Potentially $K_{1,1,t}$-graphic sequences *

Jian-Hua Yin†
Department of Applied Math, College of Information Science and Technology, Hainan University, Haikou, Hainan 570228, China.

Jiong-Sheng Li
Department of Mathematics,
University of Science and Technology of China, Hefei, Anhui 230026, China.

Wen-Ya Li
Department of Applied Math, College of Information Science and Technology, Hainan University, Haikou, Hainan 570228, China.

Abstract: Let $\sigma(K_{1,1,t}, n)$ be the smallest even integer such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(K_{1,1,t}, n)$ has a realization G containing $K_{1,1,t}$ as a subgraph, where $K_{1,1,t}$ is the $1 \times 1 \times t$ complete 3-partite graph. Recently, Lai (Discrete Mathematics and Theoretical Computer Science, 7(2005), 75-81) conjectured that for $n \geq 2t + 4$,

$$\sigma(K_{1,1,t}, n) = \begin{cases} (t+1)(n-1)+2 & \text{if } n \text{ is odd or } t \text{ is odd,} \\ (t+1)(n-1)+1 & \text{if } n \text{ and } t \text{ are even.} \end{cases}$$

In this paper, we prove that the above equality holds for $n \geq t + 4$.
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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers with $d_i \leq n - 1$ for each i is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic non-increasing sequences in NS_n is denoted by GS_n. For a

*Supported by NNSF of China (No. 10401010) and China Scholarship Council.
†E-mail: yinjh@ustc.edu

sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a graph H, a graphic sequence π is said to be *potentially H-graphic* if there exists a realization of π containing H as a subgraph.

Gould, Jacobson and Lehel [3] considered the following extremal problem on potentially H-graphic sequences: for given a graph H, determine the smallest even integer $\sigma(H, n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic. If $H = K_{r+1}$, a complete graph on $r+1$ vertices, this problem was considered by Erdös et al. [2] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for sufficiently large n. Gould et al. [3] and Li and Song [7] independently proved it for $r = 3$. Recently, Li et al. [8,9] proved that the conjecture is true for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and $n \geq \binom{r}{2} + 3$. For $H = K_{r,s}$, the $r \times s$ complete bipartite graph, Gould et al. [3] determined $\sigma(K_{2,2}, n)$ for $n \geq 4$, Yin and Li [10] determined $\sigma(K_{3,3}, n)$ for $n \geq 6$ and $\sigma(K_{4,4}, n)$ for $n \geq 8$, Yin, Li and Chen [14,11,13] further determined $\sigma(K_{r,s}, n)$ for $s \geq r \geq 1$ and sufficiently large n. For the case of $H = K_{1,1,t}$, Lai in [5] determined $\sigma(K_{1,1,2}, n)$ for $n \geq 4$. Lai [6] further determined $\sigma(K_{1,1,3}, n)$ for $n \geq 5$ and gave a lower bound for $\sigma(K_{1,1,t}, n)$. The following are his results.

Theorem 1.1 [5]

$$\sigma(K_{1,1,2}, n) = \begin{cases} 2\left[\frac{3n-1}{2}\right] & \text{if } n \geq 4 \text{ and } n \neq 6, \\ 20 & \text{if } n = 6, \end{cases}$$

where $[x]$ denotes the integer part of x.

Theorem 1.2 [6]

$$\sigma(K_{1,1,3}, n) = \begin{cases} 4n-2 & \text{if } n \geq 5 \text{ and } n \neq 6, \\ 26 & \text{if } n = 6. \end{cases}$$

Theorem 1.3 [6] Let $n \geq t+2$. Then

$$\sigma(K_{1,1,t}, n) \geq \begin{cases} (t+1)(n-1)+2 & \text{if } n \text{ is odd or } t \text{ is odd}, \\ (t+1)(n-1)+1 & \text{if } n \text{ and } t \text{ are even}. \end{cases}$$

