The variety generated by the class of K-perfect m-cycle systems

Robert Brier
Department of Mathematics
University of Queensland
Qld 4072, Australia

Abstract

A method called the standard construction generates an algebra from a K-perfect m-cycle system. Let $\mathbf{C}_m^K$ denote the class of algebras generated by K-perfect m-cycle systems. For each m and K, there is a known set Σ_m^K of identities which all the algebras in $\mathbf{C}_m^K$ satisfy. The question of when $\mathbf{C}_m^K$ is a variety is answered in [2]. When $\mathbf{C}_m^K$ is a variety it is defined by Σ_m^K. In general, $\mathbf{C}_m^K$ is a proper subclass of $\mathbf{V}(\Sigma_m^K)$, the variety of algebras defined by Σ_m^K.

If the standard construction is applied to partial K-perfect m-cycle systems then partial algebras result. Using these partial algebras we are able to investigate properties of $\mathbf{V}(\Sigma_m^K)$. We show that the free algebras of $\mathbf{V}(\Sigma_m^K)$ correspond to K-perfect m-cycle systems, so $\mathbf{C}_m^K$ generates $\mathbf{V}(\Sigma_m^K)$. We also answer two questions asked in [5] concerning subvarieties of $\mathbf{V}(\Sigma_m^K)$. Many of these results can be unified in the result that for any subset K' of K, $\mathbf{V}(\Sigma_m^{K'})$ is generated by the class of algebras corresponding to finite K-perfect m-cycle systems.

1 Introduction

An *m-circuit*, or a circuit of *length* m, in a graph G is a cyclically ordered m-tuple $(x_0, x_1, \ldots, x_{m-1})$ of vertices of G such that $x_0x_1, \ldots, x_{m-2}x_{m-1}$, $x_{m-1}x_0$ are distinct edges of G. If all the vertices of an m-circuit are distinct then it is an *m-cycle*. We do not distinguish m-circuits by their starting point or the overall direction in which the edges are traversed (forwards or backwards). Thus each cyclic permutation of $(x_0, x_1, \ldots, x_{m-1})$ or $(x_0, x_{m-1}, \ldots, x_1)$ represents the same m-circuit. Note however that the internal order of the edges makes a difference. For example, $(x_0, x_1, x_2, x_0, x_3, x_4)$ and $(x_0, x_1, x_2, x_0, x_4, x_3)$ are different 6-circuits. For $i = 0, 1, \ldots, m-1$ we say that x_i and x_{i+k} (subscripts reduced modulo m on the residues $0, 1, \ldots, m-1$) occur at *distance* k in the circuit $(x_0, \ldots, x_{m-1})$. An *m-circuit system (m-cycle system)* is a pair (A, C) where A is a set and C is a set of m-circuits (m-cycles) whose edges partition the edge set of the complete graph with vertex set A.

There is a well-known method, often called the standard construction [9], for constructing an algebra from any given m-cycle system. An additional property (defined in Section 2) which an m-circuit system may possess is that of being 2-*perfect*, and 2-perfect m-circuit systems are of special interest as the algebra obtained from an m-circuit system is a quasigroup precisely when the system is 2-perfect. A natural generalization of the notion of 2-perfect is that of K-perfect where K is some set of positive integers.

The purpose of this paper is to investigate the universal algebraic properties of the algebras which correspond to K-perfect m-cycle systems when the standard construction is applied. A few papers already provide results of this nature. For example, it is known that the class of groupoids corresponding to m-cycle systems is a variety precisely when $m \in \{3, 5\}$ [4].

Likewise, the quasigroups corresponding to 2-perfect m-cycle systems form a variety precisely when $m \in \{3, 5, 7\}$ [6]. In general, the class $\mathbf{C}_m^K$ of algebras corresponding to K-perfect m-cycle systems is a variety precisely when $m \neq 4$ and the only K-perfect m-circuit systems are cycle systems [2].

In this paper we show that the variety generated by $\mathbf{C}_m^K$ is generated by its finite members. This result answers a question posed in [5], and is proved by showing that every finite partial K-perfect m-cycle system can be embedded in a complete K-perfect m-cycle system of finite order.

A standard set of identities, denoted by Σ_m^K, which is satisfied by the algebras corresponding to K-perfect m-cycle systems is given in [2]. This is a generalization of the well-known set of identities associated with Steiner quasigroups and 2-perfect m-cycle systems, see [9].

We also investigate the free algebras generated by Σ_m^K, in particular showing that these are in $\mathbf{C}_m^K$. Along with the embedding result, this enables us to show that the variety generated by $\mathbf{C}_{m_1}^K$ and $\mathbf{C}_{m_2}^K$ is a proper subvariety of $\mathbf{C}_m^K$ where $m = \mathrm{lcm}(m_1, m_2)$, thus answering another question asked in [5].

The result that the free algebras generated by Σ_m^K are in $\mathbf{C}_m^K$ follows from a more general result concerning algebras freely generated by partial K-perfect m-cycle systems. This generalization also allows us to show that for any subset $K' \subseteq K$, $\mathbf{C}_m^{K'}$ is generated by the algebras of $\mathbf{C}_m^{K'}$ which correspond to K-perfect m-cycle systems.

