A Look at the Reciprocal of the Generating Function for p-Regular Partitions

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Abstract: The generating function for p-regular partitions is given by $\frac{(q^p;q^p)_\infty}{(q;q)_\infty}$. In this paper we will investigate the reciprocal of this generating function. Several interesting results will be presented and as a corollary of one of these we will get a parity result due to Sellers for p-regular partitions with distinct parts.

A p-regular partition of n is a partition of n in which no part is divisible by p [3, 4]. The generating function for p-regular partitions is given by $\sum_{n=0}^{\infty} b_p(n)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}$ where $b_p(n)$ is used to denote the number of p-regular partitions of n. In this paper we will investigate the reciprocal of this generating function, namely $\frac{(q;q)_\infty}{(q^p;q^p)_\infty}$. Before we begin our investigation, let's discuss what this generating function counts.

The generating function $(q;q)_\infty$ appears in Euler's Pentagonal Number Theorem [6] and counts the number of partitions of n into distinct parts where those with an even number of parts are counted with weight 1 and those with an odd number of parts are counted with weight -1. We will let Q(n) denote the number of partitions of n into distinct parts and $\Delta Q(n)$ will denote the coefficient of n in $(q;q)_\infty$. That is, $\Delta Q(n)$ will represent the difference between the number of partitions of n into an even number of distinct parts and the number of partitions of n into an odd number of distinct parts. Euler proved that $\Delta Q(n) = (-1)^k$ if $n = \frac{3k^2+k}{2}$ where k is an integer and $\Delta Q(n) = 0$ for all other values of n.

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The generating function $\frac{(q;q)_\infty}{(q^p;q^p)_\infty}$ counts the number of partitions of n into distinct parts where none of the parts are divisible by p and those with an even number of parts are counted with weight 1 and those with an odd number of parts are counted with weight −1. Using the notation of Sellers [4], we will let $b'_p(n)$ denote the number of p-regular partitions of n with distinct parts. $\Delta b'_p(n)$ will be used to represent the difference between the number of p-regular partitions of n into an even number of distinct parts and the number of p-regular partitions of n into an odd number of distinct parts. Hence $\frac{(q;q)_\infty}{(q^p;q^p)_\infty} = \sum_{n=0}^{\infty} \Delta b'_p(n) q^n$. Our first theorem looks at a sufficient condition for $\Delta b'_p(n)$ to be zero.

Theorem 1
If p is a prime with $p > 3$, then $\Delta b'_p(pk+r) = 0$ if $0 < r < p$ and $24r + 1$ is a quadratic nonresidue modulo p.

Before we prove Theorem 1 let's look at a parity result that follows from this theorem. Since $b'_p(n)$ is the sum of the number of p-regular partitions of n into an even number of distinct parts and the number of p-regular partitions of n into an odd number of distinct parts, we have $b'_p(n) \equiv \Delta b'_p(n) \pmod 2$. As a corollary we obtain the following results of Sellers on the parity of the coefficients in $\frac{(q;q)_\infty}{(q^p;q^p)_\infty}$.

Corollary 1
If p is a prime and $p > 3$, then $b'_p(pk+r) \equiv 0 \pmod 2$ if $0 < r < p$ and $24r + 1$ is a quadratic nonresidue modulo p.

The proof of Theorem 1 is essentially identical to Sellers' proof of the parity result except we are dealing with equality of two generating functions instead of a congruence. We can rewrite our generating function as

$$\sum_{n=0}^{\infty} \Delta b'_p(n) q^n = \frac{(q;q)_\infty}{(q^p;q^p)_\infty} = \frac{1}{(q^p;q^p)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2+m}{2}} \qquad (1)$$

where the last series comes from Euler's Pentagonal Number Theorem. Now on the right hand side of (1) we will get $pk + r$ as a exponent on q only if $\frac{3m^2+m}{2} \equiv r \pmod p$ for some m in $\mathbb{Z}$. Multiplying both sides of this congruence by 24 and adding 1 to both sides we get $(6m + 1)^2 \equiv 24r + 1 \pmod{p}$ which gives the desired result.

Now we will look at this generating function for specific values of p including p = 2 and p = 3. We will begin with p = 2.

Theorem 2

(1) $\Delta b_2'(2k)$ is nonnegative and equals the number of partitions of k into parts not congruent to 0, 1, 6, 7, 8, 9, 10, 15 (mod 16) and

(2) $\Delta b_2'(2k+1) \leq 0$ and $|\Delta b_2'(2k+1)|$ equals the number of partitions of k into parts not congruent to 0, 2, 3, 5, 8, 11, 13, 14 (mod 16).

