The Laplacian spectral radius of graphs with given connectivity *

Lihua Feng[a]†, Aleksandar Ilić[b]

[a]School of Mathematics, Shandong Institute of Business and Technology
191 Binhaizhong Road, Yantai, Shandong, P.R. China, 264005.

[b]Faculty of Sciences and Mathematics, University of Niš
Višegradska 33, 18000 Niš, Serbia

e-mail: `fenglihua2007@gmail.com`, `aleksandari@gmail.com`

Abstract

Motivated by the results from [J. Li, W. Shiu, W. Chan, The Laplacian spectral radius of some graphs, *Linear Algebra Appl.* 431 (2009) 99-103.], we determine the extremal graphs with the second largest Laplacian spectral radius among all bipartite graphs with vertex connectivity k.

Key words: Laplacian spectral radius; connectivity; bipartite graphs.
AMS Classifications: 05C50, 05C05.

1 Introduction

Let G be a simple connected graph. The number of vertices of G is denoted by $|G|$. The matrix $L(G) = D(G) - A(G)$ is called *the Laplacian matrix* of a graph G, where $D(G) = diag(d_u, u \in V(G))$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. The matrix $L(G)$ is a positive semi-definite and singular matrix [3]. The largest eigenvalue of $L(G)$ is called *the Laplacian spectral radius* of G and is denoted by $\lambda = \lambda(G)$. Suppose $Q(G) = D(G) + A(G)$, we call this matrix *the signless Laplacian matrix* and its largest eigenvalue is denoted by $\mu = \mu(G)$. It is well-known that $Q(G)$ is an irreducible non-negative matrix, and therefore from the Perron-Frobenius theorem, there is a unique positive unit eigenvector corresponding to $\mu(G)$. For the background on

*Supported by The Postdoctoral Science Foundation of Central South University, China Postdoctoral Science Foundation(20110491275), NNSFC (10871205, 11101245), NSFSD (Y2008A04, BS2010SF017, ZR2011AQ005), BMNSF(4112052) and Research Grant 144007 of Serbian Ministry of Science.

the Laplacian eigenvalues of a graph, the reader may refer to [11] and the references therein.

For a vertex v in G, $N(v)$ denotes the set of neighbors of v and d_v denotes the degree of v. Let $G[S]$ denotes the subgraph induced by the vertex set S, $S \subset V(G)$. For $k \geq 1$, a graph G is k-connected if either G is a complete graph K_{k+1}, or G has at least $k+2$ vertices and contains no $(k-1)$-vertex cut. Similarly, for $k \geq 1$, a graph G is k-edge-connected if it has at least two vertices and does not contain $(k-1)$-edge cut. The maximal value of k for which a connected graph G is k-connected is the connectivity of G, denoted by $\kappa(G)$. If G is disconnected, we define $\kappa(G) = 0$. The edge-connectivity $\kappa'(G)$ is defined analogously. For other notations in graph theory, we follow [2]. If G is a graph of order n, then (1) $\kappa(G) \leq \kappa'(G) \leq n-1$; (2) $\kappa(G) = n-1$, $\kappa'(G) = n-1$ and $G \cong K_n$ are equivalent.

Therefore in the sequel, we assume $1 \leq k \leq n-2$. If G is a bipartite graph, then $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. We denote by $\mathcal{V}_n^k$ the set of graphs of order n with $\kappa(G) = k \leq n-2$, and by $\mathcal{E}_n^k$ the set of graphs of order n with $\kappa'(G) = k \leq n-2$.

Let $K_{n,m}$ be a complete bipartite graph. Denote by $B_{p,k}^l$ ($p \geq 1, k+l \geq 1, l \geq 0$) the graph with $p+k+l+1$ vertices, obtained from $K_{p,k+l}$ by adding a new vertex together with edges joining this vertex to k vertices of the bipartition with $k+l$ vertices. If $l = 0$, then $B_{p,k}^0 \cong K_{p+1,k}$. For $k \leq p$, $\kappa(B_{p,k}^l) = k$.

