Distance two labelings of Cartesian products of complete graphs*

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Abstract

For two positive integers j and k with $j \geq k$, an $L(j,k)$-labeling of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least j, and the difference between labels of vertices that are distance two apart is at least k. The span of an $L(j,k)$-labeling of a graph G is the difference between the maximum and minimum integers used by it. The $\lambda_{j,k}$-number of G is the minimum span over all $L(j,k)$-labelings of G. This paper focuses on the $\lambda_{2,1}$-number of the Cartesian products of complete graphs. We completely determine the $\lambda_{2,1}$-numbers of the Cartesian products of three complete graphs K_n, K_m, and K_l for any three positive integers $n, m,$and l.

Keywords: $L(2,1)$-labeling, $\lambda_{2,1}$-number, Cartesian product.

1 Introduction

For two positive integers j and k with $j \geq k$, an $L(j,k)$-*labeling* of a graph G is an assignment L of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least j, and the difference between labels of vertices that are distance two apart is at least k. Elements of the image of L are called labels, and the *span* of L, denoted by *span*(L) is the difference between the largest and smallest labels of L. The $\lambda_{j,k}$-*number* of G, denoted $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G. If L is an $L(j,k)$-labeling with span $\lambda_{j,k}(G)$ then L is called a $\lambda_{j,k}$-labeling

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of G. We shall assume without loss of generality that the minimum label of $L(j,k)$-labelings of G is always 0.

Motivated by a special kind of channel assignment problem, Griggs and Yeh [8] first proposed and studied the $L(2,1)$-labeling of a graph. Since then the $\lambda_{2,1}$-numbers of graphs have been studied extensively, see [2, 4, 6–8, 10, 13, 15]. And $L(j,k)$-labelings were also investigated in many papers, see [3–6]. For a survey on $L(j,k)$-labelings of graphs, please see [1].

Given two graphs G and H, the *Cartesian product* of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x,y) and (x',y') are adjacent if $x = x'$ and $yy' \in E(H)$ or $y = y'$ and $xx' \in E(G)$. Let G^k denote the Cartesian product of k copies of G. Let K_n denote the complete graph on n vertices. Then $K_n^2 = K_n \times K_n$ and $K_n^3 = K_n \times K_n \times K_n$.

The $L(2,1)$-labeling of the Cartesian product of n paths, especially of the Cartesian product of n copies of P_2 (the n-cube Q_n), was investigated by Whittlesey, Georges, and Mauro [15]. In the same paper, they completely determined the $\lambda_{2,1}$-numbers of Cartesian products of two paths. Jha et al. [10] studied the $L(2,1)$-labeling of the Cartesian product of a cycle and a path. The $\lambda_{2,1}$-numbers of the Cartesian product of a cycle and a path were completely computed by Klavžar and Vesel in [11]. Partial results for the $\lambda_{2,1}$-numbers of the Cartesian products of two cycles were obtained in [11]. These partial results are completed in [14]. Georges, Mauro, and Whittlesey [7] determined $L(2,1)$-labeling numbers of Cartesian products of two complete graphs. This result was then extended by Georges, Mauro, and Stein [6] who determined the $L(j,k)$-labeling numbers of Cartesian products of two complete graphs.

Theorem 1.1 *[6] Let j, k, n, and m be integers where $2 \le m < n$ and $j \ge k$. Then*

(i) $\lambda_{j,k}(K_n \times K_m) = (n-1)j + (m-1)k$, *if* $j/k > m$;

(ii) $\lambda_{j,k}(K_n \times K_m) = (nm-1)k$, *if* $j/k \le m$.

Theorem 1.2 *[6] Let j, k, and n be integers where $2 \le n$ and $j \ge k$. Then*

(i) $\lambda_{j,k}(K_n^2) = (n-1)j + (2n-2)k$, *if* $j/k > n-1$;

(ii) $\lambda_{j,k}(K_n^2) = (n^2-1)k$, *if* $j/k \le n-1$.