Moreover, in the end of [6], Lai conjectured that the equality in Theorem 1.3 holds for $n \geq 2t+4$. Recently, Chen [15] proved that the Lai's conjecture holds for $t \geq 3$ and $n \geq 2[\frac{(t+5)^2}{2}] + 3$. In this paper, we further show that the Lai's conjecture is true for $t \geq 3$ and $n \geq t+4$. In other words, we will prove the following

Theorem 1.4 Let $t \geq 3$ and $n \geq t+4$. Then

$$\sigma(K_{1,1,t}, n) = \begin{cases} (t+1)(n-1)+2 & \text{if } n \text{ is odd or } t \text{ is odd}, \\ (t+1)(n-1)+1 & \text{if } n \text{ and } t \text{ are even}. \end{cases}$$

Remark Let $t(\geq 3)$ be odd, $n = t+3$ and $\pi = ((t+1)^{t+3})$, where the symbol x^y in a sequence stands for y consecutive terms, each equal to x. It is easy to see that π is graphic but not potentially $K_{1,1,t}$-graphic, and $\sigma(\pi) > (t+1)((t+3)-1)+2$. Therefore, Theorem 1.4 is best possible in the sense that $t+4$ cannot be replaced by a smaller integer.

2. Proof of Theorem 1.4

In order to prove Theorem 1.4, we first need the following preliminaries. In [4], Kleitman and Wang introduced the "laying off" technique as follows. Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leq k \leq n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) \\ \quad \text{if } d_k \geq k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) \\ \quad \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is the rearrangement of the $n-1$ terms in π''_k. π'_k is called the *residual sequence obtained by laying off d_k* from π. It is easy to see that if π'_k is graphic then so is π, since a realization G of π can be obtained from a realization G' of π'_k by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π'_k. Kleitman and Wang [4] also proved that if π is graphic, then there exists a realization G of π such that the vertex with degree d_k is adjacent to those vertices (other than itself) which have the largest degrees of π, and hence π'_k is graphic. Thus, they obtained the following

Theorem 2.1 [4] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leq k \leq n$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.

Theorem 2.2 [1] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any t, $1 \leq t \leq n-1$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.$$

Theorem 2.3 [10,11] Let $\pi = (d_1, \ldots, d_n) \in NS_n$, $m = \max\{d_1, \ldots, d_n\}$ and $\sigma(\pi)$ be even. The rearrangement sequence of π is denoted by $\pi^* = (d_1^*, d_2^*, \ldots, d_n^*)$, where $d_1^* \geq d_2^* \geq \cdots \geq d_n^*$ is the rearrangement of $d_1, d_2, \ldots, d_n$. If there exists an integer $n_1 \leq n$ such that $d_{n_1}^* \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left[\frac{(m+h+1)^2}{4}\right]$, then $\pi \in GS_n$.

Theorem 2.4 [12] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence, where $d_1 = r$ and $\sigma(\pi)$ is even. If $d_{r+1} \geq r-1$, then $\pi \in GS_n$.

Theorem 2.5 [3] If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Let $\pi = (d_1, \ldots, d_{t+2}, \ldots, d_n) \in GS_n$. If π has a realization G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$ and G contains $K_{1,1,t}$ as its subgraph, where $\{v_1\}$, $\{v_2\}$ and $\{v_3, \ldots, v_{t+2}\}$ is the 3-partite partition of the vertex set of $K_{1,1,t}$, then π is said to be *potentially* $A_{1,1,t}$*-graphic.* On potentially $A_{1,1,t}$-graphic sequence, we have the following

Lemma 2.1 $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ is potentially $K_{1,1,t}$-graphic if and only if it is potentially $A_{1,1,t}$-graphic.