2 Algebras corresponding to circuit systems

In this section we provide the necessary definitions for the rest of paper. It is convenient to develop the ideas mentioned in Section 1 not only for circuit systems but also for partial circuit systems.

A *partial L-circuit system* is a pair (A, C) where A is a non-empty set and C is set of edge-disjoint circuits whose vertices are in A and whose lengths are in L. The *order* of (A, C) is $|A|$, and if all the circuits in C are cycles then (A, C) is a partial L-cycle system. Note that there may be vertices in A which do not occur in any of the cycles in C. For example, this means that $(A, \emptyset)$ is a partial circuit system for any non-empty set A. If we refer to a partial circuit system as C instead of as (A, C) it should be assumed that A consists of the vertices occurring in the circuits of C. We write m-circuit system instead of $\{m\}$-circuit system.

A circuit system (A, C) is *k-perfect* if for all $u, v \in A$, u and v occur at distance k exactly once. More precisely, there is a unique $(x_0, \ldots, x_{m-1}) \in C$ with $x_i = u$, $x_{i+k} = v$ (subscripts reduced modulo m) and $k \neq \frac{m}{2}$. For a set K of integers, if an m-circuit system is k-perfect for all $k \in K$ then it is said to be *K-perfect*. Similarly, a partial circuit system (A, C) is *partially k-perfect* if for all distinct $u, v \in A$, u and v occur at distance k at most once. If (A, C) is *partially k-perfect* for all $k \in K$ then it is *partially K-perfect*. It is easy to see that a (complete) circuit system of finite order is K-perfect if it is partially K-perfect. However, there are circuit systems of infinite order which are partially K-perfect but not K-perfect (see Section 5). If a partial m-circuit system is partially k-perfect then it is also partially $(m-k)$-perfect. For this reason when talking about a K-perfect m-circuit system we consider K to be a subset of $\{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$.

Given a partial circuit system (A, C), a partial binary operation $\cdot$ on A may be defined as follows.

- For all $x \in A$, $x \cdot x = x$.
- For distinct $x, y \in A$, if there is a circuit $(\ldots, w, x, y, z, \ldots) \in C$ then $x \cdot y = z$ and $y \cdot x = w$. If no circuit in C contains the edge xy then $x \cdot y$ and $y \cdot x$ are undefined.

If (A, C) is a (partial) circuit system then $(A, \cdot)$ is a (partial) groupoid.

This construction is generally called the *standard construction* and was first introduced by Kotzig in [8]. It is well known that the standard construction on 3-cycle systems gives rise to quasigroups, which are known as Steiner quasigroups. However, given a general circuit system (A, C) the resulting groupoid $(A, \cdot)$ is a quasigroup precisely when (A, C) is 2-perfect, see [9]. One may ask what happens when this construction is applied to a K-perfect m-circuit system. Does the resulting groupoid have any structure corresponding to the K-perfect nature of the circuit system? It does, but to describe this structure we introduce a new operation, called k-division for each $k \in K$. In the case $k = 2$ this operation is the left division operation of the quasigroup.

Suppose we have a partial K-perfect m-circuit system. For each $k \in K$ the partial binary operation called *k-division* and denoted by $\backslash_k$, is defined as follows.

- For all $x \in A$, $x \backslash_k x = x$.
- For distinct $x, y \in A$, if x and y occur at distance k in some circuit,

  $$(x, x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_m)$$

  say, then $x \backslash_k y = x_1$ and $y \backslash_k x = x_{k-1}$. If x and y do not occur at distance k in any circuit of C then $x \backslash_k y$ and $y \backslash_k x$ are undefined.

So given a partial K-perfect circuit system (A, C) we can construct a partial algebra $\mathcal{A} = (A, \{\cdot\} \cup \{\backslash_k \mid k \in K\})$. If (A, C) is a (complete) circuit system the result is an algebra. This partial algebra or algebra will be referred to with a calligraphic character, such as $\mathcal{A}$, and the underlying set with the corresponding roman character, such as A. As mentioned above, we denote by $\mathbf{C}_m^K$ the class of algebras resulting from the application of the standard construction to K-perfect m-cycle systems.

It is important to make clear that when speaking of an algebra corresponding to a K-perfect m-cycle system (A, C) we mean the algebra $(A, \{\cdot\} \cup \{\backslash_k \mid k \in K\})$. For each $k \in K$ we include the operation $\backslash_k$. Given a subset $K' \subseteq K$, (A, C) is also a K'-perfect m-cycle system so we can construct the algebra $(A, \{\cdot\} \cup \{\backslash_k \mid k \in K'\})$. This is a different algebra because it has a different set of operations. In the following sections, except Section 5, context determines which algebra we obtain by applying the standard construction to (A, C). So if we refer to (A, C) as K-perfect then the resulting algebra is $(A, \{\cdot\} \cup \{\backslash_k \mid k \in K\})$. If we refer to (A, C) as K'-perfect, for some $K' \subseteq K$, then the resulting algebra is $(A, \{\cdot\} \cup \{\backslash_k \mid k \in K'\})$.