Georges, and Mauro [4] also obtained other results on $L(j,k)$-labelling numbers of Cartesian products of complete graphs. In particular, they investigated the $\lambda_{j,k}$-number of K_n^3.

Theorem 1.3 *[4] The $\lambda_{j,k}$-number of $Q_3 \cong K_2^3$ is equal to $3j$ if $j/k \le 5/2$; and $j + 5k$ if $j/k \ge 5/2$.*

Theorem 1.4 *[4] Suppose n is an odd integer, $n \geq 3$. Then*
(i) $\lambda_{j,k}(K_n^3) = (n-1)(j+3k)$, *if* $j/k \geq 3n-4$;
(ii) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \leq n-2$;
(iii) $\lambda_{j,k}(K_n^3) \leq (n-1)(j+3k)$, *if* $n-2 < j/k < 3n-4$.

Theorem 1.5 *[4] Suppose n is an even integer. Then*
(i) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \leq n/2$;

$$\text{(ii) } \lambda_{j,k}(K_n^3) \leq \begin{cases} (n^2+2n)k, & \text{if } n/2 < j/k \leq n-2, \\ n(j+3k), & \text{if } n-2 < j/k \leq 2n(n-2), \\ (n-1)j + n(2n-1)k, & \text{if } j/k > 2n(n-2)). \end{cases}$$

In the next section, we completely determine the $\lambda_{2,1}$-number of $K_n \times K_m \times K_l$ for any three positive integers n, m, l.

For two positive integers a and b with $a < b$, denote by $[a, b]$ the set of integers $a, a+1, \ldots, b$. A set of integers is called k-separated if and only if any two distinct elements of the set differ by at least k. Given a graph $G(V, E)$, a subset S of V is call 2-independent if any two vertices of it are at distance at least 3. The 2-independence number of G is the number of vertices in a maximum 2-independent set of G.

2 $\lambda_{2,1}$-numbers of $K_n \times K_m \times K_l$

This section determines the $\lambda_{2,1}$-number of $K_n \times K_m \times K_l$ for any three positive integers n, m, l. We shall always suppose that n, m and l are positive integers with $n \geq m \geq l \geq 2$.

We shall view the vertices of the graph $K_n \times K_m \times K_l$ as points in 3 Dimensional Euclidean space with coordinate (a, b, c), where a, b, c are nonnegative integers and $0 \leq a \leq n-1$, $0 \leq b \leq m-1$, $0 \leq c \leq l-1$. For $v = (a, b, c) \in V(K_n \times K_m \times K_l)$, we say that v is a vertex in the a^{th} row, b^{th} column and the c^{th} level of $K_n \times K_m \times K_l$. It is not difficult to see that two vertices are at distance k if their coordinates are different in exactly k components. In other words, two vertices on a line parallel to some coordinate axis are adjacent; two vertices on a plane parallel to some coordinate plane but not on any line parallel to some coordinate axis are at distance 2; and any two vertices not on any plane parallel to some coordinate plane are at distance 3. The diameter of $K_n \times K_m \times K_l$ is 3. The 2-independence number of $K_n \times K_m \times K_l$ is l. Thus each label can be used at most l times by any $L(2,1)$-labeling of $K_n \times K_m \times K_l$.

Theorem 2.1 *Let n, m and l be positive integers with $n \geq m \geq l \geq 2$. If $n \geq 4$ then $\lambda_{2,1}(K_n \times K_m \times K_l) = nm - 1$.*

Proof. Since $K_n \times K_m$ is of diameter 2 and has nm vertices, $\lambda_{2,1}(K_n \times K_m) \geq nm-1$. Thus $\lambda_{2,1}(K_n \times K_m \times K_l) \geq nm-1$. To prove the theorem, it suffice to give an $L(2,1)$-labeling of $K_n \times K_m \times K_l$ with span $nm-1$. We deal with the following three cases.