This theorem follows immediately by rewriting the generating function for $\Delta b_2'(n)$.

$$\sum_{n=0}^{\infty} \Delta b_2'(n) q^n = \frac{(q;q)_\infty}{(q^2;q^2)_\infty} = \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2+m}{2}}$$

$$= \frac{1}{(q^2;q^2)_\infty}\left(\left[\sum_{k=-\infty}^{\infty} q^{24k^2+2k} - \sum_{k=-\infty}^{\infty} q^{24k^2+14k+2}\right] + \left[\sum_{k=-\infty}^{\infty} q^{24k^2+26k+7} - \sum_{k=-\infty}^{\infty} q^{24k^2-10k+1}\right]\right) \quad (2)$$

$$= \frac{(q^{16};q^{16})_\infty (q^{14};q^{16})_\infty (q^2;q^{16})_\infty (q^{12};q^{32})_\infty (q^{20};q^{32})_\infty}{(q^2;q^2)_\infty}$$
$$- \frac{q(q^{16};q^{16})_\infty (q^{10};q^{16})_\infty (q^6;q^{16})_\infty (q^4;q^{32})_\infty (q^{28};q^{32})_\infty}{(q^2;q^2)_\infty} \quad (3)$$

Equality (2) follows by separating the values of m into classes modulo 4 and equality (3) follows by applying Watson's quintuple product identity [7] to each series difference. Now if we replace q by $q^{1/2}$ in the first part of (3) and look at what partitions are generated by the resulting function, we get the desired result for $\Delta b_2'(2k)$. Similarly, when we divide the second part of (3) by q and then replace q by $q^{1/2}$ we get the desired result for $\Delta b_2'(2k+1)$.

For p = 3 the theorem is

Theorem 3

(1) $\Delta b_3'(3k) \geq 0$ and equals the number of partitions of k into parts not congruent to 0, 4, 5 (mod 9),

(2) $\Delta b_3'(3k+1) \leq 0$ and $|\Delta b_3'(3k+1)|$ equals the number of partitions of k into parts not congruent to 0, 2, 7 (mod 9), and

(3) $\Delta b_3'(3k+2) \leq 0$ and $|\Delta b_3'(3k+2)|$ equals the number of partitions of k into parts not congruent to 0, 1, 8 (mod 9).

This theorem follows immediately by rewriting the generating function for $\Delta b_3'(n)$.

$$\sum_{n=0}^{\infty}\Delta b_3'(n)q^n=\frac{(q;q)_\infty}{(q^3;q^3)_\infty}=\frac{1}{(q^3;q^3)_\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{3m^2+m}{2}}$$

$$=\frac{1}{(q^3;q^3)_\infty}\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{27k^2+3k}{2}}-\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{27k^2+21k+4}{2}}-\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{27k^2-15k+2}{2}}\right) \quad (4)$$

$$=\frac{(q^{12};q^{27})_\infty(q^{15};q^{27})_\infty(q^{27};q^{27})_\infty}{(q^3;q^3)_\infty}-\frac{q^2(q^3;q^{27})_\infty(q^{24};q^{27})_\infty(q^{27};q^{27})_\infty}{(q^3;q^3)_\infty}$$

$$-\frac{q(q^6;q^{27})_\infty(q^{21};q^{27})_\infty(q^{27};q^{27})_\infty}{(q^3;q^3)_\infty} \quad (5)$$

Equality (4) follows by separating the values of m into classes modulo 3 and equality (5) follows by applying Jacobi's triple product identity [5] to each series. Now if we replace q by $q^{1/3}$ in the first part of (5) and look at what partitions are generated by the resulting function, we get the desired result for $\Delta b_3'(3k)$. Similarly, when we divide the second part of (5) by q^2 and then replace q by $q^{1/3}$ we get the desired result for $\Delta b_3'(3k+2)$ and when we divide the third part of (5) by q and then replace q by $q^{1/3}$ we get the desired result for $\Delta b_3'(3k+1)$.

For $p=5$ the theorem is

Theorem 4

(1) $\Delta b_5'(5k)\geq 0$ and equals the number of partitions of k into parts which can occur in two colors (say red and blue) and must be congruent to 1, 4 (mod 5),

(2) $\Delta b_5'(5k+1)\leq 0$ and $|\Delta b_5'(5k+1)|$ equals the number of 5-regular partitions of k,

(3) $\Delta b_5'(5k+2)\leq 0$ and $|\Delta b_5'(5k+2)|$ equals the number of partitions of k into parts which can occur in two colors and must be congruent to 2, 3 (mod 5), and

(4) $\Delta b_5'(5k+3)=\Delta b_5'(5k+4)=0$ (which is a consequence of Theorem 1).