Proof. We only need to prove that if π is potentially $K_{1,1,t}$-graphic, then it is potentially $A_{1,1,t}$-graphic. By Theorem 2.5, we may assume that π has a realization G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$ and the induced subgraph $G[\{v_1, v_2, \ldots, v_{t+2}\}]$ by $\{v_1, v_2, \ldots, v_{t+2}\}$ contains $K_{1,1,t}$ as a subgraph, where $\{v_i\}$, $\{v_j\}$ and $\{v_1, \ldots, v_{t+2}\} - \{v_i, v_j\}$ $(1 \leq i < j \leq t+2)$ is the 3-partite partition of the vertex set of $K_{1,1,t}$. Denote $H = G[\{v_1, v_2, \ldots, v_{t+2}\}]$. We consider the following cases.

Case 1. $|\{v_1, v_2\} \cap \{v_i, v_j\}| = 2$. Then $\{v_1, v_2\} = \{v_i, v_j\}$, and π is clearly potentially $A_{1,1,t}$-graphic.

Case 2. $|\{v_1, v_2\} \cap \{v_i, v_j\}| = 1$. Without loss of generality, we assume that $i = 1$ and $j > 2$. Let $A = N_H(v_j) - (\{v_2\} \cup N_H(v_2))$ and $B = N_G(v_2) - (\{v_j\} \cup N_G(v_j))$. Since $d_G(v_2) \geq d_G(v_j)$, it follows that $|B| \geq |A|$. Now choose any subset $C \subseteq B$ having $|C| = |A|$. Now form a new realization G' of π by interchanging the edges of the star centered at v_j with endvertices in A with the non-edges of the star centered at v_j with endvertices in C, and interchanging the edges of the star centered at v_2 with endvertices in C with the non-edges of the star centered at v_2 with endvertices in A. It is easy to see that G' contains $K_{1,1,t}$ as a subgraph, and $\{v_1\}$, $\{v_2\}$ and $\{v_3, \ldots, v_{t+2}\}$ is the 3-partite partition of the vertex set of $K_{1,1,t}$. In other words, π is potentially $A_{1,1,t}$-graphic.

Case 3. $|\{v_1, v_2\} \cap \{v_i, v_j\}| = 0$. Similar to the proof of Case 2, we first construct a realization G' of π containing $K_{1,1,t}$ as a subgraph such that $\{v_1\}$, $\{v_j\}$ and $\{v_1, \ldots, v_{t+2}\} - \{v_1, v_j\}$ $(j > 2)$ is the 3-partite partition of the vertex set of $K_{1,1,t}$. Then, it follows from Case 2 with G' playing the role of G that π is potentially $A_{1,1,t}$-graphic. □

Lemma 2.2 If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_1 + d_2 \geq n + t$, then π is potentially $A_{1,1,t}$-graphic.

Proof. By Theorem 2.1, we may assume that G is a realization of π with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$. Let $A = N_G(v_1) - \{v_2\}$, $B = N_G(v_2) -$

$\{v_1\}$ and $C = A \cap B$. It is easy to see that $|A-B|+|B-A|+2|C|+2 = d_1+d_2 \geq n+t$ and $|A-B|+|B-A|+|C|+2 \leq n$. Thus $|C| \geq t$. Therefore, G contains $K_{1,1,t}$ as a subgraph. By Lemma 2.1, π is potentially $A_{1,1,t}$-graphic. □

Lemma 2.3 Let $t \geq 3$, $n \geq t+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $n-2 \geq d_1 \geq d_2 = \cdots = d_{d_1+2} \geq \cdots \geq d_n$ and $d_2 \geq t+1$. Let

$$\rho_1'(\pi) = (d_2-1, d_3-1, \ldots, d_{t+2}-1, d_{t+3}-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n),$$

and denote $\rho_1(\pi) = (d_2-1, d_3-1, \ldots, d_{t+2}-1, d_{t+3}^{(1)}, d_{t+4}^{(1)}, \ldots, d_n^{(1)})$, where $d_{t+3}^{(1)} \geq d_{t+4}^{(1)} \geq \cdots \geq d_n^{(1)}$ is the rearrangement of $d_{t+3}-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n$. Let