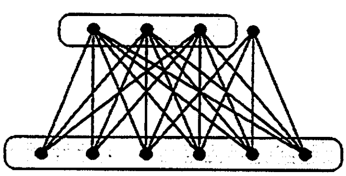

Figure 1: *The graph $B_{3,4}^2$ with 10 vertices.*

In [1], Brualdi and Solheid proposed the following problem concerning spectral radius: *Given a set of graphs $\mathcal{G}$ find an upper bound for the spectral radius of graphs in $\mathcal{G}$ and characterize the graphs in which the maximal spectral radius is attained.* This problem is well studied, see for example [4, 5, 10, 14]. For the Laplacian spectral radius of this problem, see [12, 13]. Recently, Li et al. [8] determined the extremal graphs which maximize the Laplacian spectral radius among all bipartite graphs with vertex and edge connectivity k.

In this paper, we continue to estimate the Laplacian spectral radius of graphs in $\mathcal{V}_n^k$ and $\mathcal{E}_n^k$ and we obtain the second largest value of the Laplacian spectral radius for bipartite graphs with connectivity k.

2 Bipartite graphs with $\kappa(G) = k$

Lemma 2.1 *[6] For a connected graph G, we have $\lambda(G) \leq \mu(G)$, with equality if and only if G is bipartite.*

Lemma 2.2 *[3] Let K be a spanning subgraph of a connected graph G. Then $\lambda(K) \leq \lambda(G)$.*

Since $B^l_{n-k-l-1,k}$ contains the complete bipartite graph $K_{n-k-l-1,k+l}$ as a subgraph, it follows that $\lambda(B^l_{n-k-l-1,k}) \geq \lambda(K_{n-k-l-1,k+l}) = n-1$.

Lemma 2.3 *The Laplacian spectral radius of $B^l_{n-k-l-1,k}$ is the largest root of the cubic equation $f(x) = 0$, where $f(x) = x^3 + (l+1-2n)x^2 + (n^2 - ln + kn - n - kl + l - k^2 - k)x + (kn + k^2n + kln - kn^2)$.*

Proof. By Lemma 2.1, we consider Q–matrix of $B^l_{n-k-l-1,k}$. Let X be the eigenvector of $\mu(B^l_{n-k-l-1,k})$. By symmetry, we can suppose the eigencomponents corresponding to the vertices of degrees $k, n-k-l, n-k-l-1, k+l$ are x_1, x_2, x_3, x_4, respectively. From the eigenvalue equation $QX = \mu X$, it follows

$$\begin{aligned}
\mu x_1 &= kx_1 + kx_2, \\
\mu x_2 &= (n-k-l)x_2 + x_1 + (n-k-l-1)x_4, \\
\mu x_3 &= (n-k-l-1)x_3 + (n-k-l-1)x_4, \\
\mu x_4 &= (k+l)x_4 + kx_2 + lx_3.
\end{aligned}$$

Simplifying the above equations, we have $\mu - n + k + l - \frac{k}{\mu - k} = \frac{k(n-k-l-1)}{\mu - k - l - \frac{l(n-k-l-1)}{\mu - n + k + l + 1}}$, or equivalently, $\mu^4 + (1 + l - 2n)\mu^3 + (-k - k^2 + l - kl - n + kn - ln + n^2)\mu^2 + (kn + k^2n + kln - kn^2)\mu = 0$. Since $\lambda(B^l_{n-k-l-1,k}) \geq n-1$, this implies the result. ■

Lemma 2.4 *$\lambda(B^l_{n-k-l-1,k})$ is strictly decreasing with respect to l for $0 \leq l \leq n-k-2$.*

Proof. From Lemma 2.3, $\lambda(B^{l+1}_{n-k-l-2,k})$ satisfies $g(x) = 0$, where $g(x) = x^3 + (2 + l - 2n)x^2 + (n^2 - ln + kn - 2n - kl + l - k^2 - 2k + 1)x + (2kn + k^2n + kln - kn^2)$.

