)
(122,226)(123,155)(127,172)(128,147)(130,207)(132,139)(133,196)(134,194)(138,233)
(141,153)(142,218)(146,187)(148,237)(150,241)(154,206)(156,274)(157,229)(161,210)
(162,263)(164,216)(167,252)(169,247)(175,211)(176,262)(179,235)(182,214)(183,244)
(184,188)(185,204)(193,224)(197,199)(201,219)(202,254)(215,258)(223,230)(227,250)
(232,264)(234,245)(236,265)(253,255)(259,275)(267,273).

# References


[1] E. F. Assmus, Jr. and J. D. Key, *Designs and their Codes*, Cambridge University Press, 1992, Cambridge Tracts in Mathematics, Vol. 103 (Second printing with corrections, 1993).

[2] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.

[3] L. Finkelestein, *The Maximal Subgroups of Conway's Group $Co_3$ and McLauglin's Group*, J. Algebra **25** (1973), 58–89.

[4] J. D. Key and J. Moori, *Designs, codes and graphs from the Janko groups $J_1$ and $J_2$*, J. Combin. Math. and Combin. Comput. **40** (2002), 143–159.

[5] J. D. Key, J. Moori, and B. G. Rodrigues, *On some designs and codes from primitive representations of some finite simple groups*, J. Combin. Math. and Combin. Comput. **45** (2003), 3–19.

[6] J. McLaughlin, *A simple group of order* 898,128,000, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968), Benjamin, New York, 1969, pp. 109–111.

[7] J. Moori and B. G. Rodrigues, *A self-orthogonal doubly even code invariant under* $M^cL:2$, J. Combin. Theory Ser. A **110** (2005), no. 1, 53–69.

[8] J. Moori and B. G. Rodrigues, *A self-orthogonal doubly-even code invariant under* $M^cL$, Ars Combinatoria **91** (2009), 321–332.

[9] B. G. Rodrigues, *Codes of Designs and Graphs from Finite Simple Groups*, Ph.D. thesis, University of Natal, Pietermaritzburg, 2002.

[10] I. A. Suleiman and R. A. Wilson, *The 3-modular characters of the McLaughlin group* $M^cL$ *and its automorphism group* $M^cL.2$, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, pp. 422–437.