$$\rho_2'(\pi) = (d_3-2, \ldots, d_{t+2}-2, d_{t+3}^{(1)}-1, \ldots, d_{d_2+1}^{(1)}-1, d_{d_2+2}^{(1)}, \ldots, d_n^{(1)}),$$

and denote $\rho_2(\pi) = (d_3-2, \ldots, d_{t+2}-2, d_{t+3}^{(2)}, d_{t+4}^{(2)}, \ldots, d_n^{(2)})$, where $d_{t+3}^{(2)} \geq d_{t+4}^{(2)} \geq \cdots \geq d_n^{(2)}$ is the rearrangement of $d_{t+3}^{(1)}-1, \ldots, d_{d_2+1}^{(1)}-1, d_{d_2+2}^{(1)}, \ldots, d_n^{(1)}$. If $\rho_2(\pi)$ is graphic, then π is potentially $A_{1,1,t}$-graphic.

Proof. It easily follows from the definition of $\rho_2(\pi)$ that π is potentially $A_{1,1,t}$-graphic. □

Lemma 2.4 Let $n \geq t+2$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (t+1)(n-1)+2 & \text{if } n \text{ is odd or } t \text{ is odd,} \\ (t+1)(n-1)+1 & \text{if } n \text{ and } t \text{ are even.} \end{cases}$$

Then $d_2 \geq t+1$ and $d_{t+2} \geq 2$.

Proof. If $d_2 \leq t$, then $\sigma(\pi) \leq n-1+(n-1)t = (t+1)(n-1)$, a contradiction. If $d_{t+2} \leq 1$, then by Theorem 2.2, $\sigma(\pi) = \sum_{i=1}^n d_i = \sum_{i=1}^{t+1} d_i + \sum_{i=t+2}^n d_i \leq (t(t+1) + \sum_{i=t+2}^n \min\{t+1, d_i\}) + \sum_{i=t+2}^n d_i = t(t+1) + 2\sum_{i=t+2}^n d_i \leq t(t+1) + 2(n-t-1) = (t-2)(t+1)+2n = (t-1)(t+1)+2n-(t+1) < (t-1)n+2n-(t+1) = (t+1)(n-1)$, a contradiction. □

Lemma 2.5 Let $r \geq 4$, $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence, where $d_1 = r$ and $\sigma(\pi)$ is even.

(1) If $d_{r+2} \geq r-2$, then $\pi \in GS_n$.

(2) If $r-1 \geq d_2 \geq \cdots \geq d_{r+1} \geq r-2$, then $\pi \in GS_n$.

Proof. (1) Since $\frac{1}{r-2}\left[\frac{(r+r-2+1)^2}{4}\right] = r+1+\frac{2}{r-2} \leq r+2$, by Theorem 2.3, $\pi \in GS_n$.

(2) If $d_{r+2} \geq r-2$, then by (1), $\pi \in GS_n$. Assume $d_{r+2} \leq r-3$. Let $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ be the residual sequence obtained by laying off d_1 from π. Then $\sigma(\pi_1')$ is even, and $d_1' = d_2-1, d_2' = d_3-1, \ldots, d_r' = d_{r+1}-1$. Clearly, $r-2 \geq d_1' \geq \cdots \geq d_r' \geq r-3$. It follows from Theorem 2.4 that $\pi_1' \in GS_{n-1}$, and $\pi \in GS_n$ by Theorem 2.1. □

We now prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.3, it is enough to prove that if $t \geq 3$, $n \geq t+4$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (t+1)(n-1)+2 & \text{if } n \text{ is odd or } t \text{ is odd,} \\ (t+1)(n-1)+1 & \text{if } n \text{ and } t \text{ are even,} \end{cases}$$

then π is potentially $A_{1,1,t}$-graphic. By Lemma 2.4, we have $d_2 \geq t+1$ and $d_{t+2} \geq 2$. Now use induction on t. If $t = 3$, then by Theorem 1.2 and Lemma 2.1, π is potentially $A_{1,1,3}$-graphic. In other words, Theorem 1.4 holds for $t = 3$. Assume that $t \geq 4$ and π is not potentially $A_{1,1,t}$-graphic. We have