To describe the identities satisfied by the algebras of $\mathbf{C}_m^K$ we inductively define words in the variables $\mathbf{x}$ and $\mathbf{y}$ by $W_0(\mathbf{x}, \mathbf{y}) = \mathbf{x}$, $W_1(\mathbf{x}, \mathbf{y}) = \mathbf{y}$, and $W_i(\mathbf{x}, \mathbf{y}) = W_{i-2}(\mathbf{x}, \mathbf{y}) \cdot W_{i-1}(\mathbf{x}, \mathbf{y})$ for all $i \geq 2$.

Given a circuit system (A, C) and any distinct pair $a, b \in A$, the unique circuit in C traversing the edge ab is given by

$$(W_0(a,b), W_1(a,b), \ldots, W_{m-1}(a,b))$$

where m is the length of the circuit. If (A, C) is k-perfect then

$$(W_0(a, a\backslash_k b), W_1(a, a\backslash_k b), \ldots, W_{m-1}(a, a\backslash_k b))$$

is the unique circuit in which a and b occur at distance k.

It is straightforward to show that an algebra corresponding to a K-perfect m-circuit system satisfies the following identities.

$$\begin{aligned} \mathbf{x}^2 &= \mathbf{x} \\ (\mathbf{x}\mathbf{y})\mathbf{y} &= \mathbf{x} \\ W_m(\mathbf{x}, \mathbf{y}) &= \mathbf{x} \\ \mathbf{x}\backslash_k W_k(\mathbf{x}, \mathbf{y}) &= \mathbf{y} \qquad \text{for all } k \in K \\ W_k(\mathbf{x}, \mathbf{x}\backslash_k \mathbf{y}) &= \mathbf{y} \qquad \text{for all } k \in K. \end{aligned}$$

We refer to this set of identities as Σ_m^K throughout this paper.

A class of algebras closed under the taking of homomorphic images, subalgebras and direct products is a *variety*. It is well known that if **C** is a variety then there is a set Σ of identities such that $\mathcal{A} \in \mathbf{C}$ precisely when $\mathcal{A}$ satisfies Σ, see [1]. The variety of algebras satisfying Σ is denoted $\mathbf{V}(\Sigma)$. The smallest variety containing a class **C** of algebras is denoted $\mathbf{V}(\mathbf{C})$. A class **C** *generates* $\mathbf{C}'$ if $\mathbf{C}' \subseteq \mathbf{V}(\mathbf{C})$. For more detail on the basic concepts of universal algebra see Grätzer's text [7], particularly Chapters 1 and 2.

3 Embedding finite partial K-perfect m-cycle systems

A partial K-perfect m-cycle system (A,C) *embeds* in another (A',C') if $A \subseteq A'$ and $C \subseteq C'$. Here, we show that every finite partial K-perfect m-cycle system embeds in a finite K-perfect m-cycle system. It is already known that any finite partial m-cycle system embeds in a finite m-cycle system, for example see [10]. To prove the result for K-perfect m-cycle systems we make use of edge-coloured graphs. An *edge-coloured graph* G^* is a multigraph G, without loops, and an assignment of colours to the edges of G. An edge-coloured graph G^* is *simple* if there are no parallel edges, and it is *uniform* if each colour is assigned to the same number of edges. We denote by rK_n^* the graph with n vertices and with exactly one edge of each of r colours between each pair of vertices. We will say that two edge-coloured graphs G^* and H^* are *colour-identical* if there is a bijection ϕ between the vertex sets of G^* and H^* such that there is an edge of colour α between a and b in G^* if and only if there is an edge of colour α between $\phi(a)$ and $\phi(b)$ in H^*. A *partial G^*-decomposition* is a set $\mathcal{G}^*$ of edge-disjoint edge-coloured subgraphs of rK_n^*, each colour-identical with

G^*. If the edge sets of the edge-coloured graphs in $\mathcal{G}^*$ partition the edge set of rK_n^* then $\mathcal{G}^*$ is a *G^*-decomposition*. A partial G^*-decomposition $\mathcal{G}^*$ *embeds* in a partial G^*-decomposition $\mathcal{G}'^*$ if $\mathcal{G}^* \subseteq \mathcal{G}'^*$. The following result concerning embeddings of partial G^*-decompositions was proven in [3].

Theorem 3.1. [3] *If G^* is a uniform simple edge-coloured graph then any finite partial G^*-decomposition can be embedded in a finite G^*-decomposition.*

By turning the problem of embedding partial K-perfect m-cycle systems into a problem on embedding edge-coloured graph decompositions we can use the above theorem to obtain the following result.

Theorem 3.2. *Any finite partial K-perfect m-cycle system can be embedded in a finite K-perfect m-cycle system.*

Proof Suppose $\mathcal{C}$ is a partial K-perfect m-cycle system of finite order and let $\mathcal{C} = \{C_1, \ldots, C_t\}$. Let $\{\alpha_k \mid k \in K\}$ be a set of colours and let $\mathcal{G}^* = \{G_1^*, \ldots, G_t^*\}$ be the following partial edge-coloured graph decomposition.