This theorem follows immediately by rewriting the generating function for $\Delta b_5'(n)$.

$$\sum_{n=0}^{\infty}\Delta b_5'(n)q^n=\frac{(q;q)_\infty}{(q^5;q^5)_\infty}=\frac{1}{(q^5;q^5)_\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{3m^2+m}{2}}$$

$$=\frac{1}{(q^5;q^5)_\infty}\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+5k}{2}}-\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+35k+4}{2}}+\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+65k+14}{2}}\right.$$
$$\left.+\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2-55k+10}{2}}-\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2-25k+2}{2}}\right) \quad (6)$$

$$=\frac{1}{(q^5;q^5)_\infty}\left(\left[\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+5k}{2}}+\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2-55k+10}{2}}\right]-q\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2-25k}{2}}\right.$$
$$\left.-q^2\left[\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+35k}{2}}-\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{75k^2+65k+10}{2}}\right]\right) \quad (7)$$

$$=\frac{1}{(q^5;q^{25})_\infty^2(q^{20};q^{25})_\infty^2}-\frac{q(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty}-\frac{q^2}{(q^{10};q^{25})_\infty^2(q^{15};q^{25})_\infty^2} \quad (8)$$

Equality (6) follows by separating the values of m into classes modulo 5 and equalities (7) and (8) follow by rearranging the sums and applying Jacobi's triple product identity and Watson's quintuple product identity. Now if we replace q by $q^{1/5}$ in the first part of (8) and look at what partitions are generated by the resulting function, we get the desired result for $\Delta b_5'(5k)$. Similarly, when we divide the second part of (8) by q and then replace q by $q^{1/5}$ we get the desired result for $\Delta b_5'(5k+1)$ and when we divide the third part of (8) by q^2 and then replace q by $q^{1/5}$ we get the desired result for $\Delta b_5'(5k+2)$. It should be noted that the results for p = 3 and p = 5 are the limiting cases of the Borwein conjecture [1].

As p increases the proofs get more complicated, so we now state the theorem for p = 7 without proof.

Theorem 5

(1) $\Delta b_7'(7k)\geq 0$ and equals the number of partitions of k where the parts are congruent to ±3 (mod 7) and appear in one color (say red) or the parts are congruent to ±1 (mod 7) and appear in two colors (say red or blue),

(2) $\Delta b_7'(7k+1)\leq 0$ and $|\Delta b_7'(7k+1)|$ equals the number of partitions of k where the parts can be congruent to ±1 (mod 7) and are red or the parts can be congruent to ±2 (mod 7) and are red or blue,

(3) $\Delta b_7'(7k+2)\leq 0$ and $|\Delta b_7'(7k+2)|$ equals the number of 7-regular partitions of k,

Theorem 5 (continued)

(4) $\Delta b_7'(7k+5) \geq 0$ and $|\Delta b_7'(7k+5)|$ equals the number of partitions of k where the parts can be congruent to ±2 (mod 7) and are red or the parts can be congruent to ±3 (mod 7) and are red or blue, and

(5) $\Delta b_7'(7k+3) = \Delta b_7'(7k+4) = \Delta b_7'(7k+6) = 0$.

We do not have to restrict our attention to prime values for p. We can also look at what happens for composite values of p. The following theorem, stated without proof, gives the results for $p = 4$.

Theorem 6

(1) $\Delta b_4'(4k) \geq 0$ and equals the number of partitions of k into parts not congruent to 0, 3, 10, 13, 16, 19, 22, 29 (mod 32),

(2) $\Delta b_4'(4k+1) \leq 0$ and $|\Delta b_4'(4k+1)|$ equals the number of partitions of k into parts not congruent to 0, 1, 14, 15, 16, 17, 18, 31 (mod 32),

(3) $\Delta b_4'(4k+2) \leq 0$ and $|\Delta b_4'(4k+2)|$ equals the number of partitions of k into parts not congruent to 0, 5, 6, 11, 16, 21, 26, 27 (mod 32), and

(4) $\Delta b_4'(4k+3) \leq 0$ and $|\Delta b_4'(4k+3)|$ equals the number of partitions of $k-1$ into parts not congruent to 0, 2, 7, 9, 16, 23, 25, 30 (mod 32).

Several questions should have come to mind as you read this paper—some of which I can answer and others I will leave as open problems.

(1) Is $\Delta b_p'(pk+r) \leq 0$ for all r satisfying $0 < r < p$ when p is composite? The answer is no. For $p = 6$ we have $\Delta b_6'(6k+r) \geq 0$ when $r = 4, 5$.

(2) For the values of p and r satisfying $\Delta b_p'(pk+r) = 0$, can we find an explicit bijection between those p-regular partitions with an even number of distinct parts and those with an odd number of distinct parts? Probably, since an explicit bijection exists for the zero coefficients in the generating function $(q;q)_\infty$ [2].

(3) For each prime $p > 3$, is there a value of r for which $|\Delta b_p'(pk+r)| = b_p(k)$?

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