Let $r(x) = f(x) - g(x) = -x^2 + (n + k - 1)x - kn$, where $f(x)$ is as in Lemma 2.3. Since $k \leq n-2$, it follows $r'(x) = -2x + n + k - 1 < 0$ for $x \geq n-1$. Since $\lambda(B^l_{n-k-l-1,k}) \geq \lambda(K_{n-k-l-1,k+l}) = n-1$, we have $r(\lambda(B^l_{n-k-l-1,k})) < r(n-1) = -k < 0$. This is equivalent to $f(\lambda(B^l_{n-k-l-1,k})) = 0 < g(\lambda(B^l_{n-k-l-1,k}))$, therefore the largest root of $g(x) = 0$ is less than $\lambda(B^l_{n-k-l-1,k})$, which implies the result. ■

Lemma 2.5 *[7] Let G be a connected graph and u, v be two vertices of G. Suppose $v_1, v_2, \dots, v_s \in N(v) \setminus (N(u) \cup \{u\})$ $(1 \le s \le d_v)$, and G^* is the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \le i \le s)$. Let $X = (x_1, x_2, \dots, x_n)^T$ be the principal eigenvector of $Q(G)$, where x_i corresponds to v_i $(1 \le i \le n)$. If $x_u \ge x_v$, then $\mu(G) < \mu(G^*)$.*

Theorem 2.6 *Let G be a connected bipartite graph of order n with connectivity k $(1 \le k \le \lfloor \frac{n}{2} \rfloor)$. Then we have*

(1) $\lambda(G) \le n$ with equality holding if and only if $G \cong K_{k,n-k}$.

(2) If $G \ncong K_{k,n-k}$, then $\lambda(G) \le \lambda(B^1_{n-k-2,k})$ with equality holding if and only if $G \cong B^1_{n-k-2,k}$.

Proof. The proof of the first part can be found in [8], and we will consider the second part. Let $G \ncong K_{k,n-k}$ be the graph with maximal Laplacian spectral radius among all bipartite graphs of order n with vertex connectivity k, different from $K_{k,n-k}$. Let U be a vertex cut-set of G containing k vertices, whose deletion yields the components $G_1, G_2, \dots, G_s$ of $G - U$, where $s \ge 2$.

If some component G_i of $G - U$ has at least two vertices, since G is bipartite, then $G - U$ is also bipartite, i.e., G_i is bipartite. Note that G has maximal Laplacian spectral radius, then G_i must be complete bipartite.

If some component G_i of $G - U$ is a singleton, say $G_i = \{w\}$, then w joins all vertices of U (otherwise $\kappa(G) < k$) and hence the subgraph $G[U]$ induced by U contains no edges.

Claim 1. If all components of $G - U$ are singletons, then $G \cong K_{k,n-k}$. So this is impossible.

Claim 2. If there exists one component, say G_1, of $G - U$ such that $|G_1| \ge 2$, then $G - U$ contains exactly two components.

Suppose that G_1 is complete bipartite with bipartition (V_1^1, V_1^2) and $s \ge 3$, we discuss in the following two cases.

Case 1: If there exists another component of $G - U$, say G_2, such that $|G_2| \ge 2$. Then G_2 is complete bipartite with bipartition (V_2^1, V_2^2). Since $s \ge 3$, then there is a component G_3 with $|G_3| \ge 1$.

If $|G_3| = 1$, say $G_3 = \{w'\}$, then adding all possible edges between V_2^1 (perhaps V_2^2 to make sure that the resulting graph is bipartite) and w', we can get a bipartite graph G' with more edges, and by Lemma 2.1 $\lambda(G) = \mu(G) < \mu(G') = \lambda(G')$. This contradicts to the choice of G.

If $|G_3| \ge 2$, then G_3 is complete bipartite with bipartition (V_3^1, V_3^2). Adding all possible edges between V_3^1 and V_2^2, we get a new bipartite graph with more edges and larger Laplacian spectral radius. This is again a contradiction.

Case 2: If $G_2, \ldots, G_s$ are singletons. Assume $G_i = \{w_i\}$ for $i = 2, \ldots, s$. Then $d_{w_i} = k$ for $i = 2, \ldots, s$; and $G[U]$ is an empty graph since G is bipartite. Adding all possible edges between U and V_1^1; and between w_2 and V_1^2, we get a new graph G'', and by Lemma 2.1 $\lambda(G) = \mu(G) < \mu(G'') = \lambda(G'')$. This contradicts to the choice of G.

From Claim 2, we can suppose $s = 2$.

Claim 3. G must be of the form $B^1_{n-k-2,k}$.