- For $i = 1, 2, \ldots, t$, the vertex set of G_i^* is the vertex set of C_i.
- For each $k \in K$ and each pair of vertices u and v in G_i^* there is an edge of colour α_k between u and v if and only if u and v occur at distance k in C_i.

The fact that $\mathcal{C}$ is K-perfect ensures that $\mathcal{G}^*$ is indeed a partial edge-coloured decomposition. It is clear that each G_i^* is colour-identical with G_1^*, that is $\{G_1^*, \ldots, G_t^*\}$ is a partial G_1^*-decomposition. Furthermore G_1^* is a simple edge-coloured graph. Thus by Theorem 3.1, $\{G_1^*, \ldots, G_t^*\}$ embeds in an edge-coloured G_1^*-decomposition of finite order, say $\{G_1'^*, \ldots, G_{t'}'^*\}$. Let C_i' be the unique m-cycle whose edges are colour α_1 in $G_i'^*$. Then $\{C_1', \ldots, C_{t'}'\}$ is a finite K-perfect m-cycle system in which $\mathcal{C}$ is embedded.
$\circ$

It follows from Theorem 3.2 that any algebra corresponding to a finite partial K-perfect m-cycle system can be embedded in an algebra corresponding to a finite K-perfect m-cycle system.

This enables us to answer one of the questions asked in [5].

Theorem 3.3. *The finite algebras of* $\mathbf{C}_m^K$ *generate* $\mathbf{V}(\mathbf{C}_m^K)$.

Proof Let $\mathcal{A}$ be an algebra in $\mathbf{V}(\mathbf{C}_m^K)$. Suppose $\mathcal{A}$ fails to satisfy some identity, $\mathbf{p} = \mathbf{q}$ say. Then in the K-perfect m-cycle system corresponding to $\mathcal{A}$ there is a finite collection of m-cycles, which gives rise to a partial algebra for which $\mathbf{p} = \mathbf{q}$ fails to hold. This partial subalgebra, $\mathcal{A}'$ say, can be embedded in some finite algebra, $\mathcal{B}$ say, in $\mathbf{C}_m^K$. Thus $\mathcal{B}$ fails to satisfy $\mathbf{p} = \mathbf{q}$. So any identity which does not hold in $\mathbf{C}_m^K$ does not hold in some finite algebra of $\mathbf{C}_m^K$. Thus, the finite algebras of $\mathbf{C}_m^K$ generate the same variety as $\mathbf{C}_m^K$. ○

4 The free algebras of $\mathbf{V}(\Sigma_m^K)$

In this section we discuss what is in a sense the 'largest' possible embedding of a cycle system and the resulting algebra. Let (A_0, C_0) be a partial K-perfect m-cycle system. We wish to embed this in a complete K-perfect m-cycle system. If $u, v \in A_0$ do not occur at distance k in (A_0, C_0) we want to add a cycle where they do occur at distance k. The 'easiest' way to do this is to create some new vertices $\{x_1 \ldots, x_{k-1}, x_{k+1}, \ldots, x_{m-1}\}$ and add the cycle $(u, x_1, \ldots, x_{k-1}, v, x_{k+1}, \ldots, x_{m-1})$ to the partial cycle system. By repeating this procedure for all the other missing occurrences in (A_0, C_0) we end up with the partial circuit system (A_1, C_1) in which (A_0, C_0) is embedded and in which all the missing occurrences in (A_0, C_0) are included. Similarly we can find a partial circuit system (A_2, C_2) in which (A_1, C_1) is embedded and in which all the missing occurrences in (A_1, C_1)

are included. Taking the limit of this process we end up with a K-perfect m-cycle system $(A_\omega, \mathcal{C}_\omega)$ in which $(A_0, \mathcal{C}_0)$ is embedded and where every pair of vertices occurs at distance k for all $k \in K$. We call $(A_\omega, \mathcal{C}_\omega)$ the *free completion* of $(A_0, \mathcal{C}_0)$.

The algebras arising from free completions correspond to a special sort of algebra, *freely generated* algebras. Here we give only the basic definitions needed to discuss freely generated algebras. For more detail see Chapter 4 of [7].

Given a partial algebra $\mathcal{A}$, a *relative subalgebra* of $\mathcal{A}$ is a partial algebra $\mathcal{B}$ such that $B \subseteq A$ and the operation $f(x_1, \ldots, x_n)$ is defined and equals y in $\mathcal{B}$ if and only if $x_1, \ldots, x_n, y \in B$ and $f(x_1, \ldots, x_n) = y$ in $\mathcal{A}$. We say that $\mathcal{B}$ is obtained by restricting $\mathcal{A}$ to B.