(1) $d_1 \leq n-2$ and $d_{t+2} = \cdots = d_{d_1+2}$. If $d_1 = n-1$ or there exists an integer k, $t+2 \leq k \leq d_1+1$ such that $d_k > d_{k+1}$, then the residual sequence $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ obtained by laying off d_1 from π satisfies $d_1' = d_2 - 1, d_2' = d_3 - 1, \ldots, d_{t+1}' = d_{t+2} - 1$. By Theorem 2.1, π_1' is graphic, and hence π_1' has a realization G' such that the vertex with degree d_1' is adjacent to those vertices having degrees $d_2', \ldots, d_{t+1}', \ldots, d_{d_1'+1}'$. We now form a realization G of π from G' by adding a new vertex of degree d_1 and joining it to the vertices whose degrees are reduced by one in going from π to π_1'. It is easy to see that G contains $K_{1,1,t}$ as a subgraph. Hence π is potentially $A_{1,1,t}$-graphic, a contradiction. Thus, $d_1 \leq n-2$ and $d_{t+2} = \cdots = d_{d_1+2}$.

(2) $d_2 \leq [\frac{n+t+1}{2}] - 1$. If $d_2 \geq [\frac{n+t+1}{2}]$, then $d_1 + d_2 \geq n+t$, and so π is potentially $A_{1,1,t}$-graphic by Lemma 2.2, a contradiction. Hence, $d_2 \leq [\frac{n+t+1}{2}] - 1$.

(3) $d_2 = d_3 = \cdots = d_{t+2}$. If $d_2 > d_3$, then $d_3 \leq [\frac{n+t+1}{2}] - 2$ and the residual sequence $\pi_3' = (d_1', d_2', \ldots, d_{n-1}')$ obtained by laying off d_3 from π satisfies $d_1' = d_1 - 1, d_2' = d_2 - 1, n-1 \geq (t-1)+4$ and $\sigma(\pi_3') = \sigma(\pi) - 2d_3 \geq (t+1)(n-1)+1-(n+t+1)+4 > ((t-1)+1)((n-1)-1)+2$. By the induction hypothesis, π_3' is potentially $A_{1,1,t-1}$-graphic. It is easy to get that π is potentially $K_{1,1,t}$-graphic, a contradiction. Hence $d_2 = d_3$. If there exists an integer k, $3 \leq k \leq t+1$ such that $d_k > d_{k+1}$, then $t+1 \leq d_3 \leq [\frac{n+t+1}{2}] - 1$, and it is easy to verify that the residual sequence $\pi_3' = (d_1', d_2', \ldots, d_{n-1}')$ obtained by laying off d_3 from π satisfies $d_1' = d_1 - 1, d_2' = d_2 - 1, n-1 \geq (t-1)+4$ and $\sigma(\pi_3') = \sigma(\pi) - 2d_3 \geq ((t-1)+1)((n-1)-1)+2$. By the induction hypothesis, π_3' is potentially $A_{1,1,t-1}$-graphic, and hence π is potentially $K_{1,1,t}$-graphic, a contradiction. Thus, $d_2 = d_3 = \cdots = d_{t+2}$.

By (1), (2) and (3), $\pi = (d_1, d_2, \ldots, d_n)$ satisfies $n-2 \geq d_1 \geq d_2 = d_3 = \cdots = d_{t+2} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. Let $d_2 = d_3 = \cdots = d_{d_1+2} = r(\geq t+1)$. Clearly, $d_1 + 2 \geq r+2 \geq t+3$. By the definition of $\rho_2(\pi)$, we have $\rho_2(\pi) = ((r-2)^t, d_{t+3}^{(2)}, \ldots, d_n^{(2)})$, where $r-1 \leq d_{t+3}^{(2)} \leq r$.