Case 1: If $|G_1| \geq 2$ and $|G_2| = 1$, suppose $G_2 = \{w\}$, then $G[U]$ contains no edges and $d_w = k$. Adding all possible edges between U and V_1^1, we can get $\lambda(G) \leq \lambda(B^1_{n-k-2,k})$. Thus in this case G must be of the form $B^1_{n-k-2,k}$.

Case 2: If $|G_1| \geq 2$ and $|G_2| \geq 2$, we will prove that G is of the form $B^l_{p,k}$, where $p + l + k + 1 = n$. Finally, the result follows from Lemma 2.4.

Assume that every vertex in U is adjacent to every vertex in $V_1^1 \subseteq V(G_1)$ and $V_2^1 \subseteq V(G_2)$ since G has maximal Laplacian spectral radius.

If there exists one vertex v_2 in $G - U$ such that $d_{v_2} = k$, suppose $v_2 \in V_2^1 \subseteq V(G_2)$ and $N(v_2) = \{u_1, u_2, \ldots, u_k\}$. Since G is bipartite, it follows $\{u_1, u_2, \ldots, u_k\}$ is contained in the same partition set of G. Note that $N(v_2)$ is also a vertex cut-set of G containing k vertices; $G - v_2$ is bipartite with $\{u_1, u_2, \ldots, u_k\}$ in the same partition set. Adding all possible edges in $G - v_2$ to make it complete bipartite, we get the graph $B^l_{p,k}$ and $\lambda(G) \leq \lambda(B^l_{p,k})$ from Lemma 2.1.

If all vertices from $G - U$ have degree greater than k, we take two vertices $v_1 \in V_1^1 \subseteq V(G_1)$, $v_2 \in V_2^1 \subseteq V(G_2)$. Without loss of generality, assume that $x(v_1) \geq x(v_2) > 0$, where $x(u)$ is the eigencomponent of $\mu(G)$ corresponding to the vertex u. Suppose $d_{v_2} = |V_2^2| + s > k$ since $d_{v_2} > k$, where s ($0 \leq s \leq k$) is the total number of edges joining v_2 and some vertices of U. Now we arbitrarily pick a set W of $|V_2^2| - (k - s) > 0$ vertices in V_2^2. Deleting the edges between v_2 and vertices of W, and then adding the edges between v_1 and the vertices of W, we get a bipartite graph $\widetilde{G}$ in which the degree of v_2 is k. By Lemma 2.1 and Lemma 2.5, we have $\lambda(G) = \mu(G) < \mu(\widetilde{G}) = \lambda(\widetilde{G})$. From above, it follows $\lambda(\widetilde{G}) < \lambda(B^l_{p,k})$.

From the above three claims, we get the result. ■

Corollary 2.7 *The Laplacian spectral radius of* $B^1_{n-k-2,k}$ *satisfies* $h(x) = 0$, *where* $h(x) = x^3 + (2-2n)x^2 + (n^2 - 2n + kn + 1 - k^2 - 2k)x + (2kn + k^2 n - kn^2)$. *Moreover,*

$$n - 1 < \lambda(B^1_{n-k-2,k}) < n - 1 + \frac{-T + \sqrt{T^2 + 4(n-1)T}}{2(n-1)},$$

where $T = kn - k^2 - 2k > 0$.

Proof. From Lemma 2.3, we have $B^1_{n-k-2,k}$ satisfies $h(x) = 0$. Since $B^1_{n-k-2,k}$ contains a complete bipartite graph $K_{n-k-2,k+1}$, we have $n-1 < \lambda(B^1_{n-k-2,k}) < n$. Let $\lambda(B^1_{n-k-2,k}) = n-1+t$, where $0 < t < 1$. Taking this into $h(x) = 0$, then t satisfies $m(t) = 0$, where $m(t) = t^3 + (n-1)t^2 + (kn - k^2 - 2k)t + 2k + k^2 - kn$.