Given partial algebras $\mathcal{A}$ and $\mathcal{B}$, a *homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ is a function $\phi : A \to B$ where $\phi(f(x_1, \ldots, x_n)) = f(\phi(x_1), \ldots, \phi(x_n))$ whenever $f(x_1, \ldots, x_n)$ is defined in $\mathcal{A}$. If $\phi : \mathcal{A} \to \mathcal{B}$ is a bijection and the inverse function $\phi^{-1} : B \to A$ is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$ then ϕ is an *isomorphism* and $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*.

An algebra $\mathcal{F}_{\mathbf{C}}(\mathcal{A})$ is *freely generated* by the partial algebra $\mathcal{A}$ over class $\mathbf{C}$ if

- $\mathcal{A}$ is a relative subalgebra of $\mathcal{F}_{\mathbf{C}}(\mathcal{A})$
- $\mathcal{A}$ generates $\mathcal{F}_{\mathbf{C}}(\mathcal{A})$,
- for all homomorphisms $\phi : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B} \in \mathbf{C}$ there is a homomorphism $\theta : \mathcal{F}_{\mathbf{C}}(\mathcal{A}) \to \mathcal{B}$ such that the restriction of θ to $\mathcal{A}$ is ϕ.

If no operations are defined in $\mathcal{A}$ then $\mathcal{F}_{\mathbf{C}}(\mathcal{A})$ is denoted by $\mathcal{F}_{\mathbf{C}}(A)$ and is called a *free algebra* generated by set A. For all $\mathcal{B} \in \mathbf{C}$ each map from A to B extends to a homomorphism from $\mathcal{F}_{\mathbf{C}}(A)$ to $\mathcal{B}$. If the class $\mathbf{C}$ of algebras is a variety defined by the set Σ of identities we write $\mathcal{F}_{\Sigma}(\mathcal{A})$ instead of

$\mathcal{F}_{\mathrm{C}}(\mathcal{A})$.

Lemma 4.1. *If A is a set and $\mathcal{A}$ is the algebra arising from the partial K-perfect m-cycle system $(A, \emptyset)$ then $\mathcal{F}_{\Sigma_m^K}(A) \cong \mathcal{F}_{\Sigma_m^K}(\mathcal{A})$.*

Proof The partial algebra $\mathcal{A}$ is idempotent, as are all the algebras in $\mathbf{V}(\Sigma_m^K)$, so every map from A to an algebra in $\mathbf{V}(\Sigma_m^K)$ is a homomorphism. Thus $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$ is an algebra with a free generating set A. The uniqueness of free algebras (see [7]) then means that $\mathcal{F}_{\Sigma_m^K}(A) \cong \mathcal{F}_{\Sigma_m^K}(\mathcal{A})$. ∘

Theorem 4.2. *Let (A, C) be a partial K-perfect m-cycle system and $\mathcal{A}$ its corresponding partial algebra. The algebra obtained by the free completion of (A, C) is isomorphic to $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$.*

Proof Let $(A, C) = (A'_0, C_0)$, let (A'_i, C_i) $(i \in \mathbb{N})$ be the partial K-perfect m-cycle systems that arise in the free completion of (A', C_0) and let (A'_ω, C_ω) be the free completion of (A', C_0). Let $\mathcal{A}'_i$ be the corresponding partial algebra to (A_i, C_i) and $\mathcal{A}'_\omega$ the corresponding algebra to (A'_ω, C_ω).

Since $\mathbf{V}(\Sigma_m^K)$ is a variety we know by Theorem 2 in section 28 of Gräzter [7] that $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$ exists. We define the following sequence of partial algebras.

- $A_0 = A$,
- $A_{i+1} = \{x \mid x = W_j(y, y\backslash_k z) \text{ where } y, z \in A'_i \text{ and } k \in K\}$,
- $\mathcal{A}_i$ is the relative subalgebra obtained by restricting $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$ to A'_i.

Now $\mathcal{A}'_\omega \in \mathbf{V}(\Sigma_m^K)$ so the identity map id : $\mathcal{A}_0 \to \mathcal{A}'_0$ extends to a homomorphism $\phi : \mathcal{F}_{\Sigma_m^K}(\mathcal{A}) \to \mathcal{A}'_\omega$. Let ϕ_i denote the restriction of ϕ to $\mathcal{A}_i$. We will show by induction that ϕ_i is an isomorphism of $\mathcal{A}_i$ and $\mathcal{A}'_i$ for all $i \in \mathbb{N}$. Since $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$ is the direct union of the $\mathcal{A}_i$, and $\mathcal{A}'_\omega$ is the direct union of the $\mathcal{A}'_i$, this establishes that ϕ is an isomorphism of $\mathcal{F}_{\Sigma_m^K}(\mathcal{A})$ and $\mathcal{A}'_\omega$.

First of all ϕ_0 is the identity so it is an isomorphism. Now assume that ϕ_i is an isomorphism. To show that ϕ_{i+1} is an isomorphism we need to show four things, that ϕ_{i+1} is a homomorphism, one-to-one, onto, and that ϕ_{i+1}^{-1} is a homomorphism. Firstly ϕ_{i+1} is a homomorphism. This follows easily from the fact that it is the restriction of ϕ to $\mathcal{A}_{i+1}$.