Case 1: $m \geq 4$.

We define the matrix $X = (x_{ij})_{n \times m}$ as follows. Let $x_{11} = nm-1$ and $x_{nm} = 0$. For $j = 2, 3, \ldots, m$, let $x_{1j} = \sum_{t=1}^{j-1} t$; for $1 \leq i \leq n-m+1$, let $x_{im} = m(m-1)/2 + (i-1)m$; for $n-m+1 \leq i \leq n-1$, let $x_{im} = nm - \sum_{t=1}^{n-i+1} t$. And let $x_{ij} = x_{(i-1)(j+1)}$ if $i \neq 1$ and $j \neq m$. Thus we have

$$X = \begin{pmatrix} nm-1 & 1 & 3 & 6 & \cdots & \cdots & \cdots \\ 2 & 4 & 7 & \cdots & \cdots & \cdots & \cdots \\ 5 & 8 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 9 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & nm-3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & nm-2 & 0 \end{pmatrix}_{n \times m}. \tag{2.1}$$

It is straightforward to check that each row of X is 2-separated and each column of X is also 2-separated. Furthermore, for $0 \leq a \leq n-1$ and $0 \leq b \leq m-1$, the set $\{x_{(a+q)(b+q)} : q = 0, 1, \ldots, m-1\}$ is 2-separated, where the first "+" in the subscript is taken modulo n and the second modulo m.

Define a mapping f from $V(K_n \times K_m \times K_l)$ to $\{0, 1, 2, \ldots, nm-1\}$ as:

$f((a,b,0)) = x_{(a+1)(b+1)}$, for $0 \leq a \leq n-1$, $0 \leq b \leq m-1$;

$f((a,b,c)) = f(((a+c) \bmod n, (b+c) \bmod m, 0))$, for $0 \leq a \leq n-1$, $0 \leq b \leq m-1$, $0 \leq c \leq l-1$.

We first show that if $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ are two vertices at distance 2 then $f(v_1) \neq f(v_2)$. As v_1 and v_2 are at distance 2, their coordinates are the same in exactly one component. If $c_1 = c_2 = 0$ then, from the definition of f, it is easy to see that $f(v_1) \neq f(v_2)$. For the case $c_1 \neq 0$ or $c_2 \neq 0$, suppose to the contrary that $f(v_1) = f(v_2)$. Then $f(((a_1+c_1) \bmod n, (b_1+c_1) \bmod m, 0)) = f(((a_2+c_2) \bmod n, (b_2+c_2) \bmod m, 0))$. It follows from the definition of f (for the case $c=0$) that $(a_1+c_1) \bmod n = (a_2+c_2) \bmod n$ and $(b_1+c_1) \bmod m = (b_2+c_2) \bmod m$. This implies that if the coordinates of v_1 and v_2 are the same in at least one component then they must be the same in all three components. It follows that $(a_1, b_1, c_1) = (a_2, b_2, c_2)$. A contradiction.

Now suppose $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ are two adjacent vertices in $V(K_n \times K_m \times K_l)$. Their coordinates are different in exactly one component. If $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $c_1 \neq c_2$ then $f(v_1) = x_{(a+c_1)(b+c_1)}$ and $f(v_2) = x_{(a+c_2)(b+c_2)}$. If $a_1 \neq a_2$, $b_1 = b_2 = b$ and $c_1 = c_2 = c$ then $f(v_1) = x_{(a_1+c)(b+c)}$ and $f(v_2) = x_{(a_2+c)(b+c)}$. If $a_1 = a_2 = a$, $b_1 \neq b_2$ and $c_1 = c_2 = c$ then $f(v_1) = x_{(a+c)(b_1+c)}$ and $f(v_2) = x_{(a+c)(b_2+c)}$. In all cases, from the above discussions about the properties of X, we have $|f(v_1) - f(v_2)| \geq 2$.