Let m, h and ℓ denote the term number of r, $r-1$ and $r-2$ appearing in $\rho_2(\pi)$, respectively. Clearly, $\ell \geq t$. If $m=0$, then by the definition of $\rho_2(\pi)$, it is easy to see that $h+\ell \geq (d_1+2)-2=d_1 \geq r$, and hence $\rho_2(\pi)$ is graphic by Theorem 2.4. Thus, π is potentially $A_{1,1,t}$-graphic by Lemma 2.3, a contradiction. Assume that $m \geq 1$. By the definition of $\rho_2(\pi)$, we have $h \geq d_1+r-2(t+1)$. We now consider the following cases.

Case 1. $d_1 \geq r+2$. Then $m+h+\ell \geq 1+d_1+r-2(t+1)+t = d_1+r-(t+1) \geq r+2$. By Lemma 2.5(1), $\rho_2(\pi)$ is graphic, and hence π is potentially $A_{1,1,t}$-graphic, a contradiction.

Case 2. $d_1 = r+1$. If $r \geq t+2$, then $m+h+\ell \geq r+2$, $\rho_2(\pi)$ is graphic by Lemma 2.5(1), and hence π is potentially $A_{1,1,t}$-graphic, a contradiction. If $r=t+1$ and $m \geq 2$, then $m+h+\ell \geq r+2$, $\rho_2(\pi)$ is graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $r=t+1$ and $m=1$, then $m+h+\ell \geq r+1$, $\rho_2(\pi)$ is graphic by Lemma 2.5(1) or Lemma 2.5(2), and π is potentially $A_{1,1,t}$-graphic, a contradiction.

Case 3. $d_1 = r$. If $r \geq t+2$ and $m \geq 2$, then $m+h+\ell \geq r+2$, $\rho_2(\pi)$ is graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $r \geq t+2$ and $m=1$, then $m+h+\ell \geq r+1$, $\rho_2(\pi)$ is graphic by Lemma 2.5(1) or Lemma 2.5(2), and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $r=t+1$ and $m \geq 3$, then $m+h+\ell \geq r+2$, $\rho_2(\pi)$ is graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. We now assume that $r=t+1$ and $m=2$. In this case, if $h \geq 1$ or $\ell \geq t+1$, then $m+h+\ell \geq r+2$, $\rho_2(\pi)$ is graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $h=0$ and $\ell=t$, then $\rho_2(\pi)=((t+1)^2,(t-1)^t,d_{t+5}^{(2)},\ldots,d_n^{(2)})$ (in non-increasing order), where $d_{t+5}^{(2)} \leq t-2$. By Lemma 2.5(2), the residual sequence $(t,(t-2)^t,d_{t+5}^{(2)},\ldots,d_n^{(2)})$ obtained by laying off $t+1$ from $\rho_2(\pi)$ is graphic. Thus, $\rho_2(\pi)$ is also graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. Finally, we assume that $r=t+1$ and $m=1$. In this case, $\rho_2(\pi)=((t-1)^t,t+1,d_{t+4}^{(2)},\ldots,d_n^{(2)})$, where $d_{t+4}^{(2)} \leq t$. If $t-1 \leq d_{t+4}^{(2)} \leq t$, then by Lemma 2.5(2), $\rho_2(\pi)$ is graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $1 \leq d_{t+4}^{(2)} \leq t-2$, then by Theorem 2.4, the residual sequence $((t-2)^t,d_{t+4}^{(2)}-1,d_{t+5}^{(2)},\ldots,d_n^{(2)})$ obtained by laying off $t+1$ from $\rho_2(\pi)$ is graphic, and hence $\rho_2(\pi)$ is also graphic and π is potentially $A_{1,1,t}$-graphic, a contradiction. If $d_{t+4}^{(2)}=0$, then $\rho_2(\pi)=((t-1)^t,t+1,0^{n-t-3})$ and $\pi=((t+1)^{t+3},0^{n-t-3})$. Thus, $\sigma(\pi)=(t+1)(t+3)<(t+1)(n-1)+1 \leq \sigma(\pi)$, which is impossible. □

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