Secondly ϕ_{i+1} is onto. Let x' be an element of A'_{i+1}. If $x' \in A'_i$ then it is the image of something in A_i by the assumption that ϕ_i is an isomorphism. If $x' \notin A'_i$ then there is some pair $y', z' \in A'_i$ and $k \in K$ such that x' is a vertex of the cycle where y' and z' occur at distance k. That is there is some integer j such that $x' = W_j(y', y'\backslash_k z')$. Now y' and z' are the images of some elements in $\mathcal{A}_i$, y and z say. Then x' is the image of $W_j(y, y\backslash_k z)$.

Next we need to show that ϕ_{i+1} is one-to-one. Suppose $x_1, x_2 \in A_{i+1}$ and $\phi_{i+1}(x_1) = \phi_{i+1}(x_2)$. There are three cases to consider.

i) If $x_1, x_2 \in A_i$ then

$$\phi_{i+1}(x_1) = \phi_i(x_1) = \phi_i(x_2) = \phi_{i+1}(x_2)$$

which implies that $x_1 = x_2$ by assumption.

ii) If $x_1 \notin A_i$ but $x_2 \in A_i$. There is some $k \in K$, $j \in \mathbb{Z}$ and $y, z \in A_i$ such that $x_1 = W_j(y, y\backslash_k z)$. If $\phi_{i+1}(x_1) = \phi_{i+1}(x_2)$ then $W_j(\phi_{i+1}(y), \phi_{i+1}(y)\backslash_k \phi_{i+1}(z)) = \phi_{i+1}(x_2) \in A_i$. By the construction of (A'_i, C_i) this means that $\phi_{i+1}(y)\backslash_k \phi_{i+1}(z)$ is defined in $\mathcal{A}'_i$. As ϕ_i is an isomorphism this means that $y\backslash_{k_1} z$ is defined in $\mathcal{A}_i$. But then $x_1 = W_j(y, y\backslash_k z) \in A_i$ which is a contradiction. So we do not have $x_1 \notin A_i$ and $x_2 \in A_i$.

iii) If $x_1, x_2 \notin A_i$ then there are some $k_1, k_2 \in K$, some $j_1, j_2 \in \mathbb{Z}$ and some $y_1, y_2, z_1, z_2 \in A_{i+1}$ such that $x_1 = W_{j_1}(y_1, y_1\backslash_{k_1} z_1)$ and $x_2 = W_{j_2}(y_2, y_2\backslash_{k_2} z_2)$. Because $y_1, y_2, z_1, z_2 \in A_i$, $\phi_{i+1}(y_1) \neq \phi_{i+1}(z_1)$ and $\phi_{i+1}(y_2) \neq \phi_{i+1}(z_2)$. This means that $\phi_{i+1}(x_1)$ is in the cycle where $\phi_{i+1}(y_1)$ and $\phi_{i+1}(z_1)$ occur at distance k_1. Similarly $\phi_{i+1}(x_2)$ is in the cycle where $\phi_{i+1}(y_2)$ and $\phi_{i+1}(z_2)$ occur at distance k_2. The construction of $(\mathcal{A}'_{i+1}, \mathcal{C}_{i+1})$ means that $\phi_{i+1}(x_1)$ and $\phi_{i+1}(x_2)$ are distinct unless

a) $k_1 = k_2$, $j_1 = j_2$, $\phi_{i+1}(y_1) = \phi_{i+1}(y_2)$ and $\phi_{i+1}(z_1) = \phi_{i+1}(z_2)$. Then $y_1 = y_2$ and $z_1 = z_2$ and so $x_1 = x_2$,

OR

b) $k_1 = k_2$, $j_2 = m - j_1 + k_1$, $\phi_{i+1}(y_1) = \phi_{i+1}(z_2)$ and $\phi_{i+1}(z_1) = \phi_{i+1}(y_2)$. Then $y_1 = z_2$, $z_1 = y_2$ and

$$x_1 = W_{j_1}(y_1, y_1 \backslash_{k_1} z_1) = W_{m-j_2+k_1}(z_2, z_2 \backslash_{k_1} y_2)$$

$$= W_{m-j_2+1}(y_2 \backslash_{k_1} z_2, y_2) = W_{j_2}(y_2, y_2 \backslash_{k_2} z_2) = x_2.$$

Thus $\phi_{i+1}(x_1) = \phi_{i+1}(x_2)$ implies $x_1 = x_2$, so ϕ_{i+1} is one-to-one.