Thus f is an $L(2,1)$-labeling of $K_n \times K_n \times K_l$ with span $nm - 1$.

Case 2: $m = 3$.

The matrix defined in Case 1 doesn't work when $m = 3$. This time we define the matrix X as:

$$\begin{pmatrix} 0 & n+2 & 2n+1 \\ n & 2n+2 & 1 \\ 2n & 2 & n+1 \\ \cdots & \cdots & \cdots \\ n-3 & 2n-1 & 3n-2 \\ 2n-3 & 3n-1 & n-2 \\ 3n-3 & n-1 & 2n-2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2n & n \\ 2n+1 & n+1 & 1 \\ n+2 & 2 & 2n+2 \\ 3 & 2n+3 & n+3 \\ 2n+4 & n+4 & 4 \\ \cdots & \cdots & \cdots \\ n-1 & 3n-1 & 2n-1 \end{pmatrix}$$

a. n=0 mod 3   b. n=1 mod 3

$$\begin{pmatrix} 0 & n & 2n \\ n+1 & 2n+1 & 1 \\ 2n+2 & 2 & n+2 \\ 3 & n+3 & 2n+3 \\ n+4 & 2n+4 & 4 \\ \cdots & \cdots & \cdots \\ 2n-1 & 3n-1 & n-1 \end{pmatrix}$$

c. n=2 mod 3

Similar to Case 1, one can use this matrix to define an $L(2,1)$-labeling of $K_n \times K_m \times K_l$ with span $nm - 1 = 3n - 1$.

Case 3: $m = 2$.

Clearly we have $K_n \times K_2 \times K_2 \cong K_n \times C_4$. An $L(2,1)$-labeling of $K_n \times C_4$ with span $2n - 1$ is given by the following matrix, where each row corresponds to a vertex of K_n and each column a vertex of C_4.

$$\begin{pmatrix} 2n-1 & 2 & 2n-2 & 2n-5 \\ 1 & 4 & 0 & 2n-3 \\ 3 & 6 & 2 & 2n-1 \\ 5 & 8 & 4 & 1 \\ 7 & 10 & 6 & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 2n-5 & 2n-2 & 2n-6 & 2n-9 \\ 2n-3 & 0 & 2n-4 & 2n-7 \end{pmatrix}. \tag{2.2}$$

∎

We now consider the case $n \leq 3$.

Theorem 2.2 $\lambda_{2,1}(K_2 \times K_2 \times K_2) = nm+2 = 6$ *and* $\lambda_{2,1}(K_3 \times K_2 \times K_2) = 8$.

Proof. $\lambda_{2,1}(K_2 \times K_2 \times K_2) = nm + 2 = 6$ comes from Theorem 1.3. And $\lambda_{2,1}(K_3 \times K_2 \times K_2) = \lambda_{2,1}(C_3 \times C_4) = 8$ by [12]. ∎

The next theorem deals with the case $n = m = 3$.

Theorem 2.3

$$\lambda_{2,1}(K_3 \times K_3 \times K_l) = \begin{cases} 9, & \text{if } l = 2, \\ 10, & \text{if } l = 3 \end{cases}$$

Proof. We prove the theorem by giving an $L(2,1)$-labeling of $K_3 \times K_3 \times K_2$ with span 9 and an $L(2,1)$-labeling of K_3^3 with span 10, and showing that $\lambda_{2,1}(K_3 \times K_3 \times K_2) > 8$ and $\lambda_{2,1}(K_3^3) > 9$

Following is an $L(2,1)$-labeling of $K_3 \times K_3 \times K_2$ with span 9. So $\lambda_{2,1}(K_3 \times K_3 \times K_2) \leq 9$.

$$\begin{pmatrix} 5 & 8 & 2 \\ 7 & 1 & 4 \\ 0 & 6 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 & 7 \\ 3 & 9 & 0 \\ 8 & 2 & 5 \end{pmatrix}$$

a. Labels on level 0 b. Labels on level 1

And an $L(2,1)$-labeling of K_3^3 with span 10 is given by the following three matrixes.