Suppose that the three roots of $m(t) = 0$ are t_1, t_2, t_3. If three roots are real, then we can order them as $t_1 \geq t_2 \geq t_3$. Otherwise, there is only one real root t_1 and $t_2 + t_3$ and t_2t_3 are both real numbers. From Viéta's formulas, we have

$$\begin{aligned} t_1 + t_2 + t_3 &= -n+1, \\ t_1t_2 + t_1t_3 + t_2t_3 &= kn - k^2 - 2k, \\ t_1t_2t_3 &= kn - k^2 - 2k. \end{aligned}$$

Since $n > k+2$, it follows $T = kn-k^2-2k > 0$. In order to estimate t_1, first note that $t_2t_3 = T - t_1(t_2 + t_3) = T - t_1(-n+1-t_1) > T + (n-1)t_1$, and consequently $t_1 = \frac{T}{t_2t_3} < \frac{T}{T+(n-1)t_1}$. Finally, by solving the last inequality, we have $t_1 < \frac{-T+\sqrt{T^2+4(n-1)T}}{2(n-1)}$. This completes the proof. ∎

3 Concluding remarks

In this section, we present some properties of extremal graphs with maximal Laplacian spectral radius and vertex (edge) connectivity k.

Lemma 3.1 *[9] Let G be a connected graph. Then $\lambda(G) = n$ if and only if G contains $K_{t,n-t}$ as a spanning subgraph for some t.*

Using this lemma, we have the following two results for general graphs.

Theorem 3.2 *Let G be a connected graph in $\mathcal{V}^k_n$. G has the maximal Laplacian spectral radius if and only if G contains $K_{t,n-t}$ as a spanning subgraph for some t. Moreover, $\lambda(G) = n$.*

Theorem 3.3 *Let G be a connected graph in $\mathcal{E}^k_n$. G has the maximal Laplacian spectral radius if and only if G contains $K_{t,n-t}$ as a spanning subgraph for some t. Moreover, the minimum degree of G is k and $\lambda(G) = n$.*

Proof. The proof of the first part comes from that of Lemma 3.1. For the later part, since G contains $K_{t,n-t}$ as a subgraph, we have that the diameter of G is two. From Theorem 3.22 in [2] page 77, we get $\kappa'(G)$ equals to $\delta(G)$, the minimum vertex degree of G; while $\kappa'(G) = k$, the result follows. ∎

Acknowledgement. The authors are grateful to the referees for their helpful comments.

References

[1] R.A. Brualdi, E.S. Solheid, On the spectral radius of complementary acyclic matrices of zeros and ones, *SIAM J. Algebra Discret. Method.* 7 (1986) 265-272.

[2] G. Chartrand, L. Lesniak, *Graphs and Digraphs (Third Edition)*, Chapman & Hall, 1996.

[3] D. Cvetković, M. Doob, H. Sachs, *Spectra of graphs - Theory and Application*, 3rd edition, Johann Ambrosius Barth Verlag, 1995.

[4] L. Feng, G. Yu, X.-D. Zhang, Spectral radius of graphs with given matching number, *Linear Algebra Appl.* 422 (2007) 133-138.

[5] L. Feng, Q. Li, X.-D. Zhang, Spectral radii of graphs with given chromatic number, *Appl. Math. Lett.* 20 (2007) 158-162.

[6] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, *SIAM J. Matrix Anal. Appl.* 11 (1990) 218-238.

[7] Y. Hong, X.-D. Zhang, Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees, *Discrete Math.* 296 (2005) 187-197.

[8] J. Li, W. Shiu, W. Chan, The Laplacian spectral radius of some graphs, *Linear Algebra Appl.* 431 (2009) 99-103.

[9] B. Liu, Z. Chen, M. Liu, On Graphs with Largest Laplacian Index *Czechoslovak Math. J.* 58(4) (2008) 949-960

[10] H. Liu, M. Lu, F. Tian, On the spectral radius of graphs with cut edges, *Linear Algebra Appl.* 389 (2004) 139-145.

[11] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra Appl.* 197-198 (1994) 143-176.

[12] M.Q. Zhai, J.L. Shu, Z.H. Lu, Maximizing the Laplacian spectral radii of graphs with given diameter, *Linear Algebra Appl.* 430 (2009) 1897-1905

[13] X.L. Zhang, H.P. Zhang, The Laplacian spectral radius of some bipartite graphs, *Linear Algebra Appl.* 428 (2008) 1610-1619.

[14] X.L. Zhang, H.P. Zhang, On the Laplacian spectral radius for unicyclic graphs with k pendant vertices, *Ars Combin.* 90 (2009) 345-355.