As ϕ_{i+1} is a bijection between $\mathcal{A}_i$ and $\mathcal{A}'_i$ it has an inverse ϕ_{i+1}^{-1}. Finally we need to show that ϕ_{i+1}^{-1} is a homomorphism. Now $x_1 \cdot x_2$ is defined in $\mathcal{A}_{i+1}$ if and only if $x_1, x_2 \in A_i$ or $x_1 = W_j(y, y \backslash_k z)$ and $x_2 = W_{j+1}(y, y \backslash_k z)$ for some $k \in K$, $j \in \mathbb{Z}$ and $y, z \in A_i$. Also $x'_1 \cdot x'_2$ is defined in $\mathcal{A}'_{i+1}$ if and only if $x'_1, x'_2 \in A'_i$ or $x'_1 = W_j(y', y' \backslash_k z')$ and $x'_2 = W_{j+1}(y', y' \backslash_k z')$ for some $k \in K$, $j \in \mathbb{Z}$ and $y', z' \in A'_i$. So it follows from this that $x_1 \cdot x_2$ is defined in $\mathcal{A}'_{i+1}$ if and only if $\phi_{i+1}^{-1}(x_1) \cdot \phi_{i+1}^{-1}(x_2)$ is defined in $\mathcal{A}_{i+1}$. If

$$\phi_{i+1}^{-1}(x_1) \cdot \phi_{i+1}^{-1}(x_2) \neq \phi_{i+1}^{-1}(x_1 \cdot x_2)$$

then

$$x_1 \cdot x_2 = \phi_{i+1}(\phi_{i+1}^{-1}(x_1) \cdot \phi_{i+1}^{-1}(x_2)) \neq \phi_{i+1}(\phi_{i+1}^{-1}(x_1 \cdot x_2)) = x_1 \cdot x_2$$

which is a contradiction. Thus

$$\phi_{i+1}^{-1}(x_1) \cdot \phi_{i+1}^{-1}(x_2) = \phi_{i+1}^{-1}(x_1 \cdot x_2).$$

A similar argument shows that

$$\phi_{i+1}^{-1}(x_1 \backslash_k x_2) = \phi_{i+1}^{-1}(x_1) \backslash_k \phi_{i+1}^{-1}(x_2).$$

So ϕ_{i+1}^{-1} is a homomorphism. Thus ϕ_{i+1} is an isomorphism. By the principle of mathematical induction ϕ_i is an isomorphism between $\mathcal{A}_i$ and $\mathcal{A}'_i$ for all $i \in \mathbb{N}$.

$\circ$

Theorem 4.3. *The variety* $\mathbf{V}(\Sigma_m^K)$ *is generated by* $\mathbf{C}_m^K$. *Thus the finite algebras of* $\mathbf{C}_m^K$ *generate* $\mathbf{V}(\Sigma_m^K)$.

Proof Let A be an infinite set and let $\mathcal{A}$ be the algebra corresponding to the partial K-perfect m-cycle system $(A, \emptyset)$. The algebra $\mathcal{A}_\omega$ arising from the free completion of $(A, \emptyset)$ is a free algebra by Lemma 4.1. By Theorem 4.2, $\mathcal{A}_\omega$ is a free algebra of $\mathbf{V}(\Sigma_m^K)$ generated by an infinite set, and so generates $\mathbf{V}(\Sigma_m^K)$ see [7]. Furthermore, $\mathcal{A}_\omega$ corresponds to a K-perfect m-cycle system and so $\mathbf{C}_m^K$ generates $\mathbf{V}(\Sigma_m^K)$. Theorem 3.3 then implies that the finite algebras of $\mathbf{C}_m^K$ generate $\mathbf{V}(\Sigma_m^K)$. $\circ$

We can now answer another question asked in [5].

Theorem 4.4. *If* $m_1, m_2 \geq 3$ *are distinct integers and* $m = \mathrm{lcm}(m_1, m_2)$ *then* $\mathbf{V}(\mathbf{C}_{m_1}^K \cup \mathbf{C}_{m_2}^K)$ *is a proper subvariety of* $\mathbf{V}(\mathbf{C}_m^K)$.

Proof Each of the algebras in $\mathbf{C}_{m_1}^K \cup \mathbf{C}_{m_2}^K$ satisfies Σ_m^K, so $\mathbf{V}(\mathbf{C}_{m_1}^K \cup \mathbf{C}_{m_2}^K)$ is a subvariety of $\mathbf{V}(\Sigma_m^K) = \mathbf{V}(\mathbf{C}_m^K)$ by Theorem 4.3. Now, every algebra in $\mathbf{C}_{m_1}^K \cup \mathbf{C}_{m_2}^K$ satisfies the identity $W_{m_1}(\mathbf{x}, W_{m_2}(\mathbf{x}, \mathbf{y})) = \mathbf{x}$. Thus $\mathbf{V}(\mathbf{C}_{m_1}^K \cup \mathbf{C}_{m_2}^K)$ satisfies $W_{m_1}(\mathbf{x}, W_{m_2}(\mathbf{x}, \mathbf{y})) = \mathbf{x}$. Let $\mathcal{C}$ be a K-perfect m-cycle system in which the partial K-perfect m-cycle system

$$\{(x_0, x_1, \ldots x_{m-1}), (x_0, x_{m_2}, y_2, \ldots y_{m-1})\}$$

is embedded and where $x_i \neq y_j$ for all $i, j \in \{0, \ldots, m-1\}$ with $j \geq 2$ ($\mathcal{C}$ exists by Theorem 3.2). If $\mathcal{A}$ is the algebra corresponding to $\mathcal{C}$ then