$$\begin{pmatrix} 0 & 4 & 8 \\ 5 & 9 & 1 \\ 10 & 2 & 6 \end{pmatrix} \qquad \begin{pmatrix} 9 & 1 & 5 \\ 2 & 6 & 10 \\ 4 & 8 & 0 \end{pmatrix} \qquad \begin{pmatrix} 6 & 10 & 2 \\ 8 & 0 & 4 \\ 1 & 5 & 9 \end{pmatrix}$$

a. Labels on level 0 b. Labels on level 1 c. Labels on level 2

We now show that $\lambda_{2,1}(K_3 \times K_3 \times K_2) > 8$ and $\lambda_{2,1}(K_3^3) > 9$.

First observe that, in any $L(2,1)$-labelling of $K_3 \times K_3 \times K_2$, if each of the three consecutive labels $i-1, i$ and $i+1$ is assigned to exactly two vertices of $K_3 \times K_3 \times K_2$, then the three vertices in the same level receive labels $i-1, i$,and $i+1$ respectively must lie in different rows and different columns. If this is false then the two vertices in some level labeled by $i-1$ and $i+1$ must lie in the same row (or column). Without loss of generality, suppose, in the the 0^{th} level, f assigns i to (a_0, b_0, c_0), $i-1$ to (a_1, b_1, c_0), and $i+1$ to (a_1, b_2, c_0). Then it is easy to see that, in the 1^{th} level, the only two vertices that can receive the label i are (a_2, b_1, c_1) and (a_2, b_2, c_1). If f assigns the label i to (a_2, b_1, c_1) then it is easy to see that no vertices in the 1^{th} level can receive the label $i+1$. If f assigns the label i to (a_2, b_2, c_1) then it is easy to see that no vertices in the 1^{th} level can receive the label $i-1$. Both are contradictions.

Suppose f is any $L(2,1)$-labelling of $K_3 \times K_3 \times K_2$. Then each label is used at most twice by f. From the above observation, any four consecutive labels are assigned to at most 7 vertices. As $K_3 \times K_3 \times K_2$ has 18 vertices, we must have $span(f) \geq 9$. Thus $\lambda_{2,1}(K_3 \times K_3 \times K_2) > 8$.

Now suppose f is an $L(2,1)$-labelling of K_3^3 with span 9. Then each label is used at most three times by f. Since $\lambda_{2,1}(K_3 \times K_3) = 8$, in each level of K_3^3, there is exactly one label in $[0, 9]$ not used by f. Let x_0, x_1 and x_2 be labels $[0, 9]$ not used by f in 0^{th}, 1^{th} and 2^{th} level of K_3^3, respectively. Without loss of generality, suppose $x_0 \leq x_1 \leq x_2$. Then $x_0 < x_1 < x_2$, $x_0 \leq 3$ and $x_2 \geq 6$, since, otherwise, there must exist an integer i in $[0, 6]$ and two levels of K_3^3 such that each of the four consecutive labels $i, i+1, i+2$ and $i+3$ is used twice by f in this two levels of K_3^3, which is a contradiction to the above observation. If $x_1 \leq 5$ then each of the four consecutive labels $6, 7, 8, 9$ is used twice by f in 0^{th} and 1^{th} levels of K_3^3, a contradiction. If $x_1 \geq 6$ then each of the four consecutive labels $0, 1, 2, 3$ is used twice by f in 1^{th} and 2^{th} levels of K_3^3, a contradiction. Thus $\lambda_{2,1}(K_3^3) > 9$. ∎

In this paper, we have completely determined the $\lambda_{2,1}$-numbers of the Cartesian products of any three complete graphs. It remains open to determine $\lambda_{j,k}$-numbers of the Cartesian products of any three complete graphs.

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