$W_{m_1}(x_0, W_{m_2}(x_0, x_1)) = W_{m_1}(x_0, x_{m_2}) = y_{m_1} \neq x_1$. Thus $\mathcal{A}$ does not satisfy $W_{m_1}(\mathrm{x}, W_{m_2}(\mathrm{x}, \mathrm{y})) = \mathrm{x}$ and so $\mathcal{A} \notin \mathbf{V}(\mathbf{C}^K_{m_1} \cup \mathbf{C}^K_{m_2})$. Hence $\mathbf{V}(\mathbf{C}^K_{m_1} \cup \mathbf{C}^K_{m_2})$ is a proper subvariety of $\mathbf{V}(\mathbf{C}^K_m)$. $\circ$

5 K-perfect and K'-perfect m-cycle systems

Given a partial K-perfect circuit system $(A, \mathcal{C})$ we can construct a partial algebra $\mathcal{A} = (A, \{\cdot\} \cup \{\backslash_k \mid k \in K\})$. However, for any set $K' \subseteq K$, $(A, \mathcal{C})$ is also a partial K'-perfect m-cycle system, so we can also construct a partial algebra $\mathcal{A}_{K'} = (A, \{\cdot\} \cup \{\backslash_k \mid k \in K'\})$. So $(A, \mathcal{C})$ gives rise to different partial algebras, one for each $K' \subseteq K$. We will call $\mathcal{A}_{K'}$ the *K'-reduct* of $\mathcal{A}$. We denote by $\mathbf{C}^{K,K'}_m$ the class of K'-reducts of the algebras in $\mathbf{C}^K_m$.

Theorem 5.1. *If m is a positive integer and K' and K are subsets of $\{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$ with $K' \subseteq K$, then $\mathbf{V}(\mathbf{C}^{K,K'}_m) = \mathbf{V}(\Sigma^{K'}_m)$.*

Proof We will show that the free algebras of $\mathbf{V}(\Sigma^{K'}_m)$ are in $\mathbf{V}(\mathbf{C}^{K,K'}_m)$. This is sufficient because the free algebras of a variety generate the variety. Let $\mathcal{F}$ be a free algebra of $\mathbf{V}(\Sigma^K_m)$ freely generated by a set A. Let $\mathcal{C}_0$ be the free completion of $(A, \emptyset)$ considered as a partial K'-perfect m-cycle system. By Theorem 4.2, $\mathcal{F} \cong \mathcal{A}'_0$ where $\mathcal{A}'_0$ is the algebra corresponding to $\mathcal{C}_0$. Note that $\mathcal{C}_0$ is also a partial K-perfect m-cycle system. Thus $\mathcal{A}'_0$ is the K'-reduct of $\mathcal{A}_0 = (\mathcal{A}, \{\cdot\} \cup \{\backslash_k \mid k \in K\})$, the partial algebra corresponding to $\mathcal{C}_0$ considered as a partial K-perfect m-cycle system. Let $\mathcal{C}_1$ be the free completion of $\mathcal{C}_0$ considered as a K-perfect m-cycle system and let $\mathcal{A}_1$ be the algebra corresponding to $\mathcal{C}_1$. Let $\mathcal{A}'_1$ be the K'-reduct of $\mathcal{A}_1$.

The following diagram illustrates the relationship between the four algebras.

$$\begin{array}{ccccc} & \mathcal{A}_0' & \xrightarrow{\text{extends to}} & \mathcal{A}_0 & \\ \text{is a subalgebra of} & \downarrow & & \downarrow & \text{is embedded in} \\ & \mathcal{A}_1' & \xleftarrow[\text{reduces to}]{} & \mathcal{A}_1 & \end{array}$$

Clearly $\mathcal{A}_1' \in \mathbf{C}_m^{K,K'}$. It follows easily from the construction of $\mathcal{A}_1'$ that $\mathcal{A}_0'$ is a subalgebra of $\mathcal{A}_1'$ and so $\mathcal{A}_0' \cong \mathcal{F} \in \mathbf{V}(\mathbf{C}_m^{K,K'})$. $\circ$

Finally, we obtain the following theorem which unifies some of the previous results in this paper.

Theorem 5.2. *Let m be an integer and let K and K' be subsets of $\{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$ with $K' \subseteq K$. The finite algebras of $\mathbf{C}_m^{K,K'}$ generate $\mathbf{V}(\Sigma_m^{K'})$.*

Proof The finite algebras of $\mathbf{C}_m^{K}$ generate $\mathbf{C}_m^{K}$ by Theorem 3.3. Thus, the finite algebras of $\mathbf{C}_m^{K,K'}$ generate $\mathbf{C}_m^{K,K'}$. Furthermore, $\mathbf{C}_m^{K,K'}$ generates $\mathbf{V}(\Sigma_m^{K'})$ by Theorem 5.1. So the finite algebras of $\mathbf{C}_m^{K,K'}$ generate $\mathbf{V}(\Sigma_m^{K'})$. $\circ$

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